Lecture 4

The Logistic Map \( G(x) = 4x(1-x) \)

Q. How many periodic orbits are there?

\( G(x) = 1 \ overwhelms \), \( G(1) = 0 \), \( G^2(1) = 0 \), \( G^3(1) = 1 \), \( G^4(1) = 4 \), \( G^5(1) = 6 \), \( G^6(1) = 10 \), \( G^7(1) = 16 \), \( G^8(1) = 25 \).

\( G^4(x) = 2 \) fixed points.

\( G^8(x) = 4 \) fixed points, 2 of which are fixed points of \( G^4(x) \).

2 period-2 points \( \rightarrow \) 1 period-2 orbit \( \{ p_1, p_2 \} \).

\( G^8(x) = 8 \) fixed points, 2 of which are fixed points of \( G^4(x) \).

\( G^2(p_1) = p_1 \), \( G^2(p_2) = C(p_1) = p_2 \).

\( \therefore p_1, p_2 \) are fixed points of \( G^8(x) \).

\( \therefore 6 \) period-3 points \( \rightarrow \) 2 period-3 orbit.

\( \therefore G(x) \) has periodic orbits of any period. (Sarkovskii Theorem)
Symbolic Dynamics — investigation of sensitive dependence on the initial data

Symbolic encoding of an orbit:

orbit \( x_1, x_2, \ldots \)

\( x_1, \text{ say } x_1 \in [0, \frac{1}{2}], \quad x_2 \in [0, \frac{1}{2}], \quad x_3 \in [\frac{1}{2}, 1], \quad \ldots \)

\( LLR \ldots \) — an infinite seq

i.e. assign L to any \( x_i \in [0, \frac{1}{2}] \)

R to any \( x_i \in [\frac{1}{2}, 1] \)

\( x_0 = \frac{1}{4} \), the orbit is

\( \frac{1}{4}, \frac{3}{4}, \frac{2}{4}, \frac{1}{4}, \ldots \)

\( LRRLRLRLRL\ldots \equiv LR \)

Note: Coding ambiguity: \( x = \frac{1}{2} \), orbit \( \frac{1}{2}, 1, 0, 0, \ldots \)

\( \text{coded as either } RRL \text{ or } LRL \)

\( \Rightarrow \) once an orbit lands on \( x = \frac{1}{2} \), a possible coding

(But it eventually maps to the fixed pt 0)

\( \Rightarrow \) becoming trivial.

\( \Rightarrow \) Except for these orbits, the iterates are uniquely defined.
Example: What is the set of points whose itineraries begin with e.g. LR?

The main pt: to identify the sets of all initial pts whose itineraries begin with a specific sequence of symbols.

In the case of LR, find all pts starting in the L-subinterval which are mapped to the R-subinterval after one iterate.

1. \( x \in [0, a_1] \) in the L-subinterval, mapped to \([0, \frac{1}{2}]\) after one iterate, which is the L-subinterval.

\[ \Rightarrow \forall x \in [0, a_1] \text{ has itineraries beginning with LL}. \]

2. Similarly, \( \forall x \in (a_1, \frac{1}{2}) \Rightarrow LR \),

\[ \forall x \in (\frac{1}{2}, a_2) \Rightarrow RR \]

\[ \forall x \in (a_2, 1) \Rightarrow RL \]
Q: What is the set of distinct iterates of any sequence of symbols $S_i$ = $L$ or $R$, $1 \leq i \leq k$?

Note the order of $R, L$ after the initial segment.

R: e.g. $LR$

<table>
<thead>
<tr>
<th>a₁</th>
<th>1/2</th>
</tr>
</thead>
<tbody>
<tr>
<td>b₂</td>
<td>1</td>
</tr>
</tbody>
</table>

Note the order of $R, L$ after the initial segment.

R: e.g. $RR$

<table>
<thead>
<tr>
<th>a₂</th>
<th>1/2</th>
</tr>
</thead>
<tbody>
<tr>
<td>b₂</td>
<td>1</td>
</tr>
</tbody>
</table>

Note the order of $R, L$ after the initial segment.

R: e.g. $RRR$

| a₂ | 1/2 |

For every $R$, the order of the interval is switched.

The rules

1. An interval ending in $L$ (or $R$) is divided into 2 subintervals ending in $LR$ and $L.R$, of three (or $RL$) (or $RR$) are an even number of $R$'s.

2. The order is switched if there are an odd number of $R$'s.
Q: What kind of itineraries is possible for the logistic map?

Transition Graph:

\[ \text{L} \rightarrow \text{R} \]

i.e. \( \text{L} \rightarrow \text{R} \) means the image of \( \text{L} \) contains the interval \( \text{R} \)

No. 1. For every path through this graph by directed edges, there exists an orbit with an itinerary satisfying the sequence of symbols determined by the path.

No. 2. For \( G(\alpha) \), the graph is fully connected, i.e. all possible sequences of \( \text{L} \) and \( \text{R} \) are allowed.

Q: Why is the symbolic coding easy to demonstrate the sensitive dependence on the initial conditions?

No. 1. There are \( 2^k \) choices to specify the first \( k \) symbols of an itinerary.

No. 2. When \( k \) is large, most of the \( 2^k \) subintervals are short. (No they are summed up to \( \), later we will show that any of them is bounded by \( \frac{\pi}{2k} \) in length)
3. For any such subinterval, whose symbols are, $S_1 ... S_k$

Note this subinterval contains subintervals $S_1 ... S_{k-1}, S_1 ... S_k L, S_1 ... S_k R, S_1 ... S_k RL$

4. After $k$ iterations, the orbit now stands at

$LL, LR, RR, RL$

$\Rightarrow$ sufficiently close initial pts can be mapped far apart. (e.g. the width $LR RR > \frac{1}{4}$)

Q1. Does every pt in $[0,1)$ has this sensitivity under $G(x)$?

Q2. Actually every pt in $[0,1)$ has sensitive dependence on initial data

i.e. By locating which subintervals of level $k+2$ that $x$

is in, i.e. in one of $S_1 ... S_{k-1}$

i.e. in one of $LL, LR, RR, RL$

i.e. in one of $S_1 ... S_k L, S_1 ... S_k R, S_1 ... S_k RL$

We can find a relevant $k+2$ level subinterval to

gap its pts (which are within $\frac{\epsilon}{2^{k+1}}$ of $x$) mapped $\frac{1}{4}$ away from $x$. 
For both the shift-map and logistic map,

- They have associated symbol dynamics on symbols \( \mathbb{Z}^2 \) or \( \mathbb{Z}, \mathbb{R}^2 \).

- The dynamics on symbols is 'shift and chop off the head.'

The Tent Map

\[
T(x) = \begin{cases} 
3x & x \in [0, \frac{1}{3}] \\
2(1-x) & x \in \left[\frac{1}{3}, 1\right]
\end{cases}
\]

1° Symbolic dynamics

- All subintervals of the same level are of equal length.

- For \( S_1, \ldots, S_k \)

2° Of orbits containing \( \frac{1}{2} \) are ignored, then the orbit of \( T(x) \) are in 1-1 correspondence with infinite sequences of 2 symbols.
Lagrange exponent of \( T(x) \)

\[
\lambda = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \ln |T'(x)| = \ln 2 > 0
\]

for those orbits never mapped to \( x = \frac{1}{2} \).

Q: Can we characterize an orbit as chaotic? (e.g. of periodic orbit)

1. **Asymptotically periodic**

   If an orbit \( x_0, x_1, x_2, \ldots \) converges to a periodic orbit as \( n \to \infty \),

   i.e. \( \exists \) a periodic orbit \( y_0, y_1, y_2, \ldots \), such that

   \[
   \lim_{n \to \infty} |x_n - y_n| = 0
   \]

   e.g. Any orbit attracted to a fixed point is asymptotically periodic.

2. **Eventually periodic**

   A path \( x \) is eventually periodic if there exists an integer \( N \) such that

   \[
   f^{n+p}(x) = f^n(x) \quad \text{for all } n > N
   \]

   and \( p \) is the smallest such positive integer.

   e.g. \( g(x) = 4x(1-x) \), \( x_0 = \frac{1}{2} \to 1 \to 0 \).
Chaotic orbits.

Def: An orbit \( x_n \) is chaotic if
1. It is not asymptotically periodic
2. The Lyapunov exponent \( \lambda(x_n) > 0 \).

Examples
- For the Bernoulli Shift:
  \[ \lambda > 0 \text{ for all orbits of } \sigma(x). \]
  \[ \text{all periodic orbits of } \sigma(x) \text{ are unstable} \]
  \[ \Rightarrow \text{all asymptotically periodic orbits are eventually periodic} \]
  \[ \text{any } x \text{ in } [0, 1) \text{ whose binary expansion is not eventually repeating generates a chaotic orbit.} \]

- Tent Map: \( T(x) \)
  1. \( \lambda = \ln 2 > 0 \) for any orbits avoiding \( \frac{1}{2} \).
  2. All periodic orbits are unstable.
\[ \Rightarrow \text{attracting no orbits.} \]
\[ \Rightarrow \text{Asymptotically periodic orbits of } T(x) \text{ are eventually periodic.} \]
8. An eventually periodic orbit has an eventual repeating itinerary.

There are infinitely many nonrepeating itineraries which are distinct chaotic orbits of \( T(x) \).

**Topological Conjugacy**

\[
\text{Def. The maps } f \text{ and } g \text{ are topological conjugate if } \quad f = C^{-1} \circ g \circ C
\]

for a continuous, 1-to-1 map \( C \). (Conjugacy)

For \( T(x) \) and \( G(x) \), the conjugacy is

\[
C(x) = \frac{1}{2} \left( 1 - \cos(\pi x) \right)
\]

In \( x \in [0, 1] \), \( C(x) \) is a continuous, 1-to-1.
HW. Verify \( C(x) = \frac{1}{2} (1 - \cos(x)) \) is a conjugacy \( f(x) \) \( T \circ C \circ G(x) \).

The meaning of conjugacy:

\[
\begin{array}{ccc}
\alpha & \xrightarrow{G} & G(\alpha) \\
\downarrow{C^{-1}} & \searrow{C} & \uparrow{C} \\
C^{-1}(\alpha) & \xrightarrow{T} & TCC^{-1}(\alpha)
\end{array}
\]

\( \therefore \) \( G \circ C = C \circ T \)

\[ z_{n+1} = T(z_n) \]

\[ y_{n+1} = C(T(z_n)) \]

\[ y_{n+1} = C(G(y_n)) \]

Properties of conjugacy

1. If \( \alpha \) is a fixed pt of \( f(x) \), then \( C(x) \) is a fixed pt of \( g(x) \).

\[ g(C(x)) = C(f(x)) = C(x) \]

\( \therefore \) \( g \circ C = C \circ f \)
2. If $x$ is a periodic $k$ pt of $f(x)$, then $C(x)$ is a periodic $k$ pt of $g(x)$, 

$$ f^k(x) = x, $$

**Note:**

$$ g^n = \underbrace{C \circ C^{-1} \circ \ldots \circ C^{-1}}_{n \text{ times}} = C f^n C^{-1} $$

$$ \therefore \quad g^n C = Cf^n $$

$$ \therefore \quad g^n(C(x)) = C(f^n(x)) = C(x) $$

so if $x$ is a periodic $k$ pt for $f$, then $C(x)$ is a periodic $k$ pt for $g$. If $C'(x)\neq 0$ on the periodic orbit of $f$, then

$$ (g^k)'(C(x)) = (f^k)'(x) $$

i.e. they have the same stability.

**Note:** For fixed pt:

$$ f^k = g $$

$$ \therefore \quad C(f(x)) = C(g(x)) $$

$$ \therefore \quad C'(C(x)) f'(x) = g'(C(x)) C'(x) $$

For $x$ fixed pt $\Rightarrow$

$$ C'(x) f'(x) = g'(C(x)) C'(x) $$

$$ \therefore \quad f'(x) = g'(C(x)) $$

Similarly, for higher periods.
**NB:** All periodic pts of the logistic map $G$ are unstable.

**Prf:** First exclude pts s.t. $C'(x) = 0$

$\Rightarrow x = 0, \quad \text{a fixed pt of } G(x), \quad |G(0)| = 4 > 1$

$\Rightarrow x = 0$ unstable.

For periodic orbits: $(G^k)' = 2^k$ at corresponding pts.

$\therefore (G^k)' = 2^k$

$\therefore$ all periodic orbits of $G(x)$ are unstable.

**NB:** $(G^k)' = 2^k$ is true for periodic orbits, not generally true.

---

**Invariant Measure of $G(x)$**

\begin{align*}
\tau(x): & \quad p_k(x) = 1. \\
& \quad p_a(y)dy = p(x)dx \\
\text{Let: } & \quad \frac{dy}{dx} = \frac{\frac{dy}{dx}}{1-y} \\
& \quad = \frac{\frac{dy}{dx}}{1-y} \\
& \quad = \frac{\frac{dy}{dx}}{1-y} \\
& \quad \text{No. } \sigma = \frac{1}{2} \left( -\frac{1}{2}, \frac{1}{2} \right) = \frac{\text{area of}}{2} = \frac{\text{area of } G}{2} \quad \therefore \frac{\text{area of } G}{2} = 1 - \frac{1}{2}
\end{align*}
\[ \lambda = \int_0^1 dx \left( \alpha(x) \ln |\alpha(x)| \right) \]
\[ = \int_0^1 dx \frac{1}{\sqrt{2x(1-x)}} \left( \ln 2 - 8x \right) \frac{1}{2} \]
\[ = \ln 2 > 0 \]

More detailed discussion of chaotic orbits of \( G(x) \) using \( \mathcal{C}(x) \):

Congruency of itineraries for \( T(x) \) and \( G(x) \):

1. The subintervals of \( G \) for its itineraries is the images of the tent map subintervals for the corresponding itinerary under \( C \).

Proof: The endpoints of the subintervals are those to \( x^* \) which are eventually mapped to \( x = \frac{1}{2} \).
i.e. \( \exists \text{ some integer } M, \ s.t. \ T^n(x^*_T) = \frac{1}{2} \)

\[
C^nT^n(x^*_T) = C_{C^2}
\]

\[
C^{n-1}C^n(x^*_T) = C_{C_{C^2}}
\]

\[
G^*(C^2_{C_{C^2}}) = C_{C_{C^2}}^{\frac{\pi}{2}}
\]

\[
= \frac{1}{2}(1 - \cos \frac{\pi}{2})
\]

\[
= \frac{1}{2}
\]

\[\therefore \text{ The image of the endpoints for the sub-interval for } T
\]

\[\text{ is the endpoints of } G \text{ sub-intervals.} \quad \text{Q.E.D.}
\]

2. The upper bound for the level } k \text{ sub-interval of } G \text{ is } \frac{\pi}{2^{k+1}}.

\[
\text{Pf:} \quad \text{The level } k \text{ sub-interval for } T \text{ has length } 2^{-k}.
\]

\[
[\alpha, \beta] \text{ is one of such intervals.}
\]

\[
\therefore \text{ The corresponding sub-interval for } G(x) \text{ is } [C\alpha, C\beta].
\]

The length of \([C\alpha, C\beta]\) is

\[
C(\beta) - C(\alpha) = \int_{\alpha}^{\beta} C(x) \, dx = \int_{\alpha}^{\beta} \frac{x}{2} \sin \pi x \, dx
\]

\[
\leq \frac{\pi}{2} \int_{\alpha}^{\beta} x \, dx = \frac{\pi}{2} \cdot 2^{-k} = \frac{\pi}{2^{k+1}}
\]
Lyapunov exponent of \( G \):

\[
C T = G C \quad \Rightarrow \quad C'(c(x)) T'(x) = G'(c(x)) C'(x)
\]

For an orbit \( x_{0}, x_{1}, \ldots, x_{k}, \ldots \) of \( T(x) \),

\[
T'(x_{k}) \cdots T'(x_{2}) T'(x_{1}) = \frac{G'(c(x_{k})) C'(x_{k})}{C'(x_{k+1})} \cdots \frac{G'(c(x_{2})) C'(x_{2})}{C'(x_{3})} \frac{G'(c(x_{1})) C'(x_{1})}{C'(x_{2})}
\]

\[
= \frac{G'(c(x_{k}))}{C'(x_{k+1})} G'(c(x_{k-1})) \cdots G'(c(x_{1})) C'(x_{1})
\]

Obviously, \( \frac{\ln |C'(x_{k})|}{k} \to 0 \) as \( k \to \infty \), depending on \( k \).

But \( \frac{\ln |C'(x_{k+1})|}{k} \to 0 ? \)

:. For those orbits \( \lim_{k \to \infty} \frac{\ln |C'(x_{k+1})|}{k} = 0 \), we have

\[
\lim_{k \to \infty} \sum_{k=0}^{k} \frac{1}{k} \sum_{\alpha=0}^{k} \frac{\ln |G'(c(x_{\alpha}))|}{k} = \ln 2 + \lg \alpha
\]
The logistic map \( f(x) \) has chaotic orbits.

\[ f(x) = \frac{1}{2} \ln \frac{1}{x} \]

**Proof:**

1. To show there are orbits forbidden

\[ \frac{1}{k} \ln |C(x_k)| \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty \]

1° NB: if the orbit of \( x \) never has \( 1 \) in its itinerary, then the orbit never enters \([0, \frac{1}{2}] \) and \([ \frac{1}{8}, 1 \]

\[ 4 \chi \text{ outside these two intervals,} \]

\[ \frac{1}{8} \leq \chi \leq \frac{7}{8} \quad \quad C'(\chi) = \text{fixed} \]

\[ \frac{\pi}{8} \leq \left| C'(\chi) \right| \leq \frac{\pi}{2} \]

i.e. \( \text{faded away from 0,} \)

\[ \frac{\ln \left| C'(x_k) \right|}{k} \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty \]

\[ \chi(x) = \ln 2 > 0 \quad \text{for } \chi. \]

2° NB: There are many such kinds of orbits,

if the orbit of \( T \) never has a sequence of \( \text{an consecutive L's in its itinerary,} \)

then, the orbit never enters \([0, 2^{-m}] \) or \([1-2^{-m}, 1]\)

\[ \text{for some } m \in \mathbb{N}. \]
Again: \[
\lim_{k \to \infty} \frac{1}{k} \ln |C(x_k)| = 0
\]

\[\therefore \lambda_G = \ln 2.\]

\[\therefore \text{all periodic orbits of } G \text{ are unstable.} \Rightarrow \text{attracting no orbit} \Rightarrow \text{asymptotically periodic} \rightarrow \text{periodic}\]

\[\therefore \text{Any orbit whose itinerary is not eventually repeating and which contains no } m \text{ consecutive } L \text{'s has} \]

\[\lambda_G = \ln 2 > 0.\]

\[\therefore \text{The logistic map } G \text{ has chaotic orbits.}\]

Note: Does the logistic map have a dense chaotic orbit in the unit interval?

Consider \[RL \overline{RRR} \overline{RL} \overline{LR} \overline{LR} \overline{RRR} \overline{RRL} \ldots\]

This orbit is

1. not eventually periodic, not asymptotically periodic \((G\) has no stable periodic orbit\)

2. it enters into any subinterval of level \(-k\) the length < \(\frac{R}{2^{k+1}}\)

\[\therefore \text{This orbit is a dense orbit in } [0, 1].\]
3. \( \lambda > 0 \).

By construction, one leg of each consecutive \( \xi \)'s occurs before the \( 2^n \)-th symbol of the leg. 

\( \therefore \) \( x_k \) does not exist in \( [0, 2^{-m}] \) or \( [1 - 2^{-m-1}, 1] \) for \( k < 2^m \).

\[ \frac{\pi}{2} \operatorname{min} \frac{\pi}{2^{m+1}} \leq |C'(x_k)| \leq \frac{\pi}{2} \quad (C'(x) = \frac{x}{2} \sin(x)) \]

\[ \frac{\ln \frac{\pi}{2} + \ln \frac{\pi}{2^{m+1}}}{2^m} \leq \frac{\ln |C'(x_k)|}{k} \leq \frac{\ln \frac{\pi}{2}}{k} \]

\[ \lim_{k \to \infty} \frac{1}{k} \ln |C'(x_k)| = 0 \]

\[ \therefore \text{For this orbit, } \lambda = \ln 2 > 0 \]

\[ \therefore \text{This orbit is a dense, chaotic orbit in } [0, 1]. \]

**HW.** Suppose we partition \([0, 1]\) into 2 arbitrary subintervals for the tent map, \( \begin{array}{c}
L \\
R
\end{array} \) (i.e. \( L \) is not associated \([0, 1/2]\) )

Can you construct a similar theory based on these partitions? Push your theory as far as possible.