Resonalization Group & Universality

- The logistic map:
  \[ x_{n+1} = \alpha x_n (1 - x_n) \quad (\equiv f(x_n)) \]

- More general:
  \[ x_{n+1} = f(x_n) \]
  - \( f(x) \) has a single max
  - \( x \in [0, 1] \)

- Bifurcation:

- Approach

- N.b.
  \[ \lambda > 0 \text{ vs } \lambda < 0 \]
  - Stable systems are stable.
  - All else are unstable.
  - \( \lambda = 0 \) at \( a_1, a_2 \ldots \)
  - For \( a = 4 \),
    \[ a = \log 2 \quad \text{for the logistic map.} \]
Figure 12.14 Universality in periodic windows.

The complete bifurcation diagram for the map \( g_a(x) = x^2 - a \) is shown above, while the bifurcations for \( g_a \) within a period-nine window are shown below. In the top diagram, \( x \) goes from \(-2.0 \) to \( 2.0 \) and \( a \) is between \(-0.25 \) and \( 2.0 \), while in the bottom diagram \( x \) goes from \(-0.02 \) to \( 0.02 \) and \( a \) is between \( 1.5552567 \) and \( 1.5554906 \).
Numerical observations:

1. Periodic regime: $A_0 = \text{the 4th fixed point changing from}$

\[ 2^n \rightarrow A_1 \rightarrow 2^n \]

\[ \Delta x - \Delta x_0 = -\alpha \Delta x \cdot 5^{-n} \text{ for } n \gg 1 \]

\[ \Delta x = 4.6692016091 \ldots \]

\[ A_0 - \Delta x_0 = c \cdot 5^{-n} \quad c \text{ - Const.} \]

\[ A_0 = A_0 = 3.5699456 \]

2. The distance $\Delta x_1$ of the 3rd fixed point in a $2^n$ cycle that are closest to $x = \frac{1}{2}$ has the scaling:

\[ \frac{\Delta x_n}{\Delta x_{n+1}} = -\alpha \text{ for } n \gg 1 \]

\[ \alpha = 2.5029078750 \ldots \]
Pitchfork Bifurcations

Stability

1) \( a < 1 \), \( x^* = 0 \) — stable fixed pt.
   — only one fixed pt.
2) \( 1 < a < 3 \) \( x = 0 \) unstable fixed pt.
   \[ x^2 = 1 - \frac{1}{a} \] — stable fixed pt.
3) \( a \geq 3 \), \( |f'(x^*)| = |a - 1| > 1 \)
   \[ x^* \] — unstable.
   \[ \therefore a_1 = 3 \]

Now look at \( f^2 = f(f(x)) \), \( f^4 \), ...

For \( a > a_1 \), \( f^2 \) has the following properties:

(a) \( f^2 \) has 3 extrema,
   \[ (f^2)' = 0 \] at \( x_0 = \frac{1}{2} \), \( x_1 = x_2 = f^{-1}(3) \)
   \[ \therefore f'(f(x)) f'(x) = 0 \]
   \[ x_0 = \frac{1}{2} \] makes \( f(x) = 0 \)
   \[ \text{at } x = f^{-1}(3), \quad f'(f(x)) = f'(f^{-1}(3)) = 0 \]
(c) A fixed point $x^*$ of $f(x)$ is also a fixed point of $f^2, f^3, \ldots$

(d) If a fixed point $x^*$ becomes unstable for $f^1$, it is also unstable for $f^2, f^3, \ldots$

\[ |f^2(x^*)| = |f(f(x^*))f(x^*)| = |f'(x^*)|^2 > 1 \]

\[ a < a_1 \]

\[ a = a_1 \]

\[ a > a_1 \]

At $a > 3$, two new stable fixed points $\bar{z}, \bar{z}_1$ are created via a pitchfork bifurcation (the old fixed point becomes unstable).

Any sequence of iterates starting from $x \in (0, 1)$ attracted to $\bar{z}, \bar{z}_1$

(Actually, these three fixed points are close) $\bar{z}_i = f(\bar{z}_{i-1}) \quad \bar{z}_i = f^i(x)$
They merge onto each other under $f^2$.

Any sequence will converge to an oscillating

iteration $\bar{x}_i \to \bar{x}_i' \to \bar{x}_i'' \to \bar{x}_i'^{'''} \to \ldots$

$\Rightarrow$ an attractor of period $2^{\infty}$.

(5) $A > A_2$, the fixed points of $f^2$ become unstable.

$A^* : f^2(A^*) = f'(A^*) f(A^*) = f(A^*) f(f(A^*)) = f^2(A^*)$

both become unstable at the same value of $A$.

(6) After $A > A_2$, $f^2 = f^2 \circ f^2$ creates

2 more pitchfork bifurcations.

$\Rightarrow$ Period doubling phenomena.

(7) $A_n < A < A_{n+1}$. $f$ a stable $2^n$-cycle at

a periodic orbit: $A_0, A_1, \ldots A_{2^n-1}$

such as $f_{A_n}(A_0^*) = A_1^*$, $f_{A_n}(A_1^*) = A_2^*$

Moreover:

$\left| \frac{1}{f_{A_n'}(A_0^*)} \frac{d}{dx} f_{A_n'}(A_0^*) \right| = \left| f_{A_n}(A_0^*) f_{A_n}(A_1^*) \cdots f_{A_n}(A_{2^n-1}^*) \right| < 1$
(2) At $\alpha_n$, all $\alpha$ of the $2^n$-cycle become unstable simultaneously via pitchfork bifurcations.

\[ \Rightarrow \text{ a new stable $2^n$-cycle is created.} \]

**Supercycles.**

**Stability:** i.e. $f'(x^*) = 0$. Let a fixed pt $x^*$

A $2^n$ supercycle is a stable $2^n$-cycle: i.e.

\[ \frac{d}{dx} f^{2^n}(x^*) = f'(x^*) f'(x^*) \cdots f'(x^*) = 0 \]

\[ \therefore \quad \alpha_0^* = \frac{1}{2} \text{ is always a pt within periodic orbit.} \]

\[ \therefore \quad f_2'(\frac{1}{2}) = 0 \]

**1B:**

\[ d_n = \left[ f^{2^{n-1}}(\frac{1}{2}) - \frac{1}{2} \right] \]

\[ x_0 = f^{2^n}(\frac{1}{2}) \]

**HW:** Show that $d_n = f^{2^{n-1}}_\alpha(\frac{1}{2}) - \frac{1}{2}$, i.e. the closest point to $x = \frac{1}{2}$ in $f^{2^n}_\alpha$ at $x = \frac{1}{2}$. \[ \text{at } x = \frac{1}{2} \text{ in } f^{2^n}_\alpha \]
Coordinate transform: $x \rightarrow x - \frac{1}{2}$, shifting $z = \frac{1}{2}$ to $x = 0$.

Thus: $A_n \rightarrow f^{2^{n-1}}(z)$

$A_1 = A_1$

$A_2 = A_2$

Note that the dashed square looks gizm like the big square for $f$. (inverted and resized)

No: $ap = s \underset{i}{\overset{f^2}{\in}} f$.

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Rescaling: $\lim_{n \to \infty} (-\alpha)^n d_{n+1} = d_1$ (if we stipulate $n \gg 1$ for any $\alpha$).

i.e. $\lim_{n \to \infty} (-\alpha)^n f^{2^n}(z) = d_1$

For the whole interval, this fun can be generalized to

$\lim_{n \to \infty} (-\alpha)^n f^{2^n} \left( \frac{x}{A_{n+1}, x} \right) = g_1(x)$

No: $g_1(x)$ is determined by $f^{2^n}$ around $x = 0$, universal for all $f$ in gradient max.
Renormalization / Self-similarity.

Logistic map shifted by $x \rightarrow x - \frac{1}{2}$, rescaling to make $x$ have coefficient $1$.

$f_r(x) = r - x^2$

$R_0$: at $R_0$ the map has a superstable fixed pt.

$R_0 = x^* = x^*$, \quad \frac{df}{dx} \bigg|_{x^*} = 0 \quad \Rightarrow \quad x^* = 0, \quad R_0 = 0$

$R_1$: at $R_1$ the map has a superstable fixed pt. \quad \Rightarrow \quad R_1 = 1.$

Self-similarity

$-\alpha f_r^2 \left( \frac{x}{R_1} \right)$

$\alpha \approx -2.5...$
Approximate similarity

This process can continue to \( f^n \) ...

\[
\frac{\partial}{\partial \alpha} f_{R_1}(\alpha) \approx \frac{\partial}{\partial \alpha} f_{R_2}(\alpha)
\]

\[
f_{R_0}(\alpha) \approx \alpha^n \int_{R_n} f_{R_n} \left( \frac{\alpha}{\sqrt[2n]{\alpha}} \right)
\]

\[
\lim_{n \to \infty} \alpha^n \int_{R_n} f_{R_n} \left( \frac{\alpha}{\sqrt[2n]{\alpha}} \right) = f_0(\alpha)
\]

--- a universal fun at a supersolvable pt.

For \( g(x) \) to exist \( \Rightarrow \) numerically \( x \approx -2.5029 \ldots \)

FergoBaum

\( g(x) \) --- universal, i.e. independent of \( f \) (not a quadratic max)

We can start at \( f_{R_i}(\alpha) \) instead of \( f_{R_0}(\alpha) \)

\[
f_i(x) = \lim_{n \to \infty} (-\alpha^n) \int_{R_n} f_{R_n} \left( \frac{x}{\sqrt[2n]{\alpha}} \right)
\]

\( f_i(x) \) --- a universal fun at supersolvable \( x \)-cycle.
Baby Renormalization.

\( f_\mu(x) \) — an unimodal map that undergoes a period-doubling route to chaos.

Coordinates:

Suppose the period-2 cycle is created at \( \alpha = 0 \) when \( \mu = 0 \). Then \( x, \mu \) — close to \( 0 \).

\( \mu < 0 \)

\( S_\mu = \begin{cases} \frac{\mu x}{2} + 1, & x > 2 \mu \theta \\ \frac{\mu x}{2} & x \leq 2 \mu \theta \end{cases} \)

\( \Rightarrow 15'(\infty) < 1 \)

\( \mu = 0 \), the fixed point \( x = 0 \) coming to be unstable.

\( \frac{\partial S_\mu}{\partial x} \bigg|_{x=0} = - (1 + \mu) \)

Where, rescale \( x \rightarrow \frac{x}{\alpha} \) to arrive the normal form

\( x_{n+1} = -(1 + \mu) x_n + ax_n^2 + \cdots \)

For \( \mu > 0 \), there exist period-2 pts, say \( p \) and \( g \).

As \( \mu \searrow 0 \), \( p, g \) — each period-doubling.

Then \( f^2 \) near \( p \) can be again approximated by a map with the same algebraic form as Eq (6).

Q: What are \( p, g \)?

Period-2 pts:

\( p = -(1 + \mu)g + g^2 \)

\( g = -(1 + \mu)p + p^2 \)

\( x = \frac{\mu + \sqrt{\mu^2 + 4\mu}}{2}, \quad y = \frac{\mu - \sqrt{\mu^2 + 4\mu}}{2} \)
Q: How to shift the origin to $p$?

\[ f(x) = -(1+\mu)x + x^2 \]

$p$ is a fixed pt of $f^2$.

Using a new variable $\eta$: $p + \eta_{n+1} = f^2(p + \eta_n)$

To $O(\eta^2)$, expand $\Rightarrow$

\[ \eta_{n+1} = (1 - 4\mu - \mu^2)\eta_n + C\eta_n^2 + \cdots \]

\[ C = 4\mu + \mu^2 - 3\sqrt{\mu^2 + 4\mu} \]

Note: The same algebraic form

Rescale $\eta$, $\tilde{\eta}_n = C\eta_n \Rightarrow$

\[ \tilde{\eta}_{n+1} = (1 - 4\mu - \mu^2)\tilde{\eta}_n + \tilde{\eta}_n^2 + \cdots \]

Renormalize the coefficient:

\[ -(1 + \tilde{\mu}) = (1 - 4\mu - \mu^2) \]

\[ \Rightarrow \tilde{\eta}_{n+1} = -(1 + \tilde{\mu})\tilde{\eta}_n + \tilde{\eta}_n^2 + \cdots \] (2)

Note: $\tilde{\mu} = \mu^2 + 4\mu - 2$

At $\tilde{\mu} = 0$, the map (2) loses stability for the 2-cycle and creates a 4-cycle.

i.e. $\mu^2 + 4\mu - 2 = 0 \Rightarrow \mu = -2 + \sqrt{6}$ (\(\mu < 0\) only).
This generalization can proceed forever!

\[ \mu_k \rightarrow \text{at which the map (1) creates } 2^k \text{-cycle.} \]

(ie. \( \mu_1 = 0 \) by this definition)

\[ \mu_2 = -2 + \sqrt{6} \approx 0.449 \]

In general,

\[ \mu_{k+1} = \mu_k + 4\mu_k - 2 \]

\[ \Rightarrow \mu_k = -2 + \sqrt{6 + \mu_{k-1}} \]

\[ \lim_{k \to \infty} \mu_k = \mu^* \]

\[ \therefore \mu^* = (\mu^*)^2 + 4\mu^* - 2 \]

\[ \Rightarrow \mu^* = \frac{1}{2} (3 + \sqrt{17}) \approx 0.56 \]

Note:

\( \mu = 0 \) period-2

For logistic \( \lambda = 3 \) period-2 is born

\( \mu = \mu^* \) corresponding to

\[ \lambda_0 = 3.56 \]

ct. the actual \( \lambda_0 \approx 3.57 \).
The universal constants $\delta$ and $\alpha$

1. $k \gg 1$, $\mu_k$ goes to $\mu^*$ at a rate — the universal constant $\delta$

\[
\delta \approx \frac{\mu_{k-1} - \mu^*}{\mu_k - \mu^*} \approx \left. \frac{d\mu_k}{d\mu_k} \right|_{\mu_k = \mu^*} = 2\mu^* + 4
\]
\[
= 1 + \sqrt{17} \approx 5.12
\]
\[
\mu^* = \frac{1}{2} (1 + \sqrt{17})
\]

Cf. $\delta \approx 4.67$

10% larger

2. Recall: $\tilde{\gamma} = C \eta_v$ \quad \therefore $C$ is $\alpha$ in the limit

\[
C = \frac{1 + \sqrt{17}}{2} - 2\sqrt{\frac{1 + \sqrt{17}}{2}} \approx -2.24
\]

Cf. $\alpha = 2.50$

(with 10%)
Now we switch back to \( A_n, A_{n-1}, \ldots, A_0 \) notation.

**Doubling Transformation \( T \):** (Renormalization Transformation)

Define a family of functions:

\[
g_i(x) = \lim_{n \to \infty} (-\alpha)^n \int_{A_{n+i}} \frac{x}{(-\alpha)^n} 2^n \ dx
\]

\( i = 0, 1, 2, \ldots \)

**Note:**

\[
g_i(x) = \lim_{n \to \infty} (-\alpha)^n \int_{A_{n+i-1}} \frac{x}{(-\alpha)^n} 2^{n+1} \ dx
\]

\[
= \lim_{n \to \infty} (-\alpha)^n (-\alpha)^{m+2} \int_{A_{n+i}} \left[ -\frac{1}{\alpha} \frac{x}{(-\alpha)^n} \right]
\]

\[
= \lim_{n \to \infty} (-\alpha)^n (-\alpha)^m \int_{A_{n+i}} \left[ -\frac{1}{\alpha} \frac{x}{(-\alpha)^n} \right]
\]

\[
= \lim_{n \to \infty} (-\alpha)^n (-\alpha)^m \int_{A_{n+i}} \left[ -\frac{1}{\alpha} \frac{x}{(-\alpha)^n} \right]
\]

\[
= \lim_{n \to \infty} (-\alpha)^n \int_{A_{n+i}} \left[ -\frac{1}{\alpha} \frac{x}{(-\alpha)^n} \right]
\]

\[
= \alpha g_i \left[ g_i \left( -\frac{x}{\alpha} \right) \right]
\]

i.e. They are linked by the doubling generator \( T \):

\[
g_{i+1}(x) = (-\alpha) g_i \left[ g_i \left( -\frac{x}{\alpha} \right) \right] = T g_i(x)
\]

**Def.** \( g(x) = \lim_{n \to \infty} g_i(x) \). \( g \) is a fixed pt of \( T \)-op

\[
\Rightarrow g(x) = \lim_{n \to \infty} g_i(x)
\]

\[
\Rightarrow g_i(x) - g_i(x) = -\alpha g_i \left[ g_i \left( -\frac{x}{\alpha} \right) \right]
\]

\[
\Rightarrow g(x) \text{ can be viewed as } g_i \text{ vanishing at } A_i: \text{ for } g(x) = 0 \text{ if } \frac{1}{2} \int_{A_{n+i}} 2^n \ dx
\]
\[ x \text{ is determined by } g(x) = -\alpha \frac{g(x)}{g'(x)} \]

i.e. \[ x = -\frac{g(x)}{g'(x)} \]

No: \text{ every fact: } \mu g\left(\frac{x}{\mu}\right) \text{ is also a solution if } x = T \mu.

\[ g_{T\mu} = -\frac{1}{4} \left[ g(\frac{T}{\mu} - \frac{x}{\mu}) \right] \]
\[ g_T = -4 \mu \left[ \frac{1}{\mu} g(-\frac{x}{\mu}) \right] \]

\[ f_T = -4 \mu \left[ \frac{1}{\mu} g(-\frac{x}{\mu}) \right] \]

we can fix \( g(0) = 1 \). (此次活动)

\[ x = -\frac{1}{g(0)} \]

No: \text{ No general theory of roots of the functional eq.}

\[ g(x) = -\alpha g\left(\frac{x}{\alpha} \right) \]

Reichenbach's numerical results,

\[ g(x) = 1 - 1.5276 \cdot x^2 + 0.104815 \cdot x^4 + 0.026705 \cdot x^6 - \ldots \]

\[ \alpha = 2.5029 \ldots \text{ universal value of } \alpha. \]

No: \text{ if we require } g(x) \text{ is a smooth } f, \text{ all quadratic law}

\[ \text{Taylor exp: } g(x) = 1 + 6x^2 + o(x^2) \]

FP eqn: \[ g(x) = -\alpha g\left(\frac{x}{\alpha}\right) \Rightarrow \]

\[ 1 + 6x^2 = -\alpha \left(1 + 6 \left(\frac{1+6x^2}{\alpha}ight)^2\right) \]

\[ = -\alpha \left(1 + 6 \cdot \frac{26}{\alpha} \right) x^2 + O(x^4) \]

\[ \Rightarrow \quad \beta = -2 \cdot \sqrt{12} \approx -1.866, \quad \alpha = 1261 \approx 2.75 \text{ (10% error)} \]

\[ \text{of } \alpha = 2.5029 \ldots \]
Linearization of Doubling Model

Now show the function of how along a-axis to dyn scales.

Recall, \( A = A_n \) a \( 2^n \)-cycle becomes separable

\[ f^{2^n}(x) = \frac{1}{2} \text{ lies pt of } f^{2^n} \]

The coordinate shift \( \Rightarrow f^{2^n}(0) = 0 \)

Claim:
\[ A_n - A_{n+1} \propto 5^{-n} \]

Generic \( \tau A(x) \) around \( f_A(x) \)

\[ f_A(x) = f_{A_0}(x) + (A - A_0) \frac{df_A(x)}{df_{A_0}} + \ldots \]
\[ \Rightarrow f_A(x) = f_{A_0}(x) + (A - A_0) \frac{df_A(x)}{df_{A_0}} + \ldots \]

\[ \frac{df_A(x)}{df_{A_0}} = \frac{df_A}{df_{A_0}} \]

\[ \Rightarrow f_A = f_{A_0} + (A - A_0) \frac{df_A(x)}{df_{A_0}} + \ldots \]

Linearizing \( \Rightarrow T f_A = T f_{A_0} + (A - A_0) (f_{A_0}) + \ldots \)

**Claim:**
\[ f_0 \frac{df}{dx} = -d \int f'[(A - \frac{x}{A})] df(-\frac{x}{A}) + df \frac{df}{dx}(-\frac{x}{A}) \]

\[ \Rightarrow f_0 \frac{df}{dx} = -d \int f'[(A - \frac{x}{A})] df(-\frac{x}{A}) + df \frac{df}{dx}(-\frac{x}{A}) \]

\[ \Rightarrow \frac{df_A}{df_{A_0}} = \frac{(A - A_0) f_{A_0}^2}{(A - A_0) f_{A_0}^2} \]

\[ = \frac{df_A}{df_{A_0}} \]
$$T^n f_A = T^n f_{A_0} + (A - A_0) \sum_{r=1}^{n-1} \lambda_r f_A + o(\lambda_3^n)$$

As \( n \to \infty \), \( T^n f_A \to \text{linear function} \)

$$T^n f_A(x) = (-\infty)^n f_{A_0}^2 \left( \frac{x}{\lambda_3^n} \right) \sim g(x) \quad n \gg 1$$

$$\Rightarrow \
\hat{g}(\ell) \Rightarrow \
T^n f_A \approx g(x) + (A - A_0) \lambda_3^2 \hat{g}(\ell) \quad n \gg 1$$

How can we extract scaling info. from this?

Expand \( g(x) \) with the eigenfunctions \( \phi_v \) of \( L_g \)

i.e., \( L_g \phi_v = \lambda_v \phi_v \)

$$D\phi = \sum_{\nu} c_{\nu} \phi_v$$ \( \nu = 1, 2, ... \)

NB: \( L_g \phi = \sum_{\nu} c_{\nu} \lambda_v \phi_v \)

Assume. i.e., we have the following property:

\( \lambda_v > 1, \quad |\lambda_v| < 1 \quad \nu \neq 1 \quad \text{i.e. the fixed pt } g(x) \text{ is a saddle} \)

$$\Rightarrow \hat{g}(\ell) \approx c, \quad \hat{\phi}, \quad \text{for } n \gg 1$$

$$\Rightarrow T^n f_{A_0}(x) \approx g(x) + (A - A_0) \lambda_3^2 c, \quad \phi(x) \quad n \gg 1$$
\[ x = 0: \]

\[ N \text{E: } A = A_\infty, \quad T^\infty_{A_\infty} (\omega) = g(\omega) + (A_\infty - A_\infty) \cdot \alpha \to, \quad \phi(x) \]

Recall \[ T^\infty_{A_\infty} (\omega) = (-\omega)^n P_n(\omega) = 0 \]

by def. \( x = 0 \) is on a hyperbolic orbit.

\[ g(\omega) \to 1 \]

\[ \lim_{n \to \infty} (A_\infty - A_\infty) \cdot \alpha^n = \frac{-g(\omega)}{\lambda, \phi(x)} = \text{Cosh} \]

i.e. \( A = \alpha \) and \( A_\infty - A_\infty = \alpha^{-n} \)

It can be derived from the universal eigenvalue problem

\[ L \varphi(\omega) = -\lambda \varphi(\omega) \left[ g(-\frac{x}{\alpha}) \varphi(\omega) + \varphi(\omega) \left( g(-\frac{x}{\alpha}) \right) \right] 
\]

\[ = -\lambda \varphi(\omega) \rightarrow 4.66920 \ldots \]

\[ \text{Feigenbaum} \]

\[ N \text{E: } g'(g(\omega)) = g(\omega), \quad g(0) = 1 \]

How to compute \( g'(\omega) \)?

\[ \therefore \quad g(\omega) = -x \left[ g(-\frac{x}{\alpha}) \right] 
\]

\[ \therefore \quad g'(\omega) = -\frac{1}{\alpha} \left[ g(-\frac{x}{\alpha}) \right] (\frac{x}{\alpha}) + g(\omega) \left( g(-\frac{x}{\alpha}) \right) \frac{1}{\alpha} \left[ g(-\frac{x}{\alpha}) \right] \left( \frac{x}{\alpha} \right) 
\]

\[ \therefore \quad g'(\omega) = \frac{1}{\alpha} \left[ g(-\frac{x}{\alpha}) \right] (\frac{x}{\alpha}) + g(\omega) \left( g(-\frac{x}{\alpha}) \right) \frac{1}{\alpha} \left[ g(-\frac{x}{\alpha}) \right] \left( \frac{x}{\alpha} \right) \Rightarrow 0 = -g'(\omega) \]
1. It can be shown that indeed, \( \lambda = \delta \) is the only eigenvalue of \( L \), s.t. \( \lambda > 1 \).

2. Also, it can be shown that

\[
\lambda_\infty - \lambda_\infty < \delta^{-n}, \quad n \gg 1
\]

\[
\lambda_\infty = \lambda_\infty
\]

Uniqueness: 2. fixed point of the period doubling operator:

\[
T g(x) = -\alpha g(g(-\frac{x}{\alpha})) = g(x)
\]

\[\Rightarrow\] uniqueness of \( g \).

2. Linearize \( T \)

\[
T \phi(x) = g(x) + (\lambda - \lambda_\infty) \delta^n \phi \cdot \phi(x) \quad \gg 1
\]

establishes the uniqueness of \( \delta \).

\( \delta \): \( \delta \) is the scale of \( f(x) \) is selected from \( \lambda > 1 \) all other variables other than \( \phi(x) \) remain 0.

\( j(x) \quad (: = f - g = \sum_{i=1}^n \alpha_i \phi_i \cdot T^h f \to g) \) irrelevant.
2nd scaling law for eigenfunction exponent $\lambda$.

$$A(f) = \lim_{n \to \infty} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \log |f_i(x_i)| \quad x_0 = 0.$$ 

$$\therefore \quad Tf \cdot Tf \cdots Tf = (Tf)^n = -\alpha f^{2^n}(-\frac{x}{\alpha})$$

$$\frac{d}{dx} Tf = f\left[f(-\frac{x}{\alpha})\right] f(-\frac{x}{\alpha})$$

$$\therefore \quad A(Tf) = \lim_{n \to \infty} \frac{1}{2^n} \sum_{i=1}^{2^n} \log |f_i(x_i)| = 2A(f)$$

$$\Rightarrow \quad \lambda(f) = 2^{-1} \lambda(Tf)$$

$$= 2^{-2} \lambda(T^2f)$$

$$= \cdots$$

$$= 2^{-n} \lambda(T^nf)$$

Choose $f = f_A$.

$$\therefore \quad T^n f_A(x) = g(x) + (A - A_w) \nu^n c, \varphi(x)$$

$$\therefore \quad A(f_A) = 2^{-n} A\left(g(x) + (A - A_w) \nu^n c, \varphi(x)\right)$$

Red.

$$(A - A_w) \nu^n = 1 \quad \therefore \quad A \text{ moves to } n$$

$$\therefore \quad \lambda(f_A) = 2 \frac{\log_2 (A - A_w)}{\log_2 \nu} \nu (g(x) + c, \varphi(x))$$

$$= (A - A_w)^{-1} \lambda(g(x) + c, \varphi(x))$$

$$= \frac{\log_2 (A - A_w)}{\log_2 \nu}$$

$$\beta = \frac{\log_2 (A - A_w)}{\log_2 \nu} \quad \text{critical exponent.}$$

$$\therefore \quad A \propto (A - A_w)^{-\beta}$$
Breather probing.

\[ C_{\nu}(\nu) = \lim_{s \to \infty} \frac{1}{n} \sum_{i=0}^{n} f(\nu) f^{*}(\nu) \]

\[ C_{\nu}(\nu) = \alpha^2 \lim_{s \to \infty} \frac{1}{n} \sum_{i=0}^{2^m} f(\nu) f^{*}(\nu) \]

\[ = \alpha^2 \lim_{s \to \infty} \frac{1}{n} \sum_{i=0}^{2^m} f^{2i}(\nu) f^{*}(\nu) \]

\[ \quad - \frac{1}{2} \sum_{i=0}^{2^m} f^{2i}(\nu) f^{*}(\nu) \]

\[ \Rightarrow \alpha^2 = \frac{1}{\alpha^2} C_{\nu}(\nu) \]

\[ C_{\nu}(\nu) = \alpha^2 C_{\nu}(2\nu) \]

\[ \text{since } \beta = 2\alpha, \text{ then iterating } \beta a \rightarrow \beta^2 a = g(\nu) + (A - A_0) \delta C(\nu) \]

\[ C(\nu) = \alpha^{-2n} C_{\nu}(2^{-n} \nu) \quad h = g(\nu) + (A - A_0) \delta C(\nu) \]

\[ A = A_0 \]

\[ C(\nu) = \alpha^{-2n} C_{\nu}(2^{-n} \nu) \]

\[ 2^{-n} = 1 \]

\[ C_{\nu}(\nu) = \# - \nu \quad \text{such that} \quad \gamma = \frac{\alpha^2 \log \alpha}{\log 2} \]

\[ \text{Power law decay of correlation at } \nu \]
**N:** The function space: all 1-D maps that have exactly one max or one min.

\[ W^u(g) \] — unstable invariant manifold tangent to \( \theta \), at \( g \).

\[ \Sigma: \text{Codimension 1} \]

stable manifold cutting through \( \theta \), transversally, at \( g \).

\[ P: \text{a} \circ \text{a} \circ \text{a} \circ \text{a} \text{ in the 1-parameter family} \]

\[ \text{As } a \text{ goes near } g, \text{ then along } W^u(g) \]

in the end, deciding is the closure of \( W^u(g) \)

\[ \Rightarrow \text{ a set of universal function} \]

\[ \therefore \text{ The one-parameter family corresponding to } W^u(g) \]

— a set of universal fn. (i.e. one component along irrelevant directions)
Recall: \[ T f_a(x) = -a \int_{a_{n+1}}^x (-\frac{x}{a}) \, dx = f_a(x) \]
at the period doubling.

There is a corresponding set \( g_\alpha \) for which
\[ T g_\alpha(x) = -\alpha g_\alpha(-\frac{x}{\alpha}) = g_\alpha(x) \]

Note: \( \lim_{\alpha \to 0} g_\alpha(x) = g(x) \)
i.e. \( g_\alpha(x) \) approaches \( g \) along \( W^u(g) \)

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The set of all maps \( f_a(x) \)
at the \( n \)th period-doubling bifurcation pt.
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A codimension 1

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Logistic map: family

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Surface: \( \Sigma_n \)

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Logistic

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Some other family
In the neighborhood of $q_0$, $\Sigma_n \parallel \Sigma$.

Recall under $T$, along $W^u(q_0)$, \[ \frac{D_n}{D_{n+1}} = \delta \] the eigenvalue of unstable direction.

\[ \frac{\Delta_n - \Delta_{n-1}}{\Delta_{n-1} - \Delta_0} \approx \frac{D_n}{D_{n+1}} = \frac{D_n}{D_{n+1}} \]