Notes on Computing with Transfinite Numbers

We try here to collect and summarize the results in sections 3-6 pertaining to computation with transfinite cardinals.

1. Many of the usual laws of algebra apply. In particular we have the commutative and associative laws for addition and multiplication. Thus, we have
   (a) \( a(b + c) = ab + ac \) (The distributive law.)
   (b) \( a^b a^c = a^{b+c} \)
   (c) \( (a^b)^c = a^{bc} \)
   (d) \( (ab)^c = a^c b^c \)

2. (a) Facts about \( \aleph_0 \):
   i. \( \aleph_0 + \aleph_0 = \aleph_0 \).
   ii. \( \aleph_0 \cdot \aleph_0 = \aleph_0^2 = \aleph_0 \).
   iii. \( \aleph_0 < C = 2^{\aleph_0} \).
   (b) Facts about \( C = 2^{\aleph_0} \):
   i. \( C + C = CC = C \)
   ii. \( \aleph_0 C = C \).
   (c) Facts about exponents:
      i. (\( \aleph_0 \) as exponent) If \( 2 \leq n \leq C \), we have \( n^{\aleph_0} = C \).
      11. (\( C \) as exponent) If \( 2 \leq n \leq C \), we have \( n^C = 2^C \).

3. A denumerable union of finite sets is finite or denumerable.

4. Adding and subtracting denumerable sets:
   (a) If \( a \) is transfinite, then \( a + \aleph_0 = a \).
   (b) If \( A \) is an infinite set, and \( D \subseteq A \) is denumerable, and \( A - D \) is infinite, then \( |A - D| = |A| \).

5. Preservation of non-strict inequalities. If \( a \leq b \) and \( c \leq d \), then
   \[
   a + c \leq b + d, \quad ac \leq bd, \quad a^c \leq b^d.
   \]

6. The transitive law: For cardinal numbers \( a, b, c \), if \( a \leq b \) and \( b \leq c \), then \( a \leq c \).

7. If \( a \leq b \) and \( b \leq a \), then \( a = b \). (The CBS theorem.)