Where we are: We have described the principal series of representations of $\mathrm{SL}_2(\mathbb{R})$ (the $V(\nu)$ for $\nu \in i\mathbb{R}$), some isomorphisms between them ($V(\nu) \cong V(-\nu)$) and indicated the existence of other series of representations: the complementary series $V(\nu)$ for $\nu \in (0,1)$, and the discrete series $\mathscr{D}_\nu^{\pm}$, for ν an odd integer.

1. Connection to eigenvalues of the hyperbolic Laplacian

How do we tell *which* representations $V(\nu)$ occur in $L^2(\Gamma\backslash\mathrm{SL}_2(\mathbb{R}))$?

Theorem 1. *Let $v_0 \in V(\nu)$ be the function that takes value 1 on the circle S^1. Then the space $\mathrm{Hom}(V(\nu), L^2(\Gamma\backslash\mathrm{SL}_2(\mathbb{R}))$ is canonically isomorphic to the $1/4(1-\nu^2)$-eigenspace of the Laplacian operator $\Delta := -y^2(\partial_{xx} + \partial_{yy})$, acting on $\Gamma\backslash\mathbb{H}$. The map $f \in \mathrm{Hom}(\dots) \mapsto f(v_0)$.*

Note this theorem is valid for $\nu \in i\mathbb{R} \cup (0,1)$, i.e. both for the principal series and the complementary series. Therefore, the question of which principal and complementary series occur within $L^2(\Gamma\backslash\mathrm{SL}_2(\mathbb{R}))$ is substantially equivalent to the question of eigenvalues of the hyperbolic Laplacian on $\Gamma\backslash\mathbb{H}$. The low eigenvalues, i.e. those below 1/4 correspond to complementary series.

The question of which *discrete series* occur in $L^2(\Gamma\backslash\mathrm{SL}_2(\mathbb{R}))$ is closely related to sections of powers of the canonical bundle on the Riemann surface.

Proof. We need to verify that:

(1) If $f \in \mathrm{Hom}(V(\nu), L^2(\Gamma\backslash\mathrm{SL}_2(\mathbb{R}))$, then $f(v_0)$ is an eigenfunction of the Laplacian on $\mathbb{H}$.

(2) Conversely, if φ is an eigenfunction of the Laplacian on $\Gamma\backslash\mathbb{H}$, then $v^0 \to \varphi$ extends to a unique linear isometry $V(\nu) \to L^2(\Gamma\backslash\mathrm{SL}_2(\mathbb{R}))$.

For (1) : Fix $r \geq 0$. Let Av_r be the operator on functions on the upper half-plane defined by "averaging around circles of radius r", i.e. $\mathrm{Av}_r f(z_0)$, for a function f on $\mathbb{H}$, is the average value of f on an r-circle around $z_0 \in \mathbb{H}$. (Here *average* is taken w.r.t. the length measure on this circle).

We claim that, if we set $\varphi = f(v^0)$, $\mathrm{Av}_r\varphi$ is a scalar multiple of φ. To see this, we note that the function

$$\mathrm{Av}_r\varphi = \mathrm{Av}_r f(v^0) = f\left(\int_{k\in K} ka(r).v^0 dk\right) \in V(\nu).$$

Consider $\int_{k\in K} ka(r).v^0 dk \in V(\nu)$. Because this vector is K-invariant, it must be a scalar multiple of v^0 (clearly a function on $\mathbb{R}^2$ that is homogeneous and rotation-invariant is a multiple of v^0!) Thus φ have the following beautiful properties: *they are eigenfunctions of averaging along circles.* (Compare: a harmonic function is one whose average along any circle is zero).

In any case, one can express the Laplacian operator as a limit of the operators Av_r, indeed:

$$\Delta\varphi_i = 4\lim_{r\to 0} \frac{\mathrm{Av}_r\varphi_i - \varphi_i}{r^2}$$

from where it follows that φ_i is also a Δ-eigenfunction (after taking care of suitable issues involving taking the limit). The eigenvalue is computed to be $(1+\nu^2)/4$. [1]

[1] To compute the eigenvalue: let's start with the function $f(x,y) = (x^2+y^2)^{-(1+i\nu)/2}$. We need to compute: $\int_\theta \left(\frac{1}{(e^r\cos^2(\theta) + e^{-r}\sin^2(\theta)}\right)^{-(1+i\nu)/2} d\theta$. Expanding in a power series, we get:

$$\left(1 + r^2/2 + r(\cos^2(\theta) - \sin^2(\theta))\right)^{-(1+i\nu)/2} d\theta$$

For (2): Without loss of generality, $\|\varphi\|_{L^2} = 1$. Because $V(\nu)$ is irreducible, it is spanned (topologically) by vectors of the form gv^0, for $g \in \mathrm{SL}_2(\mathbb{R})$. We shall show that:

$$\langle g_1 v^0, g_2 v^0 \rangle = \langle g_1 \varphi, g_2 \varphi \rangle, g_1, g_2 \in \mathrm{SL}_2(\mathbb{R})$$

which shows that the map $gv^0 \mapsto g\varphi$ gives a G-equivariant isometry from $V(\nu)$ to the closure of the G-span of φ. In order to verify this, it is enough to check the case $g_2 = 1$.

The functions $\langle gv^0, v^0 \rangle$ and $\langle g\varphi, \varphi \rangle$ both take the value 1 at $g = 1$, are bi-invariant under the subgroup K, and (thought of as functions on $G/K = \mathbb{H}$) are eigenfunctions of averaging-under-circles, i.e. eigenfunctions of the hyperbolic Laplacian.

Claim: uniqueness of spherical functions. There exists a unique bounded K-invariant function on $\mathbb{H}$ satisfying $f(1) = 1$ and $\Delta f = 1/4(1-\nu^2)f$. In fact, we can think of as a function of the radial variable. As a function of a single variable, it satisfies a second-order differential equation which has a two-dimensional solution space. One solution blows up (actually as radius goes to zero) and one doesn't. Therefore, the claim.

2. Decay of matrix coefficients and quantitative mixing

We are going to discuss a proof of the following:

Suppose V is a unitary representation of $\mathrm{SL}_2(\mathbb{R})$ that does not contain the identity representation or any representation $V(\nu)$, where $\nu \geq \beta > 0$. (So, if V contains no complementary series, we may take simply $\beta = 0$).

Suppose $f_1, f_2 \in V$. Then:

$$|\langle gf_1, f_2 \rangle| \leq C \log \|g\| \|g\|^{\beta - 1} \|f_1\|_{Sob} \|f_2\|_{Sob} \tag{1}$$

where:[2]

1. C is an absolute constant.
2. The "Sobolev norms" f_1, f_2 are defined by

   $$\|f\|_{Sob}^2 := \|f\|^2 + \|\Theta f\|^2$$

   where Θ is the operator $f \mapsto \frac{d}{d\theta} k(\theta) f$.

(1), together with our previous discussion, implies the quantitative mixing statement we made earlier: simply apply it to $L^2(\Gamma \backslash \mathrm{SL}_2(\mathbb{R}))$. To prove (1), it is enough to prove it for every irreducible constituent of V.

We are going to prove (1) *only for the principal series of representations*, the only one we have discussed in detail. It is somewhat more tricky to prove it for the other series: because, unlike the principal series, the complementary series and the discrete series do not have nice "models" in space of functions (except the Kirillov model).

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which equals $(1 + r^2/2)^{-(1+i\nu/2)}(1 + r^2(\cos^2\theta - \sin^2\theta)^2(1+i\nu)(3+i\nu)/8$ which equals upon θ-integration: $r^2(1+i\nu)(3+i\nu)/16 - r^2(1+i\nu)/4 = r^2(1+\nu^2)/16$. Therefore, Av_r acts on φ by the eigenvalue $r^2(1+\nu^2)/16 + O(r^3)$, which corresponds to Laplacian eigenvalue $(1-\nu^2)/4$.

[2] Clearly (1) could not be true if we replaced $\|f_1\|_{Sob}, \|f_2\|_{Sob}$ by simply $\|f_1\|, \|f_2\|$ — just take $f_2 = gf_1$ for some g very far from the identity to see that.

2.1. **Decay of matrix coefficients for the principal series.** Let $f_1, f_2 \in V(\nu)$. Then: It is enough to prove this in the case when $\|f_1\|_\infty = \|f_2\|_\infty = 1$. In that case, the inner product is bounded by:

$$\frac{1}{2}\int_{\theta \in S^1} |f_1(\theta g)||f_2(\theta)| d\theta \le \frac{1}{2}\int_\theta \left((a\cos(\theta) + b\sin(\theta))^2 + (c\cos(\theta) + d\sin(\theta))^2\right)^{-1/2} d\theta$$

Without loss of generality – because both left- and right- sides of the desired bound are unchanged if we replace g by $k_1 g k_2, k_i \in K$, we may restrict to the case when $b = c = 0$ and $a \ge 1$. The integral becomes:

$$\frac{1}{2}\int_\theta (a^2\cos^2(\theta) + a^{-2}\sin^2(\theta))^{-1/2} d\theta = 2\int_0^1 \frac{dx}{\sqrt{1-x^2}\sqrt{a^{-2} + (a^2 - a^{-2})x^2}} \ll \log(a)/a.$$

Secondly, a nice exercise: if f is a function on $S^1 = \mathbb{R}/\mathbb{Z}$,

$$\sup_x |f(x)|^2 \le C(\int_x |f(x)|^2 + \int_x |f'(x)|^2).$$

This is an example of a Sobolev estimate: average norms of derivatives control pointwise values.

3. An application of quantitative mixing: bounds on the Fourier coefficients of modular forms.

Let $\Gamma \leqslant \mathrm{PSL}_2(\mathbb{R})$. A *holomorphic modular form of weight* k is a holomorphic function $f(z)$ on $\mathbb{H}$ so that $f(z)dz^k$ is Γ-invariant; in explicit terms:

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z).$$

Fact. Suppose $\Gamma\backslash\mathrm{SL}_2(\mathbb{R})$ is cocompact. Then the space of holomorphic (resp. antiholomorphic) modular forms of weight k is canonically isomorphic to $\mathrm{Hom}(\mathscr{D}^{\pm}_{k-1}, L^2(\Gamma\backslash\mathrm{SL}_2(\mathbb{R})))$.

Nice example 1: $f(z) = e^{2\pi i z}\prod_{n=1}^\infty (1 - e^{2\pi i n z})^{24}$ is a modular form of weight 12 for $\mathrm{SL}_2(\mathbb{Z})$.

Nice example 2: If $Q(x_1, \dots, x_r)$ is a quadratic form in r variables, r even, let $r_Q(n)$ be the number of solutions to $Q(\mathbf{x}) = n$. Then $\sum_{n \ge 0} r_Q(n) e^{2\pi i n z}$ is a modular form of weight $r/2$ for some subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

Suppose Γ contains the element $n(1)$ (e.g. $\mathrm{SL}_2(\mathbb{Z})$). Then any holomorphic modular form satisfies $f(z) = f(z+1)$. Therefore, we may express:

$$f(z) = \sum_{n=0}^\infty a_n e^{2\pi i n z}, \quad a_n \in \mathbb{C}.$$

We say f is a *cusp form* if $y^{k/2}|f(z)| \to 0$ when $z \to \infty$ in $\Gamma\backslash\mathbb{H}$.

We are going to prove, by applying our quantitative mixing theorem assiduously,

Theorem 2. *There exists $\delta > 0$, depending on Γ, so that*

$$|a_n| \le C_f n^{k/2 - \delta}.$$

This theorem was proven in special cases by Kloosterman/Petersson/Rankin-Selberg; and in the general case (Γ possibly nonarithmetic) by Good. In the arithmetic case, it is maybe the first "interesting" result in the analytic theory of modular forms.

In combination with *Nice example 2* it leads to the following result:

Theorem 3. *Let* $Q(x_1, \ldots, x_r)$ *be a quadratic form in* $r \geq 4$ *variables (the hard case is* 4*). Then, for* n *big enough,* $r_Q(n) \geq 1$ *if and only there is no congruence obstruction to a solution to* $Q(\mathbf{x}) = n$*.*

This is a vast generalization of the "four squares theorem" of Legendre, of which we will thus have given a "dynamical" proof.