Lecture 5

Note Title 3/18/2006

Infinitesimal Generators

First, consider time-homogeneous diffusion process

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dW$$

$g(x)$ — twice continuously differentiable with bounded derivatives.

$$\mathcal{L}g(x) = \frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2}g(x) + \mu(x)\frac{d}{dx}g(x)$$

NB: $\quad dg(X_t) = \mathcal{L}g(X_t)\,dt + g'(X_t)\,\sigma(X_t)\,dW$

$\Rightarrow$ an associated martingale: $\quad M_t = g(X_t) - \int_0^t \mathcal{L}g(X_s)\,ds$

— a martingale.

($\because$ The drift term is removed)

$\Rightarrow$ if $X_0 = x$

$$\mathbb{E}[M_t] = \mathbb{E}\left[g(X_t) - \int_0^t \mathcal{L}g(X_s)\,ds\right] = M_0 = g(x)$$

i.e. $\quad \mathbb{E}[g(X_t)] = g(x) + \mathbb{E}\left[\int_0^t \mathcal{L}g(X_s)\,ds\right]$

$$\frac{d}{dt}\mathbb{E}\left[g(X_t)\right]\Big|_{t=0} = \lim_{t\to 0}\frac{\mathbb{E}[g(X_t)] - g(x)}{t}$$

$$= \lim_{t\to 0}\mathbb{E}\left[\frac{1}{t}\int_0^t \mathcal{L}g(X_s)\,ds\right]$$

$$= \mathcal{L}g(x) \quad [\because \text{Lebesgue dominated convergence theorem}]$$

$\mathcal{L}$ — infinitesimal generator of the Markov process $dX_t = \mu dt + \sigma dW_t$.

$\{f_n\}$ measurable

$\lim_{n\to\infty} f_n = f$ point-wise almost everywhere

$\forall n, |f_n| \le g$, g-integrable

then f is integrable and

$$\int f\,d\mu = \int \lim_{n\to\infty} f_n\,d\mu$$

For nonhomogeneous diffusion; i.e.

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t$$

For g — smooth and bounded,

the corresponding generalization is

$$M_t = g(t, X_t) - \int_0^t \left(\frac{\partial}{\partial s}g + \mathcal{L}_s g\right)(s, X_s)\,ds$$

is a martingale. (use Itô lemma)

$$\mathcal{L}_t = \frac{1}{2}\sigma^2(t,x)\frac{\partial^2}{\partial x^2} + \mu(t,x)\frac{\partial}{\partial x}$$

or with a discounting factor i.e. $e^{-rt} g(t, X_t)$

$$M_t = e^{-rt} g(t, X_t) - \int_0^t e^{-rs}\left(\frac{\partial g}{\partial s} + \mathcal{L}_s g - rg\right)(s, X_s)\,ds$$

Ref Derivatives in Financial Markets with Stochastic Volatility
Jean-Pierre Fouque, George Papanicolaou, Ronnie Sircar.

Stochastic Volatility Models

In the Black-Scholes theory,

- Stock — continuous w/o jumps (diffusion)
- Option — can be hedged continuously w/o transaction costs.
- Constant volatility.

Q: What if the volatility is not const?

Stochastic volatility $\Longrightarrow$ incomplete market.

Implied Volatility and the smile curve

Q: What is the implied volatility?

Recall BS eqn: $\mathcal{L}_{BS}(\sigma)\,P = 0$ — value of the option

$$\mathcal{L}_{BS}(\sigma) = \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} + r\frac{\partial}{\partial x} - r$$

European call option: $P(t=T, x) = (x-K)^+$

Black Scholes formula (Prices at time t and $X_t = x$)

$$C_{BS}(t,x) = x\,N(d_1) - Ke^{-r(T-t)}N(d_2)$$

$$d_1 = \frac{1}{\sigma\sqrt{T-t}}\left[\log\frac{x}{K} + \left(r+\frac{1}{2}\sigma^2\right)(T-t)\right]$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

$T-t$ = time to maturity

$$N(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-y^2/2}\,dy$$

r — interest rate.

Given an observed European call option price, C^{obs} for a contract w/ strike K and maturity T.

solve $C_{BS}(t, x; K, T; I) = C^{obs}$

$I = \sigma_{implied}$ implied volatility

Q: Does such implied volatility exist?

$$\because \frac{\partial C_{BS}}{\partial \sigma} = x e^{-d_1^2/2}\sqrt{\frac{T-t}{2\pi}} > 0$$

$C_{BS}(\sigma)$ is a monotonic fn of σ

as long as $C^{obs} > C_{BS}(t, x; K, T, \sigma = 0)$

$\Rightarrow \exists$ a unique implied vol $I > 0$.

NB: Q: Would the implied σ be different from put price of the same strike and maturity?

No! $\left\{ \begin{array}{l} \because \text{ the put-call parity:} \\ C_{BS}(t, x, K, T, \sigma) - P_{BS}(t, x, K, T, \sigma) = x - Ke^{-r(T-t)} \end{array} \right.$

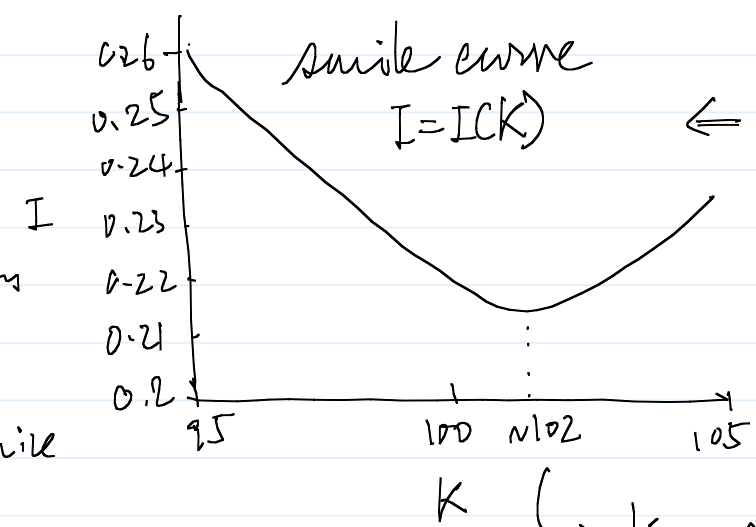
From C^{obs} $\Rightarrow$ implied vol $I = I(t, x; K, T)$

NB: if $C^{obs} = C_{BS}(t, x; K, T, \sigma)$ for some σ,

then $I = \sigma$

Smile Effect:

European options for all $T-t$, current stock price $x = 100$



$\Leftarrow$ From the B-S theory, it should be a constant.

$\hookrightarrow K_{min} \approx x$

min of I ~ near the money

i.e. $95\% \le \frac{K}{x} \le 105\%$

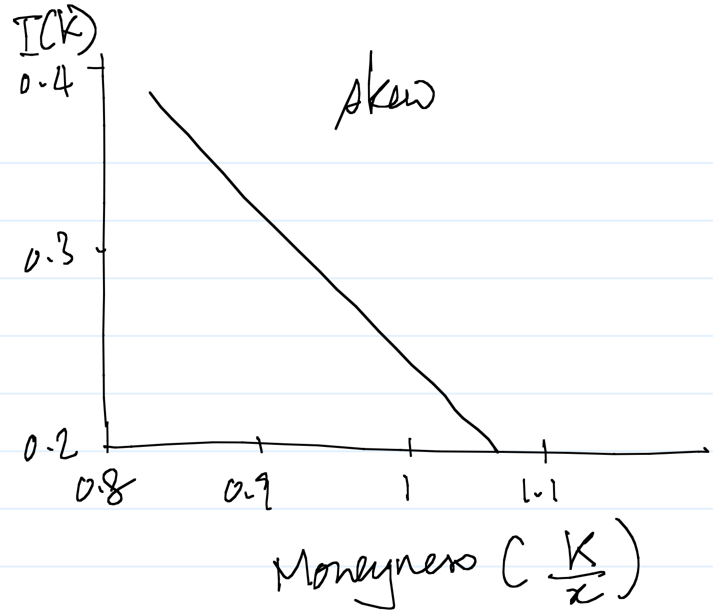
Volatility skew

S&P500

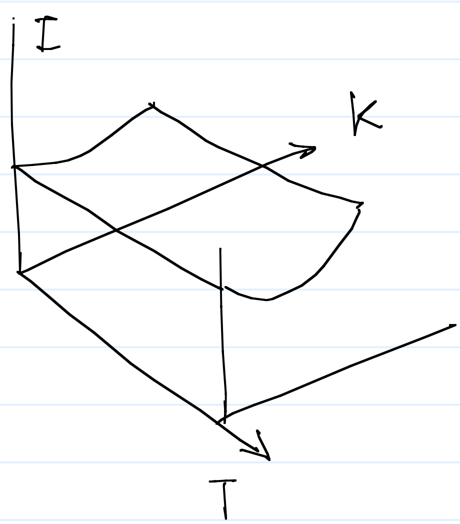
the current index value

$x = 1411.71$

$T - t = 2$ months.



In general, implied vol. surface



Q: How can we extract the local volatility surface from Europ. call option prices?

Assume $dx = rx\,dt + \sigma(x,t)\,dW$

↳ fn of x : the implied deterministic volatility

The option price $P(K,T)$

Today's stock value

$$P(K,T) = e^{-r(T-t_0)} \int_0^{\infty} (x-K)_+ \underbrace{\rho(x,T \mid x_0, t_0)}_{\text{Transition prob density fn.}} dx$$

$$\therefore \quad P(K,T) = e^{-r(T-t_0)} \int_K^{+\infty} (x-K)\, \rho(x,T \mid x_0 t_0)\, dx$$

NB: the special form of the call option payoff fn

$$\frac{\partial P}{\partial K} = -e^{-r(T-t_0)} \int_K^{\infty} \rho(x,T \mid x_0 t_0)\, dx$$

$$\frac{\partial^2 P}{\partial K^2} = e^{-r(T-t_0)} \rho(x,T \mid x_0 t_0)$$

$$\therefore \quad \rho(x,T \mid x_0 t_0) = e^{r(T-t_0)} \frac{\partial^2 V}{\partial K^2}$$

i.e. From prices $\Rightarrow$ extraction of the transition pdf. (a risk-neutral pdf)

Recall the forward Kolmogorov eqn for ρ is

$$\frac{\partial}{\partial T} \rho = \frac{1}{2} \frac{\partial^2}{\partial x^2}(\sigma^2 x^2 \rho) - \frac{\partial}{\partial x}(r x \rho)$$

Not determined yet $\sigma = \sigma(x,T)$

i.e. $= \sigma(x,t)\big|_{t=T}$

Price

$$\because \quad P(K,T) = e^{-r(T-t_0)} \int_K^{\infty} (x-K)\,\varphi\, dx$$

$$\frac{\partial P}{\partial T} = -rP + e^{-r(T-t_0)} \int_K^{\infty} (x-K)\frac{\partial \varphi}{\partial T}\, dx$$

$$= -rP + e^{-r(T-t_0)} \int_K^{+\infty} (x-K)\left[\frac{1}{2}\frac{\partial^2}{\partial x^2}\left(\sigma^2 x^2 \varphi\right) - \frac{\partial}{\partial x}\left(r x \varphi\right)\right] dx$$

integrat. by parts twice, NB. $\varphi \to 0$, $\frac{\partial \varphi}{\partial x} \to 0$ as $x \to \infty$

$$= -rP + e^{-r(T-t_0)}\left(\frac{1}{2}\sigma^2 K^2 \varphi\right) + r e^{-r(T-t_0)} \int_K^{\infty} x \varphi\, dx$$

The last term:

$$\int_K^{+\infty} x\varphi\, dx = \int_K^{+\infty} (x-K+K)\varphi\, dx = \underbrace{\int_K^{+\infty} (x-K)\varphi\, dx}_{= e^{r(T-t_0)} P} + K \underbrace{\int_K^{+\infty} \varphi\, dx}_{\Downarrow}$$

$$\frac{\partial P}{\partial K} = -e^{-r(T-t_0)} \int_K^{+\infty} \varphi\, dx$$

$$\therefore \quad \frac{\partial P}{\partial T} = \frac{1}{2}\sigma^2(K,T) K^2 \frac{\partial^2 P}{\partial K^2} - rK\frac{\partial P}{\partial K}$$

$$\therefore \quad \sigma(K,T) = \left[\frac{\frac{\partial P}{\partial T} + rK\frac{\partial P}{\partial K}}{\frac{1}{2}K^2 \frac{\partial^2 P}{\partial K^2}}\right]^{\frac{1}{2}}$$

i.e. Given a set of prices $\Rightarrow$ vol. surface $\sigma(x,t)$

NB: Interpolation (numerical) issues:

— ill-posedness / regularization

NB: $\sigma(K,T)$ is not a prediction about future volatility,

For example, a few days later, $\sigma(K,T)$ may need refit.

<u>But</u>. Often today's options are priced using yesterday's option data

NB: Use $\sigma(K,T)$ to price nontraded (e.g. exotic) contracts

to be consistent with all instruments

i.e. price exotic consistent with vanillas

with the same volatility structure and

simultaneously hedged with these vanillas.

$\Rightarrow$ Reduction of exposure to model errors

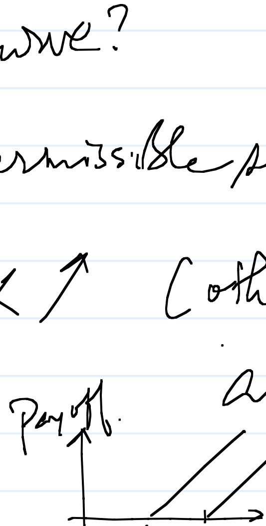
Q: How to interpret the smile curve?

First, some bounds on the permissible slope of $I(K)$

NB: $C^{obs}(K) \searrow$ as $K \nearrow$ (otherwise, there is arbitrage opportunity)

if $C^{obs}(K_1) < C^{obs}(K_2)$ $\quad K_1 < K_2$

buy $C^{obs}(K_1)$ sell $C^{obs}(K_2)$



$$\therefore \frac{\partial C^{obs}}{\partial K} = \frac{\partial C_{BS}}{\partial K} + \frac{\partial C_{BS}}{\partial \sigma}\frac{\partial I}{\partial K} \leq 0$$

$$\Rightarrow \quad \frac{\partial I}{\partial K} \leq - \frac{\left(\frac{\partial C_{BS}}{\partial K}\right)}{\left(\frac{\partial C_{BS}}{\partial \sigma}\right)} \qquad \left(\text{NB: } \frac{\partial C_{BS}}{\partial \sigma} > 0\right)$$

From put-call parity,

$$\frac{\partial I}{\partial K} \geq - \frac{\left(\frac{\partial P_{BS}}{\partial K}\right)}{\left(\frac{\partial P_{BS}}{\partial \sigma}\right)}$$

Using the BS formula, $\Rightarrow$ Bounds:

$$-\frac{1}{x}\sqrt{\frac{2\pi}{T-t}}\,(1-N(d_2))\,e^{-r(T-t)+\frac{d_1^2}{2}} \leq \frac{\partial I}{\partial K} \leq \frac{1}{x}\sqrt{\frac{2\pi}{T-t}}\,N(d_2)\,e^{-r(T-t)+\frac{d_1^2}{2}}$$

$$\text{where } \begin{cases} d_1 = \left.\frac{1}{\sigma\sqrt{T-t}}\left[\log\frac{x}{K} + (r+\frac{1}{2}\sigma^2)(T-t)\right]\right|_{\sigma=I} \\ d_2 = \left.\left(d_1 - \sigma\sqrt{T-t}\right)\right|_{\sigma=I} \end{cases}$$

i.e. the slope cannot be too positive or too negative.

NB: Empirical observation; Prices tend to go down, when the volatility goes up and vice versa. $\Rightarrow$ negative correlation

NB: Smiles $\Rightarrow$ The Black-Scholes theory needs modification

Implied Deterministic Volatility

$\sigma = \sigma(t, X_t)$

and the stock price $dX_t = \mu X_t dt + \sigma(t, X_t) X_t dW_t$

then, the generalized Black-Scholes PDE:

$$\frac{\partial}{\partial t} P + \frac{1}{2} \sigma^2(t,x) x^2 \frac{\partial^2 P}{\partial x^2} + rx \frac{\partial P}{\partial x} - rP = 0$$

NB: 1° the derivation is identical to that of const. σ.

2°. Hedging Ratio $\Delta = \frac{\partial P}{\partial x}$ (the argument is the same as const. σ)

3°. The market is still complete

(∵ the randomness of the volatility is a fn of the randomness of the lognormal model

$\exists!$ risk-neutral measure $\mathbb{P}^*$ under which the underlying is a geometric Brownian motion with drift r,

i.e

$$dX_t = rX_t dt + \sigma(t, X_t) X_t dW_t^*$$

W_t^* is a $\mathbb{P}^*$-Brownian motion

NB: the way we construct the vol. surface $\Longrightarrow$ the implied deterministic vol.

Recall. Special case: $\sigma(t,x) = \sigma(t)$ — time-dependent vol.

↳ a deterministic fn of t.

$$\overline{\sigma^2} = \frac{1}{T-t}\int_t^T \sigma^2(s)\,ds$$ ← Note time average of σ^2 [not σ]

NB: 1° the BS formula holds with volatility parameter $\sqrt{\overline{\sigma^2}}$

i.e. the root-mean-square volatility.

2° For fixed t, T, Options priced using the BS formula with time-averaged volatility don't exhibit smile across strike prices

3°. But there is change of implied volatility with time to maturity

$$\left[\because \quad \overline{\sigma^2} = \frac{1}{T-t}\int_t^T \sigma^2(s)\,ds \text{ changes as } T \text{ changes}\right]$$

⇒ The volatility surface changes along T-axis

Stochastic Volatility Models

In general, the asset X_t : $dX_t = \mu X_t dt + \sigma_t X_t dW_t$

σ_t — volatility process — diffusion process
jump process
a Markov chain

NB: 1° $\sigma_t > 0$ to be volatility.

2° NB: The implied deterministic volatility is perfectly correlates w/ W_t, $\because \sigma = \sigma(t, X_t)$

σ_t can have its own random component

Mean-reverting Stochastic volatility models

$\sigma_t = f(Y_t)$ f is some positive fn

the rate of mean reversion

$$dY_t = \alpha(m - Y_t)dt + \cdots d\hat{Z}_t$$

Brownian motions: $\hat{Z}_t$ and W_t can be correlated.

m: the long-run mean level of Y_t.

Example: Ornstein-Uhlenbeck process:

$$d Y_t = \alpha(m - Y_t)\,dt + \beta\, d\hat{Z}_t$$

soln: $$Y_t = m + (y-m)e^{-\alpha t} + \beta \int_0^t e^{-\alpha(t-s)}\, d\hat{Z}_s$$

↑ initial y

$$\mathbb{E}\left[\left(\int_a^b g\,dW\right)^2\right] = \int_a^b \mathbb{E}(g^2)\,ds$$

$\therefore \mathrm{Var}\, Y_t$

$$= \mathbb{E}\left[\left(Y_t - (m + (y-m)e^{-\alpha t})\right)^2\right]$$

$$= \int_0^t \mathbb{E}\left(\beta e^{-\alpha(t-s)}\right)^2 ds$$

$$= \beta^2 \int_0^t e^{-2\alpha(t-s)}\, ds$$

$$= \frac{\beta^2}{2\alpha}(1 - e^{-2\alpha t})$$

$$Y_t \sim N\left[m + (y-m)e^{-\alpha t},\ \frac{\beta^2}{2\alpha}(1 - e^{-2\alpha t})\right]$$

$t \to \infty$, $\Rightarrow$ invariant distribution

$$N\left(m, \frac{\beta^2}{2\alpha}\right)$$

NB: no y-dependence

i.e. forgot.

$\hat{Z}_t$ and W_t:

$$d\,\mathbb{E}(W_t Z_t) = \rho\, dt, \quad -1 \le \rho \le 1$$

$$d(Z_t W_t) = Z_t\, dW_t + W_t\, dZ_t + d\,\mathbb{E}(W_t Z_t)$$

$$\hat{Z}_t = \rho W_t + \sqrt{1-\rho^2}\, Z_t$$

W_t, Z_t are indep't Brownian motions

NB: $\rho < 0$ from financial data.

more general $\rho = \rho(t)$ — depending on time

Feller process (CIR)

$$dY_t = \kappa(m' - Y_t)\,dt + \nu\sqrt{Y_t}\,d\hat{Z}_t$$

Models of Volatility:

	ρ:	$f(y)$:	Y-process
Hull-White:	$\rho=0$,	$f(y)=\sqrt{y}$,	Lognormal (not mean-reverting)
Scott:	$\rho=0$	$f(y)=e^y$	Mean-reverting OU
Stein-Stein	$\rho=0$	$f(y)=\lvert y\rvert$	"
Ball-Roma	$\rho=0$	$f(y)=\sqrt{y}$	CIR
Heston	$\rho\neq 0$	$f(y)=\sqrt{y}$	CIR

— Qualitative Effects on the stock-price distribution under stochastic volatility.

exponential OU:

$f(y)=e^y$, Y_t — a mean reverting OU.

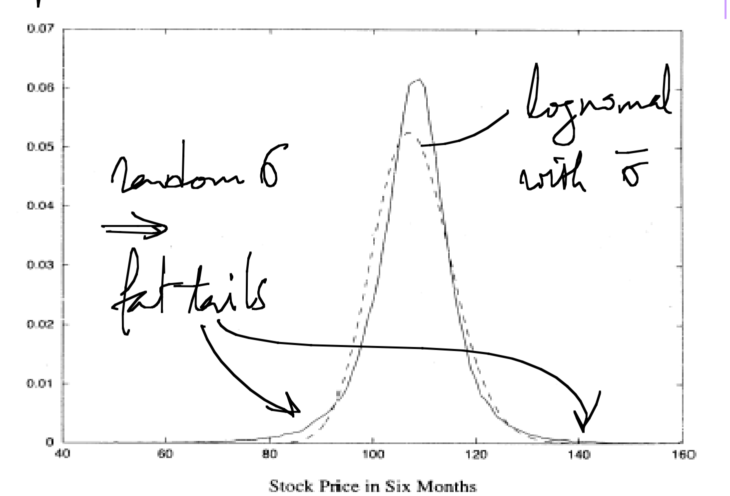


Figure 2.3. Density functions for the stock price (under the subjective measure) in six months when the present value is 100. The solid line is estimated from simulation of an expOU stochastic volatility model with $\alpha = 1$, $\beta = \sqrt{2}$, long-run average volatility $\bar{\sigma} = 0.1$, and negative correlation $\rho = -0.2$. The dotted line is the corresponding Black–Scholes lognormal density function with volatility $\bar{\sigma}$. The mean growth rate of the stock is $\mu = 0.15$.

$\rho < 0 \Rightarrow$ asymmetric tails left — fatter.

Lecture 6

Note Title 3/20/2006

Derivative Pricing with Stochastic Volatility

$$dX_t = \mu X_t\,dt + \sigma_t X_t\,dW_t$$

$$\sigma_t = f(Y_t)$$

$$dY_t = \alpha(m - Y_t)\,dt + \beta\,d\hat{Z}_t$$ — a mean reverting OU process.

— Recall the BS theory, the uncertainty introduced by dW can be hedged away using the underlying asset.

No arbitrage principle determines the price of the option.

— Note there is an additional uncertainty introduced by Z_t.

Q: How can we price?

Use the underlying and an option with different maturity.

$P^{(1)}(t,x,y)$ — the price of a European derivative with maturity T_1 and payoff function $h(X_{T_1})$

$P^{(2)}(t,x,y)$ — the price of another Europ option with the same payoff h but different maturity.

$T_2 > T_1 > t$.

In addition, we have risky asset X_t,

and riskless bond $\beta_t = e^{rt}$

r – short-term Const. interest rate.

Replication:

$$P^{(1)}(T_1, X_{T_1}, Y_{T_1}) = a_{T_1} X_{T_1} + b_{T_1} \beta_{T_1} + c_{T_1} P^{(2)}(T_1, X_{T_1}, Y_{T_1})$$

Self-financing:

$$\left.dP^{(1)}(t, X_t, Y_t) = a_t\, dX_t + b_t\, r e^{rt} dt + c_t\, dP^{(2)}(t, X_t, Y_t)\right\} \text{Eq}(*)$$

No arbitrage $\Rightarrow$

$$\left.P^{(1)}(t, X_t, Y_t) = a_t X_t + b_t e^{rt} + c_t P^{(2)}(t, X_t, Y_t) \quad \forall t < T_1\right\} \text{Eq}(**)$$

Q: Can we find such processes $\{a_t, b_t, c_t\}$?

Recall Itô Lemma

$$dg(t, X_t, Y_t) = \frac{\partial g}{\partial t} dt + \frac{\partial g}{\partial x} dX_t + \frac{\partial g}{\partial y} dY_t + \frac{1}{2}\left(\frac{\partial^2 g}{\partial x^2} d\langle X\rangle_t + 2\frac{\partial^2 g}{\partial x \partial y} d\langle XY\rangle_t + \frac{\partial^2 g}{\partial y^2} d\langle Y_t\rangle\right)$$

$$\langle X_t\rangle \equiv \int_0^t \sigma_x^2\, ds \qquad \langle Y_t\rangle \equiv \int_0^t \sigma_y^2\, ds$$

$$d\langle XY\rangle_t = \mathrm{Cov}(dX_t dY_t)$$

For our case, we assume $dW_t d\tilde{Z}_t = \rho\, dt$ i.e. correlation between the uncertainty of stock and the uncertainty of vol.

LHS of Eq(*):

$$dP^{(1)}(t, X_t, Y_t) = \left[\frac{\partial p^{(1)}}{\partial t} + \underbrace{\left(\frac{1}{2} f^2(y) x^2 \frac{\partial^2}{\partial x^2} + \frac{1}{2}\beta^2 \frac{\partial^2}{\partial y^2} + \rho\,(f(y)x)\,\beta \frac{\partial^2}{\partial x \partial y} \right)}_{\equiv M_1} p^{(1)} \right] dt$$

$$+ \frac{\partial p^{(1)}}{\partial x} dX_t + \frac{\partial p^{(1)}}{\partial y} dY_t$$

RHS of Eq(*) (apply the Itô Lemma to

$$= \left[c_t \left(\frac{\partial}{\partial t} + M_1 \right) p^{(2)} + b_t r e^{rt} \right] dt$$

$$+ \left(a_t + c_t \frac{\partial p^{(2)}}{\partial x} \right) dX_t + c_t \frac{\partial p^{(2)}}{\partial y} dY_t$$

NB: Equating dY_t terms (i.e. equating $d\tilde{Z}_t$ terms) $\Rightarrow$

$$\frac{\partial p^{(1)}}{\partial y} = c_t \frac{\partial p^{(2)}}{\partial y} \qquad \text{i.e.} \quad c_t = \frac{\partial p^{(1)}/\partial y}{\partial p^{(2)}/\partial y}$$

Similarly, from dX_t terms (i.e. dW_t terms) $\Rightarrow$

$$a_t = \frac{\partial p^{(1)}}{\partial x} - c_t \frac{\partial p^{(2)}}{\partial x}$$

NB. Eq(**) $\Rightarrow$ $b_t = (P_t^{(1)} - ax - c_t P_t^{(2)})\, e^{-rt}$

Substituting all these ($a_t = \cdots$, $b_t = \cdots$, $c_t = \cdots$) into dt-terms

$$\Rightarrow \quad \frac{\partial p^{(1)}}{\partial t} + M_1 p^{(1)} = \underbrace{\left(\frac{\partial p^{(1)}/\partial y}{\partial p^{(2)}/\partial y}\right)}_{a_t}\overset{= c_t}{} \left(\frac{\partial}{\partial t} + M_1\right) p^{(2)}$$

$$+ \left(p_t^{(1)} - \overbrace{\left(\frac{\partial p^{(1)}}{\partial x} - \underbrace{\frac{\partial p^{(1)}/\partial y}{\partial p^{(2)}/\partial y}}_{= c_t} \cdot \frac{\partial p^{(2)}}{\partial x}\right) x - \frac{\partial p^{(1)}/\partial y}{\partial p^{(2)}/\partial y}\, p_t^{(2)}}\right) e^{-rt} \cdot r e^{rt}$$

a little rearrangement $\Rightarrow$

$$\underbrace{\left(\frac{\partial p^{(1)}}{\partial y}\right)^{-1} M_2 p^{(1)}(t,x,y)}_{\text{depending on } T_1 \text{ only}} = \underbrace{\left(\frac{\partial p^{(2)}}{\partial y}\right)^{-1} M_2 p^{(2)}(t,x,y)}_{\text{depending on } T_2 \text{ only}}$$

$$M_2 = \frac{\partial}{\partial t} + M_1 + \left(r \cdot x\frac{\partial}{\partial r} - r\right)$$

NB: $p^{(1)}$ and $p^{(2)}$ can be arbitrary fn's, (i.e. different payoff fns)

$\therefore$ it must be equal to a function that is indep't of maturity (in general, contract type)

$$\therefore \frac{1}{\frac{\partial P}{\partial y}}\left\{\frac{\partial P}{\partial t} + \frac{1}{2} f^2(y) x^2 \frac{\partial^2 P}{\partial x^2} + \rho\beta x f(y) \frac{\partial^2 P}{\partial x \partial y} + \frac{1}{2}\beta^2 \frac{\partial^2 P}{\partial y^2} + rx\frac{\partial P}{\partial x} - rP\right\}$$

$$= -\left[\underbrace{\alpha(m-y) - \beta\Lambda(t,x,y)}\right] \longleftarrow \text{The form is explained later}$$

since $\gamma(t,x,y)$ is an arbitrary fn

$$\Lambda = \rho\frac{(\mu - r)}{f(y)} + \gamma(t,x,y)\underbrace{\sqrt{1-\rho^2}}$$

$\boxed{\text{Why } \sqrt{1-\rho^2} \text{ later!}}$

Final condition $P(T,x,y) = h(x)$

NB: Y_t — OU process. $\Rightarrow$ $y \in (-\infty, +\infty)$

Rewrite the above equation:

$$\underbrace{\left[\frac{\partial}{\partial t}+\frac{1}{2}f^2(y)x^2\frac{\partial^2}{\partial x^2}+\left(rx\frac{\partial}{\partial x}-r\right)\right]}_{\mathcal{L}_{BS}(f(y))}P+\underbrace{\rho\beta x f(y)\frac{\partial^2}{\partial x\partial y}}_{\text{correlation}}P$$

$$+\underbrace{\left[\frac{1}{2}\beta^2\frac{\partial^2}{\partial y^2}+\alpha(m-y)\frac{\partial}{\partial y}\right]}_{\mathcal{L}_{OU}}P-\underbrace{\beta\Lambda\frac{\partial}{\partial y}P}_{\text{premium}}=0 \qquad Eq(P)$$

$\mathcal{L}_{OU}$ i.e. the infinitesimal generator of OU process

premium $\downarrow$ market price of volatility risk

NB: $\gamma(t,x,y)$ — the risk premium factor from $d\tilde{Z}_t$.

NB: $|\rho|=1$ perfect correlation w/ dW_t

there is no such contribution $(\because \sqrt{1-\rho^2}=0)$

NB: Why our particular form of $\alpha(y-m)-\beta\Lambda$?

$$d\hat{Z}_t=\rho\, dW+\sqrt{1-\rho^2}\,dZ$$

Itô lemma $\Rightarrow$

$$dP(t,X_t,Y_t)=\left(\frac{\partial}{\partial t}+\frac{1}{2}f^2x^2\frac{\partial^2}{\partial x^2}+\rho\beta f(y)\frac{\partial^2}{\partial x\partial y}+\frac{1}{2}\beta^2\frac{\partial^2}{\partial y^2}+\mu x f(y)\frac{\partial}{\partial x}+\alpha(m-y)\frac{\partial}{\partial y}\right)P\,dt$$

$$+\left(xf(x)\frac{\partial P}{\partial x}+\beta\rho\frac{\partial P}{\partial y}\right)dW+\beta\sqrt{1-\rho^2}\frac{\partial P}{\partial y}dZ_t$$

$\rightarrow$ the difference is $\mu x\frac{\partial P}{\partial x}+rP-rx\frac{\partial P}{\partial x}$

NB: P satisfies $\left(\frac{\partial}{\partial t}+\mathcal{L}_{BS}(f(y))+\beta\rho f(y)\frac{\partial^2}{\partial x\partial y}+\mathcal{L}_{OU}\right)P=\beta\Lambda\frac{\partial}{\partial y}P$

NB: $\Lambda=\rho\,\frac{\mu-r}{f(y)}+\gamma(t,x,y)\sqrt{1-\rho^2}$

excess return-to-risk ratio: $\frac{\mu - r}{f(y)}$

riskfree return

$\beta \nearrow$ i.e. volatility risk increases $\Rightarrow$ the increase of return rate

$$\therefore\ dP(t,x,y) = \left[\frac{\mu-r}{f(y)}\left(xf(y)\frac{\partial P}{\partial x} + \beta\rho\frac{\partial P}{\partial y}\right) + rP + \gamma\beta\sqrt{1-\rho^2}\frac{\partial P}{\partial y}\right]dt$$

$$+\left(xf(y)\frac{\partial P}{\partial x} + \beta\rho\frac{\partial P}{\partial y}\right)dW_t + \beta\sqrt{1-\rho^2}\frac{\partial P}{\partial y}\,dZ_t$$

NB: Market Price of Volatility Risk

1° NB: $\gamma(t,x,y)$ is not observed in the process of X_t, Y_t

— it can be only seen from the derivative prices P.

2° Econometric methods (maximum likelihood, moment methods)

to find, α, β, m, ρ. (after choosing an $f(y)$.)

Then use derivative data to estimate γ. (say, assuming γ is const)

3°. Cross-sectional fitting

$$\min_{\substack{\alpha,\beta,m,\rho,\sigma_0\\ \gamma,\mu}} \sum_{\substack{(K,T)\in\\ \text{some set } \mathcal{K}}} \left(C(K,T,\alpha,\beta,m,\rho,\sigma_0,\gamma,\mu) - C^{obs}(K,T)\right)^2$$

model predicted call option prices

$\Rightarrow$ This method — Computationally very expensive.

Special Case: $(\rho = 0)$

NB: In equity markets, $\rho < 0$

Foreign-exchange data $\rho \approx 0$.

Hull-White Model

$$dX = \mu X\,dt + \sigma X\,dW$$
$$d\sigma = \mu_V \sigma\,dt + \sigma_V \sigma\,dZ$$
$$\mu_V = (X, \sigma, t), \quad \sigma_V = \sigma_V(X, \sigma, t)$$

e.g. mean-reverting — volatility of σ

$$dW\,dZ = \rho\,dt$$

NB: $f(\gamma) = \gamma$

$$\frac{\partial}{\partial t}P + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 P}{\partial x^2} + \rho \sigma_V^2 \sigma^2 x^2 \frac{\partial^2 P}{\partial x \partial \sigma} + \frac{1}{2}\sigma_V^2 \sigma^2 \frac{\partial^2 P}{\partial \sigma^2} + r x \frac{\partial P}{\partial X} + (\mu_V - \lambda \sigma_V)\sigma \frac{\partial P}{\partial \sigma} - rP = 0$$

Assumptions
1) no correlation b/w stock and volatility
2) risk-neutral dynamics, i.e. $\lambda = 0$
3) μ_V, σ_V independent of stock x.

Def: rand variable

$$V \equiv \overline{\sigma^2} = \frac{1}{T-t}\int_t^T \sigma_\tau^2\,d\tau \quad \text{— averaged volatility.}$$

then, the option value — BS formula w/ vol. $= V$

$$P(x_t, \sigma_t^2) = \int C(V)\, p(V \mid \sigma_t^2)\, dV$$

Conditional distribution of V given σ_t^2 at time t.

Intuitive understanding:

Payoff function, e.g. $(X_T - K)_+$

$$P(X_t, \sigma_t^2, t) = e^{-r(T-t)} \int f(X_T)\, p(X_T \mid X_t, \sigma_t^2)\, dX_T$$

Conditional distribution of X_T given X_t, σ_t^2 at t.

$p(X_T \mid X_t, \sigma_t)$ depends on the process of X, σ.

Recall $p(x|y) = \int g(x|z)\, h(z|y)\, dz$ for 3 rand variables

p, g, h — conditional pdf.

$$\therefore\ p(X_T \mid \sigma_t^2) = \int g(X_T \mid v)\, h(v \mid \sigma_t^2)\, dv$$

$$\therefore\ P(X_t, \sigma_t^2, t) = e^{-r(T-t)} \iint f(X_T)\, g(X_T|v)\, h(v|\sigma_t^2)\, dX_T\, dv$$

$$= \int_0^\infty \Big[\underbrace{e^{-r(T-t)} \int f(X_T)\, g(X_T|v)\, dX_T}_{} \Big] h(v|\sigma_t^2)\, dv$$

i.e. $C(v) \equiv$ Black-Scholes price w/ volatility v

this is true when $\rho = 0$, μ_v, σ_v — indep't of X_t.

Lemma: Suppose that, in a risk-neutral world,

$$dX_t = r X_t\, dt + \sigma_t X_t\, dW_t$$

$$d\sigma_t = \mu_t^v \sigma_t\, dt + \sigma_t^v \sigma_t\, dZ_t$$

assume r — constant, μ_t^V, σ_t^V — indep't of X_t

dW, dZ — indep't Wiener processes.

Then the distribution of $\log\frac{X_T}{x}$ (when $X_0 = x$) conditional on V

is $N(rT - \frac{VT}{2}, VT)$ $\quad \left(V = \frac{1}{T}\int_0^T \sigma_t^2\, dt\right)$

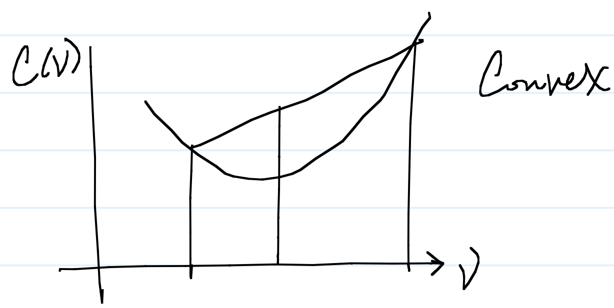
NB: if X_t and σ_t are correlated, the lemma does not hold.

Financial implication

NB: $P = \mathbb{E}[C(V)]$

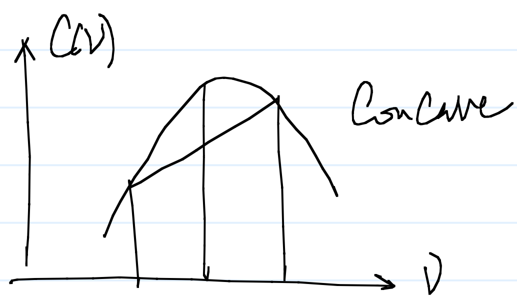
1) if $C(V)$ is convex, i.e.

$\mathbb{E}[C(\cdot)] > C(\mathbb{E}[\cdot])$



2) if $C(V)$ is concave

$\mathbb{E}[C(\cdot)] < C(\mathbb{E}[\cdot])$



NB: BS price for a call is (as a fn of V) $\begin{cases} \text{convex for small } V \\ \text{concave for high values of } V \end{cases}$

To determine when BS price is too high or too low,

examine the curvature of $C(v)$

$$C''(v) = \frac{x\sqrt{T-t}}{4v^{3/2}} N'(d_1)(d_1 d_2 - 1)$$

$$d_{1,2} = \frac{1}{\sqrt{v(T-t)}}\left(\log\frac{X}{K} + \left(r \pm \frac{v}{2}\right)(T-t)\right)$$

The sign of $C''(v)$ is determined by the sign of $(d_1 d_2 - 1)$

the inflection pt: $d_1 d_2 = 1$

i.e. $v = v_{critical} = \frac{2}{T-t}\left[\sqrt{1+\left[\log\frac{X}{K} + r(T-t)\right]^2} - 1\right]$

when $v < v_{critical}$ $\quad C'' > 0 \quad C$ — Convex in v

$v > v_{critical}$ $\quad C'' < 0 \quad C$ — Concave in v

1) at $X = Ke^{-r(T-t)}$, $\quad v_{critical} = 0$

$\boxed{\text{NB: } K = Xe^{r(T-t)} \text{ forward price of a stock.}}$

$\because v > 0$ (By default)

$\Rightarrow C$ — Concave always.

$\Rightarrow$ the actual option price $<$ the BS price.

2) at large $\left|\log\frac{X}{K}\right|$ i.e. X — very small or very large

deep out of money $\quad$ deep in the money

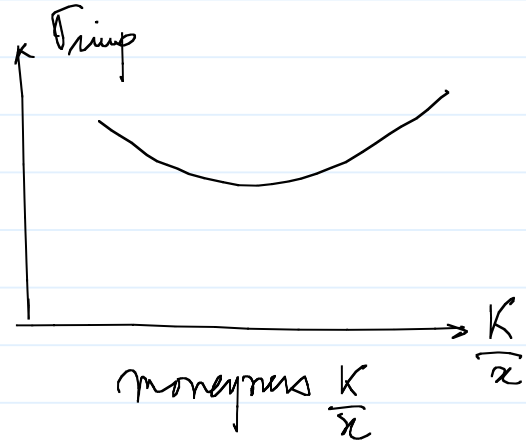
as $\left|\log\frac{X}{K}\right| \to \infty$, $\Rightarrow v_{critical}$ becomes arbitrarily large

$\therefore \quad C \to$ Convex

$\Rightarrow$ the actual price (i.e. $\mathbb{E}[C(\cdot)]$) > the BS price

This picture is consistent with

NB: high price $\to$
high implied vol.



Summary

Features of the stochastic volatility approach.

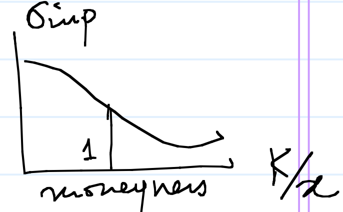
1°. more realistic return distributions,
— tails are fatter than lognormal
— asymmetry of the distribution with noise sources correlated.

2°. Smile effects in option prices in stochastic volatility models. (the correlation controls skewness)

$$dX_t = rX_t dt + X_t \sigma_t (\sqrt{1-\rho^2}\, dW_t + \rho\, dZ_t)$$
$$dY_t = \left[\alpha(m - Y_t) - \beta\left(\rho \frac{\mu - r}{f(Y_t)} + \gamma_t \sqrt{1-\rho^2}\right)\right]dt + \beta\, dZ_t$$

dW, dZ indept. $\sigma_t = f(Y_t)$

Numerical computation from call:



But. 1°. Volatility is not directly observable.

— Difficult in estimating parameters in any model.

2°. No clear cut in choosing a right stochastic vol. model.

3°. Incomplete market.

— i.e. derivatives cannot be perfectly hedged with the underlying asset.

a vol. risk premium has to be estimated from option prices.

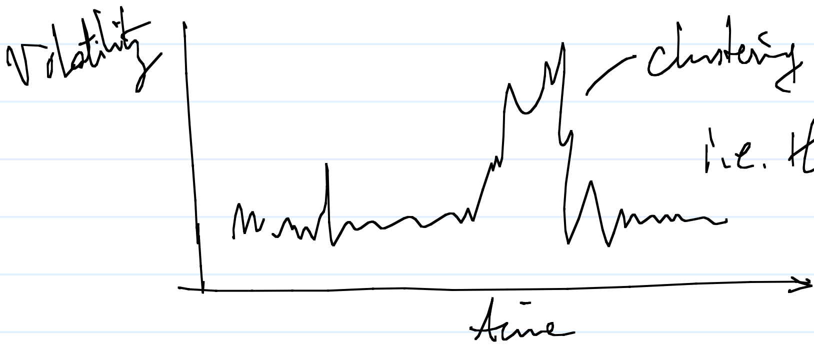
Lecture 7

Note Title 4/9/2006

Scales in mean-reverting Stochastic Volatility

NB: Volatility is not directly observed.

Empirical studies use implied vol. often.



clustering or burstiness.

i.e. the tendency of high volatility comes in bursts.

$\Rightarrow$ a feature of mean reversion.

Simple Examples

Markov chain

Y_t : a 2-state Markov Chain

$$Y_t \in \{-1, +1\}$$

low vol. ↑ high vol ↑ } crude representations of volatility

Random holding time

In dt: switching prob. $\alpha\, dt$

not switching prob $1 - \alpha\, dt$

} α - Const. NB: exponentially waiting time.

Transition prob. matrix
$$\underline{\underline{P}}(t) = \begin{pmatrix} P(Y_{s+t}=-1 \mid Y_s=-1) & P(Y_{s+t}=+1 \mid Y_s=-1) \\ P(Y_{s+t}=-1 \mid Y_s=+1) & P(Y_{s+t}=+1 \mid Y_s=+1) \end{pmatrix}$$

then for small δt increment, we have

$$\underline{\underline{P}}(\delta t) = \begin{pmatrix} 1-\alpha\delta t & \alpha\delta t \\ \alpha\delta t & 1-\alpha\delta t \end{pmatrix} + o(\delta t)$$

$$\therefore \underline{\underline{P}}(t+\delta t) - \underline{\underline{P}}(t) = \underline{\underline{P}}(\delta t)\underline{\underline{P}}(t) - \underline{\underline{P}}(t)$$

$$= \left(\underline{\underline{P}}(\delta t) - I\right)\underline{\underline{P}}(t) \qquad I: 2\times 2 \text{ identity matrix}$$

$$= \begin{pmatrix} -\alpha\delta t & \alpha\delta t \\ \alpha\delta t & -\alpha\delta t \end{pmatrix} \underline{\underline{P}}(t) + o(\delta t)$$

$\delta t \to 0$.

$$\frac{d\underline{\underline{P}}}{dt}(t) = \begin{pmatrix} -\alpha & \alpha \\ \alpha & -\alpha \end{pmatrix} \underline{\underline{P}}(t) \equiv \mathcal{L}\underline{\underline{P}} \qquad \mathcal{L} = \begin{pmatrix} -\alpha & \alpha \\ \alpha & -\alpha \end{pmatrix}$$

NB: The holding time b/w switches is

$$f(\tau) = \begin{cases} \alpha e^{-\alpha\tau} & \tau \geq 0 \\ 0 & \tau < 0 \end{cases}$$

and the mean $\quad \mathbb{E}[\tau] = \frac{1}{\alpha} \qquad \alpha$ – rate of mean reversion

$\therefore$ large α, rapid switching

<u>invariant distribution of Y_t.</u>

i.e. find an initial distribution for Y_0 s.t.

for $\forall t > 0$, Y_t has the same distribution.

Q. How to find the invariant distribution?

g – arbitrary fn $\quad \frac{d}{dt}\mathbb{E}\{g(Y_t)\} = \frac{d}{dt}\mathbb{E}\{\mathbb{E}[g(Y_t) \mid Y_0]\} = 0$

(*)

Suppose the initial distribution $p_0 = \begin{pmatrix} p_0^- \\ p_0^+ \end{pmatrix}$

then Eq (*) $\Rightarrow$ $\frac{d}{dt} \underbrace{p_0^{T} \underline{\underline{P}}(t)\, g}$ $= 0$ (**)

transpose

$= \left(P(+|-) g(+) + P(-|-) g(-) \right) p_0^-$
$+ \left(P(+|+) g(+) + P(-|+) g(-) \right) p_0^+$

$\because$ $\frac{d}{dt} \underline{\underline{P}}(t) = \mathcal{L}\, \underline{\underline{P}}(t)$

$\therefore$ Eq (**) $\Rightarrow$ $p_0^T \mathcal{L} \underbrace{\underline{\underline{P}}(t)\, g} = 0$ $\forall g$

the range of $\mathcal{L}$.

$\therefore$ p_0 is orthogonal to the range of $\mathcal{L}$

$\therefore$ $\mathcal{L}^T p_0 = 0$ i.e. p_0 is in the null space of $\mathcal{L}^T$

($\mathcal{L}^T$ — adjoint operator of $\mathcal{L}$

Here is just a transpose.

(1) $A_{m \times n}$; any vector in its row space is orthogonal to any vector in its null space.

NB: $V \perp W$ if $\forall v \in V, \forall w \in W$
$v^T w = 0$

i.e. the row space is orthogonal to the null space.

(2) the null space of A^T is orthogonal to the range of A

(3) $Ax = b$ is consistent iff $b^T y = 0$ $\forall y$ s.t. $A^T y = 0$
i.e. b is orthogonal to every vector that is orthogonal to the column vector

NB: The invariant distribution solves the adjoint equ.

the soln of $\mathcal{L}^{T} p_0 = 0$ is

$$\begin{pmatrix} -\alpha & \alpha \\ \alpha & -\alpha \end{pmatrix} \begin{pmatrix} p_0^- \\ p_0^+ \end{pmatrix} = 0 \Rightarrow P\{Y_0 = -1\} = P\{Y_0 = +1\} = \frac{1}{2}$$

NB: $\mathcal{L}\phi = 0 \Rightarrow \phi = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$. $c_1 = c_2$ i.e. $\phi = c\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Const. soln.

i.e. the null vectors of the generator of the Markov chain are constant.

$\mathcal{L}$: generator: i.e

$$\mathcal{L}g(y) = \lim_{t \searrow 0} \frac{\mathbb{E}[g(Y_{\Delta t})] - g(y)}{\Delta t}, \quad \forall g$$

Ornstein-Uhlenbeck Process

a standard Brownian motion

$$dY_t = \alpha(m - Y_t)\,dt + \beta\, d\hat{Z}_t$$

NB: Y_t is a Gaussian process.

The corresponding infinitesimal generator of the Markov process Y_t:

$$\mathcal{L} = \alpha(m-y)\frac{\partial}{\partial y} + \frac{1}{2}\beta^2\frac{\partial^2}{\partial y^2}$$

Q: What is the invariant distribution?

$\because \mathcal{L}$ – infinitesimal generator. $\therefore$ $\frac{d}{dt}\mathbb{E}[g(Y_t)] = \mathbb{E}[\mathcal{L}g(Y_t)]$

if $Y_0 \sim$ invariant distribution, then $\frac{d}{dt}\mathbb{E}[g(Y_t)] = 0$

initial

$$\therefore \mathbb{E}[\mathcal{L}g(Y_0)] = 0 \quad \forall g$$

Let $\Phi(y)$ — the pdf of invariant distribution

$$\therefore \int_{-\infty}^{+\infty} \Phi(y) \mathcal{L} g(y)\, dy = 0$$

$$\text{i.e.} \int_{-\infty}^{+\infty} \Phi(y) \left[\alpha(m-y)\frac{\partial}{\partial y} + \frac{1}{2}\beta^2 \frac{\partial^2}{\partial y^2} \right] g(y)\, dy = 0$$

Integration by parts $\Rightarrow$

NB: $\Phi, \Phi' = 0$ as $|y| \to \infty$

$$\int_{-\infty}^{+\infty} g(y) \left[-\alpha \frac{\partial}{\partial y}(m-y)\Phi(y) + \frac{1}{2}\beta^2 \frac{\partial^2}{\partial y^2}\Phi(y) \right] dy = 0 \qquad \text{Eq}(\Phi)$$

$$\text{i.e.} \int_{-\infty}^{+\infty} g(y) \mathcal{L}^\dagger \Phi(y)\, dy = 0$$

the adjoint of $\mathcal{L}$.

$$\mathcal{L}^\dagger = -\alpha \frac{\partial}{\partial y}\left[(m-y)\,\cdot\,\right] + \frac{1}{2}\beta^2 \frac{\partial^2}{\partial y^2}$$

$\because$ Eq(Φ) is valid for any $g(y)$

$$\therefore \mathcal{L}^\dagger \Phi = \frac{1}{2}\beta^2 \frac{d^2}{dy^2}\Phi - \alpha \frac{d}{dy}\left[(m-y)\Phi\right] = 0$$

$$\therefore \frac{1}{2}\beta^2 \frac{d}{dy}\Phi - \alpha(m-y)\Phi = C$$

$$\therefore \Phi = C' e^{-\frac{\alpha(m-y)^2}{\beta^2}}$$

$$\because \int_{-\infty}^{+\infty} \Phi\, dy = 1$$

$$\therefore \Phi(y) = \frac{1}{\sqrt{2\pi\nu^2}} e^{-\frac{(y-m)^2}{2\nu^2}} \sim N(m, \nu^2)$$

$$\nu^2 \equiv \frac{\beta^2}{2\alpha}$$

NB: $E\left[(Y_t - m)(Y_s - m)\right] = \nu^2 e^{-\alpha|t-s|}$

$\frac{1}{\alpha}$ — correlation time. Burst durations if $\sigma_t = e^{Y_t}$

NB: Q: What is the null space of the generator $\mathcal{L}$?

i.e. $$\mathcal{L}\phi = \frac{1}{2}\beta^2\frac{d^2}{dy^2}\phi + \alpha(m-y)\frac{d}{dy}\phi = 0$$

The soln is
$$\phi(y) = C_1\int_{-Y}^{y} e^{\frac{(m-z)^2}{2\nu^2}}\,dz + C_2$$

rapid growing soln as $y\to\pm\infty$. excluded

i.e. well-behaved soln

$\therefore\ \phi(y) = C_2$

i.e. the null space of $\mathcal{L}$ is a const.

— a common property of generators of Markov processes.

NB: Ergodic process. i.e.

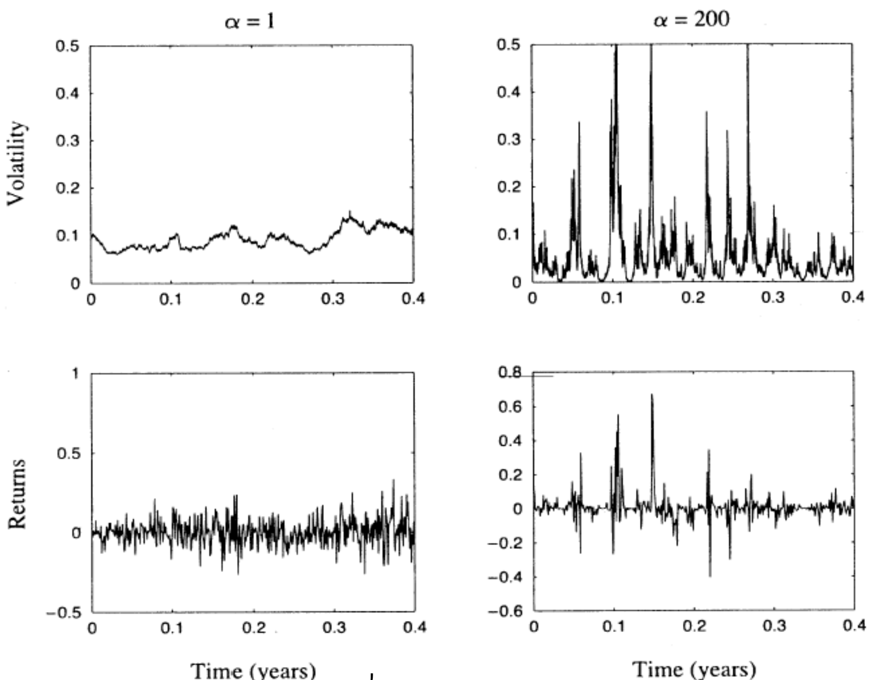
inv. distribution

$$\underbrace{\lim_{t\to\infty}\frac{1}{t}\int_0^t g(Y_s)\,ds}_{\text{the long time average of a bounded function } g} = \langle g\rangle \equiv \underbrace{\int_{-\infty}^{+\infty} g(y)\,\Phi(y)\,dy}_{\text{statistical average wrt its invariant distribution}}$$

The Returns Process

$$\underbrace{\frac{dX_t}{X_t} - \mu dt}_{\text{the de-meaned returns process}} = \sigma_t dW \quad \text{— volatility model.}$$

For Y_t — OU process, $\sigma_t = f(Y) = e^Y$

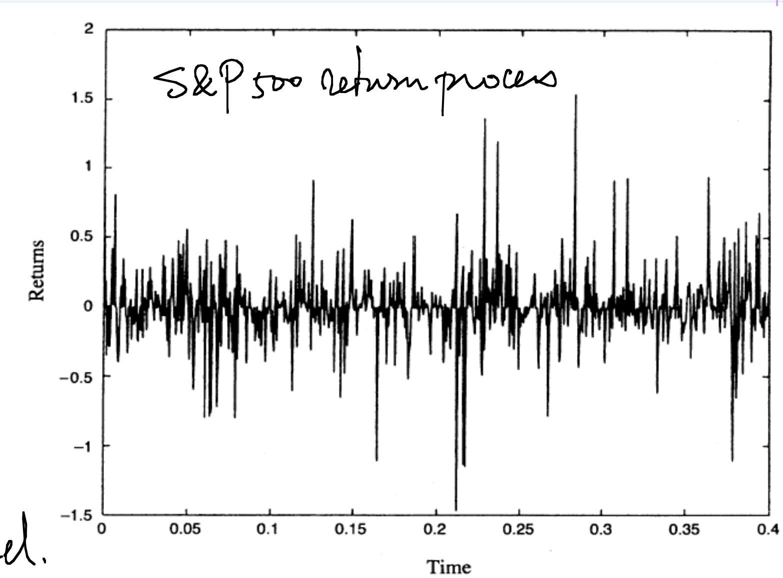


For fast mean-reverting, ← bursty for large α.

size of fluctuations ~ constant

The S&P 500 return process ←

~ fast mean-reverting stochastic vol. model.



[redacted] 1996 S&P 500 returns computed from half-hourly data.

Asymptotics for Pricing European Derivatives

Scalings: 1° The rate of mean reversion — α

$$\varepsilon = \frac{1}{\alpha} \text{ — correlation time}$$

— a small parameter for fast mean reversion

2° The variance of the invariant distribution Y_t is ν^2 — the long-run size of the volatility fluctuations

Assumption $\nu^2 = \text{Const}$ as $\varepsilon \to 0$

Rescale: $dY_t = \alpha(m - Y_t)dt + \beta d\hat{Z}_t$

$$\nu^2 = \frac{\beta^2}{2\alpha} \quad \therefore \quad \beta^2 = 2\nu^2\alpha$$

$$\therefore \quad \beta = \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}$$

f is bounded away from 0 to avoid degenerate diffusion i.e. 0 volatility

$$\therefore \quad dX_t^\varepsilon = rX_t^\varepsilon\, dt + f(Y_t^\varepsilon)X_t^\varepsilon\, dW_t \quad \leftarrow \text{stock}$$

$$dY_t^\varepsilon = \frac{1}{\varepsilon}(m - Y_t^\varepsilon)dt + \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}\, d\hat{Z}_t$$

$$\hat{Z}_t = \rho W_t + \sqrt{1-\rho^2}\, Z_t, \quad W_t, Z_t \text{ — indep}^t \text{ Brownian Motion}$$

The Rescale Pricing Equation

a European derivative with payoff $h(x)$ and maturity T

the price $P^\varepsilon(t, x, y)$

$Y_t^\varepsilon = y$ — the process driving vol.

$X_t^\varepsilon = x$ — present stock value

The PDE is rescaled as

$$\begin{cases} \dfrac{\partial}{\partial t} P^\varepsilon + \dfrac{1}{2} f(y)^2 x^2 \dfrac{\partial P^\varepsilon}{\partial x^2} + \underbrace{\rho \dfrac{\nu\sqrt{2}}{\sqrt{\varepsilon}}}_{\beta} x f(y) \dfrac{\partial^2 P^\varepsilon}{\partial x \partial y} + \underbrace{\dfrac{\nu^2}{\varepsilon}}_{\frac{\beta^2}{2}} \dfrac{\partial^2 P^\varepsilon}{\partial y^2} \\ \quad + r\left(x \dfrac{\partial P^\varepsilon}{\partial x} - P^\varepsilon\right) + \left[\dfrac{1}{\varepsilon}(m-y) - \dfrac{\nu\sqrt{2}}{\sqrt{\varepsilon}} \Lambda(y)\right] \dfrac{\partial P^\varepsilon}{\partial y} = 0 \\ \text{for } t < T \\ P^\varepsilon(T, x, y) = h(x) \\ (\text{final data}) \end{cases}$$

where $\Lambda(y) = \rho \dfrac{\mu - r}{f(y)} + \gamma(y)\sqrt{1-\rho^2}$

the market price of vol. risk.

[assumed to be a bounded fn. of y alone]

The Notation. NB: Three scales. $\frac{1}{\varepsilon}, \frac{1}{\sqrt{\varepsilon}}, 1$.

$\frac{1}{\varepsilon}$: $\quad \mathcal{L}_0 = \nu^2 \dfrac{\partial^2}{\partial y^2} + (m-y)\dfrac{\partial}{\partial y}$

$\frac{1}{\sqrt{\varepsilon}}$: $\quad \mathcal{L}_1 = \sqrt{2}\rho\nu x f(y) \dfrac{\partial^2}{\partial x \partial y} - \sqrt{2}\nu\Lambda(y)\dfrac{\partial}{\partial y}$

(1): $\quad \mathcal{L}_2 = \dfrac{\partial}{\partial t} + \dfrac{1}{2} f(y)^2 x^2 \dfrac{\partial^2}{\partial x^2} + r x \dfrac{\partial}{\partial x} - r = \mathcal{L}_{BS}(f(y))$.

NB: 1°. $\mathcal{L}_0$ — infinitesimal generator of the OU process

2° $\mathcal{L}_1$ — due to correlation ρ. and market price of risk.

3°. $\mathcal{L}_2$ — the Black-Scholes operator w/ vol. $f(y)$.

$\therefore$ the pricing PDE $\Rightarrow$

$$\begin{cases} \left(\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2\right)P^\varepsilon = 0 \\ P^\varepsilon(T, x, y) = h(x) \end{cases} \qquad (*)$$

NB: It is a <u>singular perturbation</u> problem.

Reason: $\varepsilon \frac{\partial}{\partial t}$ $\quad \varepsilon \to 0$.

The Formal Expansion

We are mainly interested in these 2 terms

$$\begin{cases} P^\varepsilon = \underbrace{P_0 + \sqrt{\varepsilon}P_1} + \varepsilon P_2 + \varepsilon\sqrt{\varepsilon}P_3 + \cdots \\ P_0(T, x, y) = h(x) \end{cases}$$

$P_0, P_1 \cdots$ fns of (t, x, y)

NB: $P_1(T, x, y) = 0$

Plug into Eq (*) $\Rightarrow$

$$\frac{1}{\varepsilon}\mathcal{L}_0 P_0 + \frac{1}{\sqrt{\varepsilon}}(\mathcal{L}_0 P_1 + \mathcal{L}_1 P_0)$$
$$+ (\mathcal{L}_0 P_2 + \mathcal{L}_1 P_1 + \mathcal{L}_2 P_0)$$
$$+ \sqrt{\varepsilon}(\mathcal{L}_0 P_3 + \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1) + \cdots = 0$$

$\mathcal{L}_0$ = generator of OU. $\mathcal{L}_0 = \nu^2 \frac{\partial^2}{\partial y^2} + (m-y)\frac{\partial}{\partial y}$ acting on y-variable only

$O(\frac{1}{\varepsilon})$: $\mathcal{L}_0 P_0 = 0$

$\therefore$ P_0 is a Const with respect to y-variable.

(Recall the null space of the Markov Process in general is a Constant)

$\therefore$ $P_0 = P_0(t,x)$

$O(\frac{1}{\sqrt{\varepsilon}})$: $\mathcal{L}_0 P_1 + \mathcal{L}_1 P_0 = 0$, with $P_0 = P_0(t,x)$

$\mathcal{L}_1$ has the derivative $\frac{\partial}{\partial y}$.

$\left(\mathcal{L}_1 = \sqrt{2}\rho\nu x f(y)\frac{\partial^2}{\partial x \partial y} - \sqrt{2}\nu\Lambda(y)\frac{\partial}{\partial y} \right)$

$\therefore$ $\mathcal{L}_0 P_1 = 0$

Again $P_1 = P_1(t,x)$. i.e. function of (t,x) only

$\therefore$ The 1st 2 terms in the expansion

$P_0 + \sqrt{\varepsilon} P_1$ will not depend on the present vol.

($\because$ They are y-independent)

$O(1)$: $\mathcal{L}_0 P_2 + \underbrace{\mathcal{L}_1 P_1}_{=0} + \mathcal{L}_2 P_0 = 0$

$= 0$ ($\mathcal{L}_1$ has $\frac{\partial}{\partial y}$, $P_1 = P_1(t,x)$ only)

$\therefore$ $\mathcal{L}_0 P_2 + \mathcal{L}_2 P_0 = 0$ (1)

For fixed x, $\mathcal{L}_2 P_0$ is a fn of y ($\because f(y)$)

Consider only y dependence. Eq (1) $\Rightarrow$

$$\mathcal{L}_0 \chi + g = 0 \quad \text{— Poisson eqn for } \chi.$$

$$\chi = \chi(y)$$

Q: When does this eqn have a soln?

For soln to exist,

the centering condition, i.e.

$$\langle g \rangle = \int_{-\infty}^{+\infty} g(y)\,\Phi(y)\,dy = 0$$

the inv. distribution of the Markov process whose infinitesimal generator is $\mathcal{L}_0$

must be satisfied — a necessary condition for $\mathcal{L}_0 \chi + g = 0$ to have solutions

NB: For $\mathcal{L}_0$, $\Phi(y) = \dfrac{1}{\sqrt{2\pi}\,\nu} e^{-\frac{(y-m)^2}{2\nu^2}}$

Why this condition?

Pf: $\because g = -\mathcal{L}_0 \chi$

$\therefore \langle g \rangle = -\langle \mathcal{L}_0 \chi \rangle$

$$= -\int (\mathcal{L}_0 \chi)\,\Phi(y)\,dy$$

integration by parts ($\Phi, \Phi' \to 0$ as $|y| \to \infty$)

$$= \int \chi(y)\,\mathcal{L}_0^{\dagger}\,\Phi(y)\,dy$$

$$= 0 \quad (\because \mathcal{L}_0^{\dagger}\,\Phi(y) = 0)$$

QED

Formal soln of $\mathcal{L}_0 \chi + g = 0$:

$$\chi(y) = \int_0^{+\infty} \mathbb{E}\left(g(Y_t) \mid Y_0 = y\right) dt$$

Check: $\mathcal{L}_0 \int_0^{\infty} \mathbb{E}(g(Y_t) \mid Y_0 = y)\, dt = \int_0^{\infty} \mathcal{L}_0 \mathbb{E}\{g(Y_t) \mid Y_0 = y\}\, dt$

$$= \int_0^{\infty} \frac{d}{dt} \mathbb{E}\{g(Y_t) \mid Y_0 = y\}\, dt$$ (Definition of infinitesimal generator)

$$= + \mathbb{E}\{g(Y_t) \mid Y_0 = y\}\Big|_{\infty} - \mathbb{E}\{g(Y_t) \mid Y_0 = y\}\Big|_{t=0}$$

$$\lim_{t \to \infty} \mathbb{E}\{g(Y_t) \mid Y_0 = y\} = \langle g \rangle \overset{\text{||}}{0}$$ (Centering Condition)

i.e. long-run average is $\langle g \rangle$ i.e. wrt invariant distribution

$$= -\mathbb{E}\{g(Y_t) \mid Y_0 = y\}\Big|_{t=0}$$

$$= -g(y)$$

i.e. $\mathcal{L}_0 \chi = -g(y)$. QED.

NB: all solns of $\mathcal{L}_0 \chi + g(y) = 0$ is

$$\chi(y) = \int_0^{\infty} \mathbb{E}\{g(Y_t) \mid Y_0 = y\}\, dt + \text{Const}.$$

i.e. different solns differ by a const.

The 0th order Term

NB: $\mathcal{L}_0 P_2 + \mathcal{L}_2 P_0 = 0$

the centering condition now is

$$\langle \mathcal{L}_2 P_0 \rangle = 0$$

$\because$ $P_0 = P_0(t, x)$ i.e. indep't of y

$$\therefore \langle \mathcal{L}_2 \rangle P_0 = 0$$

$$\because \mathcal{L}_2 = \frac{\partial}{\partial t} + \frac{1}{2} f^2(y) x^2 \frac{\partial^2}{\partial x^2} + r x \frac{\partial}{\partial x} - r = \mathcal{L}_{BS}(f(y))$$

$$\therefore \langle \mathcal{L}_2 \rangle = \frac{\partial}{\partial t} + \frac{1}{2} \langle f^2(y) \rangle x^2 \frac{\partial^2}{\partial x^2} + r x \frac{\partial}{\partial x} - r = \mathcal{L}_{BS}(\bar{\sigma})$$

$$\bar{\sigma}^2 = \int_{-\infty}^{+\infty} f^2(y) \Phi(y) dy$$

$\therefore$ the zero-order $P_0(t, x)$ is

the solu of the BS equation with effective $\bar{\sigma}$.

$$\begin{cases} \mathcal{L}_{BS}(\bar{\sigma}) P_0 = 0 \\ P_0(T, x) = h(x) \quad \text{final data} \end{cases}$$

NB: $\because$ $\langle \mathcal{L}_2 P_0 \rangle = 0$ (i.e. centering condition)

$$\therefore \mathcal{L}_2 P_0 = \mathcal{L}_2 P_0 - \langle \mathcal{L}_2 P_0 \rangle = (\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle) P$$

$$= \frac{1}{2}\left(f^2(y) - \bar{\sigma}^2\right) x^2 \frac{\partial^2}{\partial x^2} P_0$$

From $\mathcal{L}_0 P_2 + \mathcal{L}_2 P_0 = 0 \implies$

$$P_2(t,x,y) = -\mathcal{L}_0^{-1}\mathcal{L}_2 P_0$$

$$= -\frac{1}{2}\mathcal{L}_0^{-1}\left(f^2(y) - \bar{\sigma}^2\right)x^2\frac{\partial^2}{\partial x^2}P_0$$

if $\mathcal{L}_0 \phi = f^2(y) - \bar{\sigma}^2$

i.e. ϕ is a soln

then soln $= \mathcal{L}_0^{-1}(f^2(y) - \bar{\sigma}^2)$

$= \phi_0(y) + \text{Const.}$

$$= -\frac{1}{2}\left(\phi(y) + c(t,x)\right)x^2\frac{\partial^2}{\partial x^2}P_0$$

↑ i.e. the const. wrt y which can be (t,x)-dependent.

Q: What is ϕ? $\because \mathcal{L}_0\phi = f^2(y) - \bar{\sigma}^2$

$\therefore \phi$ satisfies $\quad \nu^2\frac{d^2}{dy^2}\phi + (m-y)\frac{d}{dy}\phi = f^2(y) - \bar{\sigma}^2 \quad$ Eq(ϕ)

NB: $\frac{1}{\Phi}\frac{d}{dy}\left(\Phi\frac{d}{dy}\phi\right) = \frac{1}{\Phi}\left(\frac{d}{dy}\Phi\right)\left(\frac{d}{dy}\phi\right) + \frac{d^2}{dy^2}\phi$

$\Phi = \frac{1}{\sqrt{2\pi}\,\nu}e^{-(y-m)^2/2\nu^2}$

$\therefore \frac{\Phi'}{\Phi} = -\frac{y-m}{\nu^2}$

$$= -\frac{y-m}{\nu^2}\frac{d}{dy}\phi + \frac{d^2}{dy^2}\phi$$

$$\overset{\text{Eq}(\phi)}{=} \frac{1}{\nu^2}\left(f^2(y) - \bar{\sigma}^2\right)$$

$$\therefore \quad \Phi(y)\frac{d}{dy}\phi = \frac{1}{\nu^2}\int_{-\infty}^{y}\left(f^2(z) - \bar{\sigma}^2\right)\Phi(z)\,dz$$

NB: $y \to -\infty$, 0 both sides

$$\therefore \quad \frac{d}{dy}\phi = \frac{1}{\nu^2\Phi(y)}\int_{-\infty}^{y}\left(f^2(z) - \bar{\sigma}^2\right)\Phi(z)\,dz$$

The 1^{st}-order Correction

$O(\sqrt{\varepsilon})$-terms in the expansion:

$$\mathcal{L}_0 P_3 + \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1 = 0$$

Poisson eqn. $\Rightarrow$ Centering Condition:

$$\langle \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1 \rangle = 0$$

Recall $P_2(t,x,y) = -\frac{1}{2}\left(\phi(y) + c(t,x)\right) x^2 \frac{\partial^2}{\partial x^2} P_0$

and $P_1 = P_1(t,x)$ indep't of y.

$$\langle \mathcal{L}_2 \rangle = \mathcal{L}_{BS}(\bar{\sigma})$$

$$\Rightarrow \quad -\frac{1}{2}\left\langle \mathcal{L}_1(\phi(y)+c)\right\rangle x^2 \frac{\partial^2 P_0}{\partial x^2} + \underbrace{\langle \mathcal{L}_2 f(y)\rangle}_{\mathcal{L}_{BS}(\bar{\sigma})} P_1 = 0$$

($\mathcal{L}_1$ has $\frac{\partial}{\partial y}$ only, $\therefore \mathcal{L}_1 c = 0$)

$$\therefore \quad \mathcal{L}_{BS}(\bar{\sigma}) P_1 = \frac{1}{2}\langle \mathcal{L}_1 \phi(y)\rangle x^2 \frac{\partial}{\partial x^2} P_0$$

$\mathcal{L}_1 = \sqrt{2}\rho\nu x f(y) \frac{\partial^2}{\partial x \partial y} - \sqrt{2}\nu \Lambda(y) \frac{\partial}{\partial y}$

$$= \frac{1}{2}\left(\sqrt{2}\rho\nu\langle f(y)\phi'(y)\rangle x\frac{\partial}{\partial x} - \sqrt{2}\nu\langle \Lambda(y)\phi'(y)\rangle\right) x^2 \frac{\partial^2}{\partial x^2} P_0$$

$$\therefore \begin{cases} \mathcal{L}_{BS}(\bar{\sigma}) P_1 = \frac{\sqrt{2}}{2}\rho\nu\langle f\phi'\rangle x^3 \frac{\partial^3}{\partial x^3} P_0 \\ \qquad + \sqrt{2}\rho\nu\langle f\phi'\rangle x^2 \frac{\partial^2}{\partial x^2} P_0 - \frac{\sqrt{2}}{2}\nu\langle \Lambda\phi'\rangle x^2 \frac{\partial^2}{\partial x^2} P_0 \\ P_1(T,x) = 0 \end{cases}$$

— the equation for P_1

the 1^{st} small correction

$\therefore$ $\tilde{P}_1(t,x) \equiv \sqrt{\varepsilon} P_1(t,x)$ is the soln of

$$\begin{cases} \mathcal{L}_{BS}(\bar{\sigma}) \tilde{P}_1 = H(t,x) \\ \tilde{P}_1 = 0 \end{cases}$$

the source: $H(t,x) = V_2 x^2 \frac{\partial^2 P_0}{\partial x^2} + V_3 x^3 \frac{\partial^3 P_0}{\partial x^3}$

$$\left.\begin{aligned} V_2 &= \frac{\nu}{\sqrt{2\alpha}} \left(2\rho \langle f\phi' \rangle - \langle \Lambda \phi' \rangle \right) \\ V_3 &= \frac{\rho \nu}{\sqrt{2\alpha}} \langle f \phi' \rangle \end{aligned}\right\} \text{small corrections}$$

Q: How to solve for $\tilde{P}_1$?

NB. $\mathcal{L}_{BS}(\bar{\sigma}) \left(-(T-t) H \right) = H - (T-t) \mathcal{L}_{BS}(\bar{\sigma}) H$

$\mathcal{L}_{BS}(\bar{\sigma}) H$ contains only terms of the type:

$$\mathcal{L}_{BS}(\bar{\sigma}) \left(x^n \frac{\partial^n}{\partial x^n} P_0 \right) = x^n \frac{\partial^n}{\partial x^n} \mathcal{L}_{BS}(\bar{\sigma}) P_0 = 0$$

(check yourself)

$\therefore$ $\mathcal{L}_{BS}(\bar{\sigma}) \left(-(T-t) H \right) = H$

$\therefore$ $\tilde{P}_1(t,x) = -(T-t) \left(V_2 x^2 \frac{\partial^2}{\partial x^2} P_0 + V_3 x^3 \frac{\partial^3}{\partial x^3} P_0 \right)$

$\therefore$ The corrected price is

$$P_0(t,x) - (T-t) \left(V_2 x^2 \frac{\partial^2}{\partial x^2} P_0 + V_3 x^3 \frac{\partial^3}{\partial x^3} P_0 \right)$$

with $\mathcal{L}_{BS}(\bar{\sigma}) P_0 = 0$, $P_0(T,x) = h(x)$

i.e. P_0 — the BS price w/ $\bar{\sigma}$.

Universality of the correction:

observation: Any stochastic volatility model will gives rise a 1^{st} correction of this type (if the stochastic vol. model is driven by an ergodic diffusion process)

NB: V_2 and V_3 determine the 1^{st} correction.

NB: $$H(t,x) = \frac{\varepsilon}{\alpha}\langle \mathcal{L}_1 \phi(y)\rangle x^2 \frac{\partial^2}{\partial x^2} P_0$$

$$= \frac{1}{2\sqrt{\alpha}}\langle \mathcal{L}_1 \mathcal{L}_0^{-1}(f^2(y) - \bar{\sigma}^2)\rangle x^2 \frac{\partial^2}{\partial x^2} P_0$$

$$= \frac{1}{\sqrt{\alpha}}\langle \mathcal{L}_1 \mathcal{L}_0^{-1}(\mathcal{L}_2 - \langle \mathcal{L}_2\rangle)\rangle P_0$$

Recall
$$(\mathcal{L}_2 - \langle \mathcal{L}_2\rangle) P_0 = \frac{1}{2}(f^2(y) - \bar{\sigma}^2) x^2 \frac{\partial^2 P_0}{\partial x^2}$$

NB: $$\left.\begin{aligned} dY_t^\varepsilon &= \frac{1}{\varepsilon}\mu_Y(t,Y_t)\,dt + \frac{1}{\sqrt{\varepsilon}}\sigma_Y(t,Y_t)\,dZ_t \\ \mathcal{L} &= \frac{1}{\varepsilon}\mu_Y(t,y)\frac{\partial}{\partial y} + \frac{1}{2\varepsilon}\sigma_Y^2(t,y)\frac{\partial^2}{\partial y^2} \end{aligned}\right\}$$ a general diffusion proc.

the pricing PDE $\Rightarrow$ $$\left(\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2\right)P^\varepsilon = 0$$

$$\mathcal{L}_0 = \frac{1}{2}\sigma_Y^2\frac{\partial^2}{\partial y^2} + \mu_Y\frac{\partial}{\partial y},$$

$$\mathcal{L}_1 = \rho\sigma_Y x f(y)\frac{\partial^2}{\partial x\partial y} - \sigma_Y\Lambda(y)\frac{\partial}{\partial y}$$

$$\mathcal{L}_2 = \mathcal{L}_{BS}(f(y))$$

Following the perturbation procedure above, $\Rightarrow$

$$\therefore \mathcal{L}_{BS}(\bar{\sigma})\tilde{P}_1 = \frac{1}{\sqrt{\alpha}}\langle \mathcal{L}_1 \mathcal{L}_0^{-1}(\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle)\rangle P_0$$

$$= A P_0 \qquad A = \underbrace{V_2 x^2 \frac{\partial^2}{\partial x^2} + V_3 x^3 \frac{\partial^3}{\partial x^3}}$$

this structure is universal. detailed expressions don't matter.

For example, if Y_t — OU process.

$$V_2 = \frac{\nu}{\sqrt{2\alpha}}\left(2\rho\langle f\phi' \rangle - \langle \Lambda \phi' \rangle\right), \quad V_3 = \frac{\rho\nu}{\sqrt{2\alpha}}\langle f\phi' \rangle$$

However, we are going to use the universal form for parameterization.

NB: the market price of vol. risk is in V_2

Put-Call Parity

$$C_0(t,x) - P_0(t,x) = x - Ke^{-r(T-t)}$$

NB: $$\tilde{C}_1 - \tilde{P}_1 = -(T-t)\left(V_3 x^3 \frac{\partial^3}{\partial x^3} + V_2 x^2 \frac{\partial^2}{\partial x^2}\right)(C_0 - P_0)$$

$$= -(T-t)\left(V_3 x^3 \frac{\partial^3}{\partial x^3} + V_2 x^2 \frac{\partial^2}{\partial x^2}\right)(x - Ke^{-r(T-t)})$$

$$= 0$$

i.e. Put-Call parity preserved.

$$\therefore \left(C_0(t,x) + \tilde{C}_1(t,x)\right) - \left(P_0(t,x) + \tilde{P}_1(t,x)\right) = x - Ke^{-r(T-t)}$$

The Skew Effect

$V_3 \propto \rho$. correlation.
e.g. OU process
$V_3 = \frac{\rho \nu}{\sqrt{2\alpha}} \langle f \phi' \rangle$

$$\because \quad \mathcal{L}_{BS}(\bar{\sigma}) \tilde{P}_1 = V_2 x^2 \frac{\partial^2 P_0}{\partial x^2} + V_3 x^3 \frac{\partial^3}{\partial x^3} P_0$$

if $\rho = 0$, then $\frac{\partial^3}{\partial x^3}$-term vanishes.

$$\therefore \begin{cases} \mathcal{L}_{BS}(\bar{\sigma})(P_0 + \tilde{P}_1) = V_2 x^2 \frac{\partial^2 P_0}{\partial x^2} + V_3 x^3 \frac{\partial^3}{\partial x^3} P_0 \\ (P_0 + \tilde{P}_1)(T, x) = h(x) \end{cases}$$

$$\to V_2 x^2 \frac{\partial^2 (P_0 + \tilde{P}_1)}{\partial x^2} - V_2 x^2 \frac{\partial^2 \tilde{P}_1}{\partial x^2}$$

$\hookrightarrow$ This can be absorbed by $\tilde{\sigma}$

if V_2 is sufficiently small. $\tilde{\sigma} \equiv \sqrt{\bar{\sigma}^2 - 2V_2}$

$$\therefore \quad \underbrace{\mathcal{L}_{BS}(\tilde{\sigma})(P_0 + \tilde{P}_1)}_{\tilde{P} \text{ corrected price}} = \underbrace{-V_2 x^2 \frac{\partial^2 \tilde{P}_1}{\partial x^2}}_{O(\varepsilon)\ (\because\ \tilde{P}_1 \sim \sqrt{\varepsilon},\ V_2 \sim \sqrt{\varepsilon})} + \underbrace{V_3 x^3 \frac{\partial^3}{\partial x^3} P_0}_{O(\sqrt{\varepsilon})}$$

$$\begin{cases} \mathcal{L}_{BS}(\tilde{\sigma}) \tilde{P} \approx V_3 x^3 \frac{\partial^3}{\partial x^3} P_0 \\ \tilde{P}(T, x) = h(x) \end{cases}$$

$\therefore$ V_2 merely corrects the volatility level.

V_3 cannot be absorbed into the original infinitesimal operator.

$\Longrightarrow$ skew. i.e. 3rd moment of stock price return if $\rho = 0. \implies V_3 = 0$

$$\therefore \mathcal{L}_{BS}(\tilde{\sigma})\tilde{p} \simeq 0$$

vol. $\tilde{\sigma}$, including the correction of market price of vol. risk

Implied volatility and Calibration

Q: How to link V_2, V_3 with observed prices or implied volatilities?

Suppose $h(x) = (x-K)^+$

Recall $P_0 = C_{BS} = xN(d_1) - Ke^{-r(T-t)}N(d_2)$

$$d_{1,2} = \frac{1}{\sigma\sqrt{T-t}}\left[\log\left(\frac{x}{K}\right) + \left(r \pm \frac{1}{2}\sigma^2\right)(T-t)\right]$$

We can check: $\dfrac{\partial d_{1,2}}{\partial x} = \dfrac{1}{x\sigma\sqrt{T-t}}$

$$N'(d) = \frac{1}{\sqrt{2\pi}} e^{-d^2/2}$$

$$e^{-d_2^2/2} = e^{-d_1^2/2}\left(\frac{x e^{r(T-t)}}{K}\right)$$

and the Delta $\dfrac{\partial P_0}{\partial x} = N(d_1)$

the Gamma $\dfrac{\partial^2 P_0}{\partial x^2} = \dfrac{e^{-d_1^2/2}}{x\sigma\sqrt{2\pi(T-t)}}$

"the Epsilon" $\dfrac{\partial^3 P_0}{\partial x^3} = -\dfrac{e^{-d_1^2/2}}{x^2 \bar{\sigma}\sqrt{2\pi(T-t)}}\left(1 + \dfrac{d_1}{\bar{\sigma}\sqrt{T-t}}\right)$

$$\therefore H(t,x) = \frac{x e^{-d_1^2/2}}{\bar{\sigma}\sqrt{2\pi(T-t)}}\left(V_2 - V_3\left(1 + \frac{d_1}{\bar{\sigma}\sqrt{T-t}}\right)\right)$$

$$\therefore \tilde{P}_1(t,x) = -(T-t)\,H(t,x)$$

the 1st correction $\displaystyle = \frac{x e^{-d_1^2/2}}{\bar{\sigma}\sqrt{2\pi}}\left(V_3 \frac{d_1}{\bar{\sigma}} + (V_3 - V_2)\sqrt{T-t}\right)$

Q: How can we extract V_2, V_3 from the market-observed prices?

Recall $C_{BS}(t, x; K, T; I) = C^{obs}(K,T)$

implied vol

Expand $I = \bar{\sigma} + \sqrt{\varepsilon} I_1 + \cdots$

$$\therefore C_{BS}(t, x; K, T; I) = C_{BS}(t, x; K, T; \bar{\sigma}) + \sqrt{\varepsilon}\, I_1 \frac{\partial C_{BS}}{\partial \sigma}(t, x; K, T, \bar{\sigma}) + \cdots$$

$$= P_0(t,x) + \tilde{P}_1(t,x)$$

$$\therefore \tilde{P}_1(t,x) = \sqrt{\varepsilon}\, I_1 \frac{\partial C_{BS}}{\partial \sigma}(t, x; K, T, \bar{\sigma})$$

$O(\varepsilon)$

$$\therefore I = \bar{\sigma} + \tilde{P}_1(t,x)\left[\frac{\partial C_{BS}}{\partial \sigma}(t, x; K, T; \bar{\sigma})\right]^{-1} + O\left(\frac{1}{\alpha}\right)$$

Recall the Vega: $\dfrac{\partial C_{BS}}{\partial \sigma} = x e^{-d_1^2/2}\sqrt{\dfrac{T-t}{2\pi}}$

$$\therefore I = \bar{\sigma} + \tilde{P}_1(t,x) \cdot \frac{1}{x} e^{d_1^2/2}\sqrt{\frac{2\pi}{T-t}} + O\left(\frac{1}{\alpha}\right)$$

$$\therefore \quad I = \bar{\sigma} + \underbrace{\frac{x e^{-d_1^2/2}}{\bar{\sigma}\sqrt{2\pi}}\left(V_3 \frac{d_1}{\bar{\sigma}} + (V_3 - V_2)\sqrt{T-t}\right)}_{\tilde{P}_1(t,x)} \cdot \frac{1}{x} e^{d_1^2/2}\sqrt{\frac{2\pi}{T-t}} + O\left(\frac{1}{\alpha}\right)$$

$$= \bar{\sigma} + \frac{V_3 d_1}{\bar{\sigma}^2\sqrt{T-t}} + \frac{V_3 - V_2}{\bar{\sigma}} + O\left(\frac{1}{\alpha}\right)$$

use d_1's expression

$$\Rightarrow \quad I = \bar{\sigma} + \frac{V_3}{\bar{\sigma}^3}\left(r + \frac{3}{2}\bar{\sigma}^2\right) - \frac{V_2}{\bar{\sigma}} - \frac{V_3}{\bar{\sigma}^3}\boxed{\frac{\log\left(\frac{K}{x}\right)}{(T-t)}} + O\left(\frac{1}{\alpha}\right)$$

moneyness

$\|$ LMMR

log-moneyness-to-Maturity-Ratio

$$\therefore \quad I = a\left[\frac{\log\left(\frac{\text{strike price}}{\text{asset price}}\right)}{\text{Time to maturity}}\right] + b + O\left(\frac{1}{\alpha}\right)$$

NB: a, b, Can be extract from the market.

$$\begin{cases} a = -\dfrac{V_3}{\bar{\sigma}^3} \\[2ex] b = \bar{\sigma} + \dfrac{V_3}{\bar{\sigma}^3}\left(r + \dfrac{3}{2}\bar{\sigma}^2\right) - \dfrac{V_2}{\bar{\sigma}} \end{cases}$$

$$\Rightarrow \begin{cases} V_3 = -a\bar{\sigma}^3 \\ V_2 = \bar{\sigma}\left((\bar{\sigma} - b) - a\left(r + \frac{3}{2}\bar{\sigma}^2\right)\right) \end{cases}$$

NB: The implied volatility surface i.e. $I = I(K, T-t)$ is

$$I \cong a \cdot \text{LMMR} + b$$

i.e. I is a fn of $\left(\dfrac{\log\frac{K}{x}}{T-t}\right)$ Combined.

NB: if $K = x$, $I \approx b$ i.e. b is the at-the-money implied vol.

Why don't we go to higher order corrections?

NB: Loss of universality, i.e. the correction & calibration procedure depend on specific model of Y_t.

Dividends:

For simplicity, Continuous dividend yield D_0

$\therefore$ $$dX_t = (\mu - D_0) X_t\,dt + f(Y_t) X_t\,dW$$ with the same volatility process.

Recall: the corresponding BS-operator:

$$\mathcal{L}_{BS}^{D}(\bar\sigma) = \frac{\partial}{\partial t} + \frac{1}{2}\bar\sigma^2 x^2 \frac{\partial^2}{\partial x^2} + (r - D_0) x \frac{\partial}{\partial x} - r$$

In our calculation:

only change: $\mathcal{L}_2 \Rightarrow \mathcal{L}_2^D = \frac{\partial}{\partial t} + \frac{1}{2} f(y)^2 x^2 \frac{\partial^2}{\partial x^2} + (r - D_0) x \frac{\partial}{\partial x} - r$

$$= \mathcal{L}_{BS}^{D}(f(y))$$

For P_0: we still have
$$\begin{cases} \mathcal{L}_2^D P_0 = \mathcal{L}_{BS}^D(\bar\sigma) P_0 = 0 \\ P_0(T, x) = h(x) \end{cases}$$

$\because$ $\mathcal{L}_{BS}(\bar\sigma) \tilde P_1 = \frac{1}{\sqrt{\alpha}} \langle \mathcal{L}_1 \mathcal{L}_0^{-1} (\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle) \rangle P_0$ in general

$$= \left(V_2 x^2 \frac{\partial^2}{\partial x^2} + V_3 x^3 \frac{\partial^3}{\partial x^3} \right) P_0$$

$\because$ $(\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle) P_0 = (\mathcal{L}_{BS}^D - \langle \mathcal{L}_{BS}^D \rangle) P_0$: There is no depend on D_0 in the operator

NB: $\tilde P_1(t, x) = -(T - t)\left(V_2 x^2 \frac{\partial^2}{\partial x^2} P_0 + V_3 x^3 \frac{\partial^3}{\partial x^3} P_0 \right)$ all dependence on D_0 is in P_0.

$\therefore$ V_2, V_3 remain the same. (no D_0-dependence)

For a European call,

$$\therefore \quad P_0(t,x) = x e^{-D_0(T-t)} N(d_{10}) - K e^{-r(T-t)} N(d_{20})$$

$$d_{10} = \frac{1}{\bar{\sigma}\sqrt{T-t}}\left[\log\left(\frac{x}{K}\right) + \left(r - D_0 + \frac{1}{2}\bar{\sigma}^2\right)(T-t)\right]$$

$$d_{20} = d_{10} - \bar{\sigma}\sqrt{T-t}.$$

$$\tilde{P}_1(t,x) = \frac{x e^{-D_0(T-t) - d_{10}^2/2}}{\bar{\sigma}\sqrt{2\pi}}\left(V_3 \frac{d_{10}}{\bar{\sigma}} + (V_3 - V_2)\sqrt{T-t}\right)$$

The calibration formula become.

$$I = \bar{\sigma} + \frac{V_3 d_{10}}{\bar{\sigma}^2\sqrt{T-t}} + \frac{V_3 - V_2}{\bar{\sigma}} + O\left(\frac{1}{\alpha}\right)$$

$$= \bar{\sigma} + \frac{V_3}{\bar{\sigma}^3}\left(r - D_0 + \frac{3}{2}\bar{\sigma}^2\right) - \frac{V_2}{\bar{\sigma}} - \frac{V_3}{\bar{\sigma}^3}\left[\frac{\log\frac{K}{x}}{T-t}\right] + O\left(\frac{1}{\alpha}\right)$$

$$I = a\left[\frac{\log\left(\frac{\text{strike price}}{\text{asset price}}\right)}{\text{time to maturity}}\right] + b + O\left(\frac{1}{\alpha}\right)$$

$$V_2 = \bar{\sigma}\left((\bar{\sigma} - b) - a\left(r - D_0 + \frac{3}{2}\bar{\sigma}^2\right)\right)$$

$$V_3 = -a\bar{\sigma}^3$$

$\Rightarrow$ From calibrated a, b to get V_2, V_3

Implementation

Model & Data

Mean-reverting, Stochastic Volatility

e.g.
$$dX_t = \mu X_t\,dt + f(Y_t)\,X_t\,dW_t$$
$$dY_t = \alpha(m - Y_t)\,dt + \beta\left(\rho\,dW_t + \sqrt{1-\rho^2}\,dZ_t\right)$$

W_t and Z_t — indep't.

Q: How to estimate the parameters?

1). the rate of mean reversion α.

2) the long-run mean m of Y_t

3) the volatility of volatility, β

4) the correlation coefficient ρ.

NB: only the stock price X_t is directly observable.

Discrete Data:

NB: Tick-by-tick observations — $O(10^3)$ points per day.
— unevenly spaced.

Suppose we average over 5 minutes intervals

1 hour 12 pts. 6-hour trading days 251 trading days

$\Rightarrow$ 72 data pts. $\Rightarrow 72 \times 251 = 18072$ pts/yr

$\bar{X}_n$ — n^{th} 5 mins average of the asset price at $t_n = n\Delta t$

$\Delta t = 5$ mins

Normalized fluctuation of the data:

$$\bar{D}_n = \frac{2(\bar{X}_n - \bar{X}_{n-1})}{\sqrt{\Delta t}\,(\bar{X}_n + \bar{X}_{n-1})} \quad \text{i.e.} \quad \frac{\text{increment}\,(X_n - X_{n-1})}{\text{average}\left(\frac{X_n + X_{n-1}}{2}\right)\sqrt{\Delta t}}$$

$\therefore$ $\bar{D}_n$ — the observed realization of return

$$\therefore \quad \frac{1}{\sqrt{\Delta t}}\frac{\Delta X_t}{X_t} = f(Y_t)\frac{\Delta W_t}{\sqrt{\Delta t}} + \mu\sqrt{\Delta t} \quad \text{(from stoch. vol model)}$$

NB: $\mu\sqrt{\Delta t}$ is negligibly small.

$$\bar{D}_n = f(Y_n)\,\varepsilon_n$$

iid Gaussian rand. var. $\sim N(0,1)$

NB: $\frac{\Delta W_t}{\sqrt{\Delta t}}$ has var 1.

e.g. if $f(y) = e^y$, y — OU process.

then $$L_n = \log|\bar{D}_n| = \underbrace{\log f(Y_n)}_{\text{the OU process.}} + \log|\varepsilon_n|$$

log fluctuations

$$\therefore \quad \mathbb{E}[Y_{n+j} Y_n] = \frac{\beta^2}{2\alpha} e^{-\alpha j \Delta t}$$

Variogram analysis

variogram (empirical structure function)

$$V_j^N = \frac{1}{N}\sum_{n=1}^{N} (L_{n+j} - L_n)^2$$

N: the total # of obs.

j — lag

Assume $\rho = 0$ i.e. no correlation b/w volatility and asset prices

then $\mathbb{E}(L_{n+j} - L_n)^2$

$= \mathbb{E}(L_n - L_0)^2$ (stationarity)

$\rho=0$ $= \mathbb{E}[\log f(Y_j) - \log f(Y_0)]^2 + \mathbb{E}(\log|\varepsilon_j| - \log|\varepsilon_0|)^2$

$= 2\mathbb{E}[\log f(Y)]^2 - 2\mathbb{E}[\log f(Y_j)\log f(Y_0)] + 2\underbrace{\mathrm{Var}(\log|\varepsilon|)}_{(\text{NB: } \mathbb{E}\log|\varepsilon| = 0)}$

$\cong 2\nu^2(1 - e^{-j\alpha\Delta t}) + 2C^2$

$C^2 \equiv \mathrm{Var}(\log|\varepsilon|)$

$\nu^2 =$ variance of Y

NB: For more general f, we still have

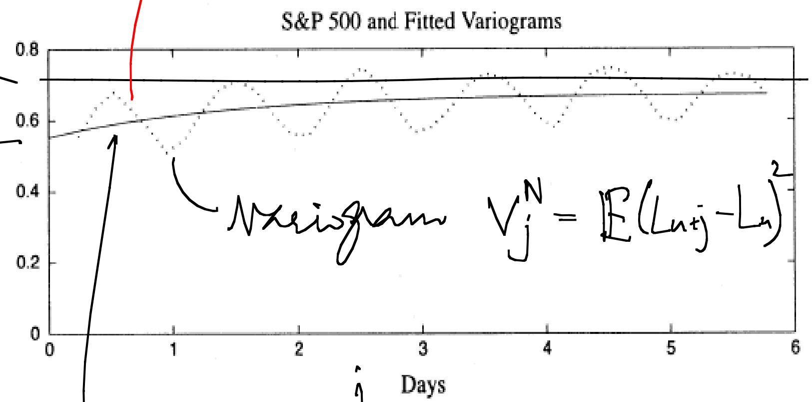
$\mathbb{E}[f(Y_{n+j})f(Y_n)] \sim \nu_f^2 e^{-j\alpha\Delta t}$ as $\alpha \to \infty$ ν_f^2 — var of $(\log f(Y))$

$\therefore$ $\nu^2 \to \nu_f^2 \Rightarrow \mathbb{E}(L_{n+j} - L_n)^2 = 2\nu_f^2(1 - e^{-j\alpha\Delta t}) + 2C^2$

oscillations: day effects
(not in the model)

Asymptote $\Rightarrow 2\nu_f^2$

intercept: $(j=0) \Rightarrow 2C^2$ of the $\log|\varepsilon_n|$



Variogram $V_j^N = \mathbb{E}(L_{n+j}-L_n)^2$

j

from the exponential decay $\Rightarrow \alpha$

i.e. $V_j^N \simeq 2C^2 + 2\nu_f^2\left(1 - e^{-\alpha|j|}\right)$ if ignoring correlations b/w $\log|\varepsilon_n|$ and Y_n

e.g. For S&P500 vol. the characteristic time of mean-reversion is 1.5 days

$\Rightarrow$ In terms of calendar years, $\frac{1}{\alpha} \sim 0.004$.

$\Rightarrow$ establish fast-mean-reversion

Now Let's turn to the calibration of the price:

(1) Estimate $\bar{\sigma}$ (effective historical volatility) from stock-price returns

(2). Use variogram analysis of historical stock-price returns to establish that vol. is fast mean-reverting.

(3). Fit
$$I = a\left(\frac{\log\left(\frac{\text{strike price}}{\text{stock price}}\right)}{\text{time to maturity}}\right) + b$$

to the implied volatility surface I across the strikes and maturities for liquid options.

$\Rightarrow$ estimation of a, b,

(4)
$$V_2 = \bar{\sigma}\left((\bar{\sigma} - b) - a\left(r + \frac{3}{2}\bar{\sigma}^2\right)\right)$$
$$V_3 = -a\bar{\sigma}^3$$
r: interest rate assumed to be const.

(5) Correct the price P_0 of a European option w/ payoff $h(X_T)$ by
$$P_0 - (T-t)\left(V_2 x^2 \frac{\partial^2 P_0}{\partial x^2} + V_3 x^3 \frac{\partial^3 P_0}{\partial x^3}\right)$$
P_0 — BS price w/ const vol $\bar{\sigma}$ and interest rate r

due to stochastic volatility.

NB: V_2, V_3 — market constants (arising from stoch. vol.)
Can be used to price other types of derivatives
e.g. Asian, barrier, American etc. and for hedging

Features of this approach

1°. Model independence.

i.e. no specific stochastic volatility model is assumed

except: fast mean reversion.

2°. Parsimony of Parameters

NB. Model parameters:

mean level of vol.
variance of vol.
Rate of mean-reversion of vol.
correlation ρ
volatility risk premium
(NB. the difficulty of estimating ρ and risk premium)

} at least 5 parameters

But only mean level of vol $\bar{\sigma}$
a, b
} 3 parameters need to price.

3° Stability of Parameter Estimates.

A posteriori, through daily fitting, say. $\Rightarrow$ stable.

S&P500 Europ. calls $\Rightarrow$ $a = -0.154 \pm \underline{0.032}$ $\quad b = 0.149 \pm \underline{0.007}$

$\bar{\sigma} = 0.1, \; r = 0.02$

$V_2 = -0.0044, \quad V_3 = 0.000154$

Hedging Strategies

NB:

In an incomplete market, perfect hedge — not possible.

Aim of the hedge: find a reasonable trade-off b/w the risk of a failed hedge and the cost of implementing hedge.

Review: BS Delta Hedging

Under a const volatility $\bar{\sigma}$,

$$dX_t = \mu X_t dt + \bar{\sigma} X_t dW_t$$

$$\Delta = \frac{\partial P_0}{\partial x}$$

Hedged portfolio: Δ units of the risky asset

$e^{-rt}(P_0 - X_t \Delta)$ units of riskfree asset.

NB: a time t: this portfolio $= P_0$

a time T: it replicates $h(X_T)$. at maturity.

NB: $$dP_0 = \left(\frac{\partial P_0}{\partial t} + \frac{1}{2}\bar{\sigma}^2 x^2 \frac{\partial^2 P_0}{\partial x^2} + \mu x \frac{\partial P_0}{\partial x} \right) dt + \frac{\partial P_0}{\partial x} x \sigma dW$$

$$\overset{\text{BS eqn}}{=} \left(-rx \frac{\partial P_0}{\partial x} + rP_0 \right) dt + \mu x \frac{\partial P_0}{\partial x} dt + \frac{\partial P_0}{\partial x} x \sigma dW$$

$$= \frac{\partial P_0}{\partial x} dX + r\left(P_0 - X \frac{\partial P_0}{\partial x} \right) dt = \Delta dX + r(P_0 - X\Delta) dt$$

i.e. Price variation = the variation due to market. (self-financing)

The strategy and its cost

Suppose we follow the same BS strategy in a stochastic volatility environment:

$$\begin{cases} dX_t = \mu X_t\,dt + f(Y_t) X_t\,dW_t \\ dY_t = \alpha(m - Y_t)\,dt + \beta\,d\hat{Z} \end{cases}$$

$\therefore$ The portfolio has $a_t = \frac{\partial P_0}{\partial x}(t, x)$ stocks

$b_t = e^{-rt}\left(P_0(t,x) - X_t \frac{\partial P_0}{\partial x}(t, X_t)\right)$ bonds

at time t.

$\therefore$ its value at time t:

$$a_t X_t + b_t e^{rt} = P_0(t, X_t)$$

at time T. $P_0(T, X_T) = h(X_T)$

$\therefore$ the strategy replicates the derivative at maturity

Q: Is it self-financing?

$$dP_0(t, X_t) = \left(\frac{\partial P_0}{\partial t}(t, X_t) + \frac{1}{2} f^2(Y_t) X_t^2 \frac{\partial^2 P_0}{\partial x^2}(t, X_t)\right) dt + \underbrace{\frac{\partial P_0}{\partial x}(t,x)\,dX_t}_{=\, a_t dX_t}$$

But the change due to the market is

$$a_t dX_t + r b_t e^{rt} dt$$

(i.e. the self-financing part)

$\therefore$ the infinitesimal cost of the strategy is

$$dP_0(t, X_t) - (a_t dX_t + r b_t e^{rt} dt)$$

$$= \left(\frac{\partial P_0}{\partial t} + \frac{1}{2} f^2(Y_t) X_t^2 \frac{\partial^2 P_0}{\partial x^2}\right) dt + a_t dX_t - (a_t dX_t + r b_t e^{rt} dt)$$

$$= \left(\frac{\partial P_0}{\partial t} + \frac{1}{2} f^2(Y_t) X_t^2 \frac{\partial^2 P_0}{\partial x^2}\right) dt - r\left(P_0 - X_t \frac{\partial P_0}{\partial x}\right) dt$$

NB. $\mathcal{L}_{BS}(\bar{\sigma}) P_0 = 0$

$$= \frac{1}{2}\left(f^2(Y_t) - \bar{\sigma}^2\right) X_t^2 \frac{\partial^2 P_0(t, X_t)}{\partial x^2} \cdot dt$$

$\therefore$ The cumulative cost up to time t is

$$E_0(t) = \frac{1}{2} \int_0^t \left(f^2(Y_s) - \bar{\sigma}^2\right) X_s^2 \frac{\partial^2 P_0}{\partial x^2}(s, X_s)\, ds$$

$\therefore$ In addition to the initial cost $P_0(0, X_0)$, the additional total cost is

$$E_0(T) = \frac{1}{2} \int_0^T \left(f^2(Y_s) - \bar{\sigma}^2\right) X_s^2 \frac{\partial^2 P_0}{\partial x^2}(s, X_s)\, ds$$

$E_0(T)$ — further financed.

Averaging Effect

$\because$ Y_t is running on the fast time scale (i.e. $\alpha \ll 1$)

$$\therefore \quad \frac{1}{2} \int_0^t f^2(Y_s) X_s^2 \frac{\partial^2 P}{\partial x^2}(s, X_s)\, ds \overset{\text{averaging}}{\approx} \frac{1}{2} \langle f^2 \rangle \int_0^t X_s^2 \frac{\partial^2 P}{\partial x^2}(s, X_s)\, ds$$

$$= \frac{1}{2} \bar{\sigma}^2 \int_0^t X_s^2 \frac{\partial^2 P}{\partial x^2}(s, X_s)\, ds$$

$\therefore$ $E_0(T)$ will be small.

Q: How to evaluate this small cost?

Recall the Poisson eqn $\mathcal{L}_0 \Phi = f^2(y) - \langle f^2 \rangle$

$$\therefore \quad f^2(Y_s) - \bar{\sigma}^2 = (\mathcal{L}_0 \Phi)(Y_s)$$

$\mathcal{L}_0$ — infinitesimal generator of Y.

$$\mathcal{L}_0 = (m-y)\frac{\partial}{\partial y} + \nu^2 \frac{\partial^2}{\partial y^2}$$

Itô formula $\Rightarrow$

$$d\Phi(Y_s) = \alpha (\mathcal{L}_0 \Phi)(Y_s)\, ds + \beta\, \Phi'(Y_s)\, d\hat{Z}_s \qquad \nu^2 = \frac{\beta}{2\alpha}$$

$$= \alpha(\mathcal{L}_0 \Phi)(Y_s)\, ds + \nu\sqrt{2\alpha}\, \Phi'(Y_s)\, d\hat{Z}_s$$

$$\therefore \quad (\mathcal{L}_0 \Phi)(Y_s)\, ds = \frac{1}{\alpha}\left[d\Phi(Y_s) - \nu\sqrt{2\alpha}\, \Phi'(Y_s)\, d\hat{Z}_s \right]$$

$\therefore$ The cumulative cost:

$$E_0(t) = \frac{1}{2\alpha}\int_0^t \underbrace{X_s^2 \frac{\partial^2 P_0}{\partial x^2}(s, X_s)\left[d\Phi(Y_s)\right.}_{\text{integration by parts.}} \left. - \nu\sqrt{2\alpha}\, \Phi'(Y_s)\, d\hat{Z}_s \right]$$

Recall: $d\left(X_s^2 \frac{\partial^2 P_0}{\partial x^2}(s, X_s)\, \Phi(Y_s) \right)$

$$= X_s^2 \frac{\partial^2 P_0}{\partial x^2}(s, X_s)\, d\Phi(Y_s) + \Phi(Y_s)\, d\left(X_s^2 \frac{\partial^2 P_0}{\partial x^2}(s, X_s) \right)$$

$$+ \left(X_s^2 \frac{\partial^3 P_0}{\partial x^3}(s, X_s) + 2X_s \frac{\partial^2 P_0}{\partial x^2}(s, X_s) \right) f(Y_s) X_s\, \Phi'(Y_s)\, \beta \rho\, dt$$

↑ correlation term $\qquad \beta = \nu\sqrt{2\alpha} \qquad dW\, d\hat{Z}_t$

$$\therefore\ E_0(t) = \frac{1}{2\alpha}\left\{ X_t^2 \frac{\partial^2 P_0}{\partial x^2}(t, X_t)\,\Phi(Y_t) - X_0^2 \frac{\partial^2 P_0}{\partial x^2}(0, X_0)\,\Phi(Y_0) \right. \to O\left(\frac{1}{\alpha}\right)$$

$$\left. - \int_0^t \Phi(Y_s)\, d\left(X_s^2 \frac{\partial^2 P_0}{\partial x^2}\right) \right\}$$

Correlation

$$- \frac{\rho\nu}{\sqrt{2\alpha}} \int_0^t f(Y_s)\Phi'(Y_s)\left(2X_s^2 \frac{\partial^2 P_0}{\partial x^2} + X_s^3 \frac{\partial^3 P_0}{\partial x^3}\right) ds \to O\left(\frac{1}{\sqrt{\alpha}}\right)$$

$$- \frac{\nu}{\sqrt{2\alpha}} \underbrace{\int_0^t X_s^2 \frac{\partial^2 P_0}{\partial x^2} \Phi'(Y_s)\, d\hat{Z}_s}_{\text{martingale}}$$

$\therefore$ $E_0(t)$ has the form

drift. mean-0 martingale

$$E_0(t) = \frac{1}{\sqrt{\alpha}}(B_t + M_t) + O\left(\frac{1}{\alpha}\right)$$

$$B_t \equiv -\frac{\rho\nu}{\sqrt{2}} \int_0^t f(Y_s)\Phi'(Y_s)\left(2X_s^2 \frac{\partial^2 P_0}{\partial x^2} + X_s^3 \frac{\partial^3 P_0}{\partial x^3}\right) ds$$

$$M_t \equiv -\frac{\nu}{\sqrt{2}} \int_0^t X_s^2 \frac{\partial^2 P_0}{\partial x^2} \Phi'(Y_s)\, d\hat{Z}_s$$

$\therefore$ The B-S strategy $\Rightarrow$ not mean self-financing to $O\left(\frac{1}{\sqrt{\alpha}}\right)$

(i.e. $E[B_t] \neq 0$)

NB: a better hedging strategy should have the mean-cost vanish.

Mean Self-Financing Hedging Strategy

Q: How to push B_T to the next order to achieve 0-mean?

Consider

$$a_t = \frac{\partial(P_0+\tilde{Q}_1)}{\partial x}(t,X_t) \quad \text{shares of risky asset.}$$

$$b_t = e^{-rt}\left(P_0(t,X_t)+\tilde{Q}_1(t,X_t) - X_t\frac{\partial(P_0+\tilde{Q}_1)}{\partial x}(t,X_t)\right): \quad \text{bonds}$$

Recall $\quad V_3 = \frac{\rho\nu}{\sqrt{2\alpha}}\langle f\phi'\rangle$

$$B_t \equiv -\frac{\rho\nu}{\sqrt{2}}\int_0^t \underline{\underline{f(Y_s)\phi'(Y_s)}}\left(2X_s^2\frac{\partial^2 P_0}{\partial x^2} + X_s^3\frac{\partial^3 P_0}{\partial x^3}\right)ds$$

Try
$$\begin{cases} \mathcal{L}_{BS}(\bar{\sigma})\tilde{Q}_1 = V_3\left(2x^2\frac{\partial^2 P_0}{\partial x^2} + x^3\frac{\partial^3 P_0}{\partial x^3}\right) \\ \tilde{Q}_1(T,x) = 0 \quad \text{— replication at maturity.} \end{cases}$$

$$\therefore \quad \tilde{Q}_1(t,x) = -(T-t)V_3\left(2x^2\frac{\partial^2 P_0}{\partial x^2} + x^3\frac{\partial^3 P_0}{\partial x^3}\right)$$

NB: $\tilde{Q}_1 \sim O\left(\frac{1}{\sqrt{\alpha}}\right)$

$$\therefore \quad a_t = \frac{\partial P_0}{\partial x} - \frac{V_3(T-t)}{x}\left(4x^2\underbrace{\frac{\partial^2 P_0}{\partial x^2}}_{\text{Gama}} + 5x^3\underbrace{\frac{\partial^3 P_0}{\partial x^3}}_{\text{Epsilon}} + x^4\underbrace{\frac{\partial^4 P_0}{\partial x^4}}_{\text{Kappa} = \frac{\partial^4 P_0}{\partial x^4}}\right)$$

$b_t = e^{-rt}(P_0 - a_t X_t)$

$\therefore$ the new cost of this hedging strategy is

$$dE_1^Q(t) = d(P_0 + \tilde{Q}_1) - a_t dX_t - r b_t e^{rt} dt$$

$\therefore$ the total cost

$O(\frac{1}{\alpha})$ ($\because \tilde{Q}_1 \sim O\frac{1}{\sqrt{\alpha}}$ and averaging $\Rightarrow O(\frac{1}{\sqrt{\alpha}})$)

$$E_1^Q(T) = \frac{1}{2}\int_0^T \left(f^2(Y_t) - \bar{\sigma}^2\right) X_t^2 \frac{\partial^2 P_0}{\partial x^2}\,dt + \frac{1}{2}\int_0^T \underbrace{\left(f^2(Y_t) - \bar{\sigma}^2\right)}_{\text{averaging effect } \sim O(\frac{1}{\sqrt{\alpha}})} X_t^2 \frac{\partial^2 \tilde{Q}_1}{\partial x^2}\,dt$$
$$+ \underbrace{\int_0^T V_3 \left(2X_t^2 \frac{\partial^2 P_0}{\partial x^2} + X_t^3 \frac{\partial^3 P_0}{\partial x^3}\right) dt}_{\text{III}}$$

NB: the 1st term is the original $E_0(T)$ (under BS strategy)

which is $E_0(T) = \frac{1}{\sqrt{\alpha}}(B_t + M_t) + O(\frac{1}{\alpha})$

NB. $\frac{1}{\sqrt{\alpha}} B_t = -\frac{\rho \nu}{\sqrt{2\alpha}} \int_0^t f(Y_s)\Phi'(Y_s)\left(2X_s^2 \frac{\partial^2 P_0}{\partial x^2} + X_s^3 \frac{\partial^3 P_0}{\partial x^3}\right) ds$

$\therefore$ the B_t-term combines with term (III) to give

$$\frac{\rho \nu}{\sqrt{2\alpha}} \underbrace{\int_0^T \left[2X_s^2 \frac{\partial^2 P_0}{\partial x^2} + X_s^3 \frac{\partial^3 P_0}{\partial x^3}\right]\left[\langle f\Phi' \rangle - f(Y_s)\Phi'(Y_s)\right] ds}_{\text{integral } O(\frac{1}{\sqrt{\alpha}})} \sim O(\frac{1}{\alpha})$$

[which is again a form of averaging.]

Recall $V_3 = \frac{\rho \nu}{\sqrt{2\alpha}} \langle f\Phi' \rangle$

$$\therefore \quad E_1^Q(T) = \frac{1}{\sqrt{\alpha}} M_T + O\left(\frac{1}{\alpha}\right)$$

$$= -\frac{\nu}{\sqrt{2\alpha}} \underbrace{\int_0^T X_s^2 \frac{\partial^2 P_0}{\partial x^2} \phi'(Y_s)\, d\hat{Z}_s}_{\text{zero-mean (}\therefore\text{ the bias is removed in hedging)}} + O\left(\frac{1}{\alpha}\right)$$

NB: $\frac{1}{\sqrt{\alpha}} M_T$ — nonhedgeable.
due to stochastic volatility

NB: Now the new strategy is mean self-financing to order $O\left(\frac{1}{\alpha}\right)$.

NB: Implementation of this hedging strategy need only V_3, and $\bar{\sigma}$.