The goal of the course is to discuss two key fields where ideas from ergodic theory have impacted number theory, and the connection between them. I will not assume background beyond basic understanding of measure theory and of manifolds. Unfortunately there is at present no satisfactory slogan as to why ergodic methods have had such importance.

The style of the course will be to give complete proofs in special cases and sketches of proofs, with references, in the general case.

Szemeredi's theorem asserts that, given a subset $S \subset \mathbb{Z}$ with positive upper density, there exist arithmetic progressions of arbitrary finite length contained in S. There are three different proofs, using ideas from graph theory (Szemeredi) ergodic theory (Furstenberg) and harmonic analysis (Gowers). A remarkable fact is that they exhibit the same internal structure. We shall focus on the ergodic approach.

Take irrational α and consider the sequence of fractional parts $\{P(n)\alpha\}_{n\in\mathbb{N}}$, where P is a polynomial with integral coefficients. Weyl showed that this is dense in $[0, 1)$ (and a little bit more). Ratner's theorem is a far-reaching *nonabelian* generalization of that statement.

There is a deep connection – the "nilmanifolds theorem" – between Szemeredi's theorem and a particular (much easier) special case of Ratner's theorem. This has been established by Host/Kra, Ziegler, and Green/Tao in different aspects. To see a hint of it consider the set $S = \{n : \{n^2\alpha\} \in [0.1, 0.2]\}$. If x_1, x_2, x_3, x_4 are in arithmetic progression, then $x_1^2, x_2^2, x_3^2, x_4^2$ satisfy a nontrivial linear constraint. This means that the set S behaves "peculiarly" from the point of view of four-term arithmetic progressions, and indeed such sets S play an important role in Szemeredi's theorem.

Syllabus:

(1) Roth's theorem.
(2) Proof of Szemeredi's theorem (after Furstenberg).
(3) Generalizations without proof:
    - density Hales-Jewett theorem, Bergelson-Liebman theorem;
    - the characteristic factors theorem of Host/Kra and Ziegler
    - the quantitative inverse theorem and "higher Fourier analysis" of Gowers and Green-Tao

    An open question or two.
(4) Ratner's thoerem.
(5) Proof of Ratner's theorem in a special case (after Einsiedler, Witte)
(6) Applications of Ratner's theorem to arithmetic and analytic number theory.
(7) Some more open questions.