Due May 10, 2006 *(You must submit your work in the pdf format. Email to cai@cims.nyu.edu with subject line "Final exam, PDE in Finance". I will acknowledge with a receipt to your submission)*

1. Consider a random walker:
$$dy = f dt + g dW$$
on the interval $[a, b]$.
   (a) Let $G(x, y, t)$ be the probability of the walker, starting from x at time 0, being at y at time t without having yet hit the boundary. What version of the forward Kolmogorov equation does G solve?
   (b) Express, as an integral involving G_t, the first passage time density to the boundary, i.e., the probability that the process, starting from $a < x < b$, first hits the boundary at time t.
   (c) Using your answers to (a) and (b), show that the first passage time density to the boundary can be expressed as
$$-\frac{1}{2}\frac{\partial}{\partial y}\left(g^2 G(x, y, t)\right)\Big|_{y=b} + \frac{1}{2}\frac{\partial}{\partial y}\left(g^2 G(x, y, t)\right)\Big|_{y=a}.$$

2. Consider scaled Brownian motion with jumps:
$$dy = \sigma dW + J dN$$
starting at $y(0) = x$. Assume the jump occurrences are Poisson with rate λ and the jumps have mean 0 and variance δ^2
   (a) Find $\mathbb{E}\left[y^2(T)\right]$.
   (b) What backward Kolmogorov equation does part (a) solve?

3. Suppose
$$dX_t = B(X_t)\, dt + \sqrt{A(X_t)} dW_t$$
is an SDE in one dimension.
   (a) Show that, for $a < x < b$, the solution $u(a, b, x)$ of
$$\frac{1}{2} A(x) u_{xx} + B(x) u_x = 0, \quad u(a) = 1, u(b) = 0$$
gives the probability that the solution to the SDE starting from x exits from a before exiting from b.
   (b) Calculate $u(x)$ explicitly for the geometric Brownian motion
$$dX_t = \mu X_t dt + \sigma X_t dW_t.$$
   (c) Calculate
$$u(a, x) = \lim_{b\to\infty} u(a, b, x).$$
   (d) Under what conditions on μ and σ, is it true that
$$\lim_{a\to 0} u(a, x) = 0$$
for any $x > 0$?

4. For a diffusion process $dy_t = f(y_t)\,dt + g(y_t)\,dW_t$ starting at x at time 0, show that the k-th moment of the arrival time τ to the boundary, i.e., $\mu^{(k)} \equiv \mathbb{E}\left[\tau^k\right]$, satisfies the recursive system

$$\frac{1}{2}g^2(x)\,\mu_{xx}^{(k)} + f(x)\,\mu_x^{(k)} = -k\mu^{(k-1)}, \qquad a < x < b$$

with $\mu^{(k)} = 0$ at $x = a, b$ for $k \geq 2$.

5. For the usual lognormal process

$$dy = \mu y ds + \sigma y dW,$$

consider an Asian option whose payoff function is $\Phi\left(y(T), \frac{1}{T}\int_0^T y(s)\,ds\right)$.

(a) Show that, if $z(s) = \int_t^s y(u)\,du$, then $(y(s), z(s))$ solves

$$\begin{aligned} dy &= \mu y dt + \sigma y dW \\ dz &= y ds \end{aligned}$$

with initial condition $y(t) = x, \quad z(t) = 0$.

(b) Show that

$$u(x, \zeta; t) = \mathbb{E}_{y(t)=x, z(t)=\zeta}\left[\Phi\left(y(T), \frac{1}{T}z(T)\right)\right]$$

solves the backward Kolmogorov equation

$$u_t + \mu x u_x + x u_\zeta + \frac{1}{2}\sigma^2 x^2 u_{xx} = 0$$

for $t < T$, with final value $u(x, \zeta, T) = \Phi(x, \zeta/T)$ at $t = T$.

(c) Conclude that $\mathbb{E}_{y(0)=x}\left[\Phi\left(y(T), \frac{1}{T}\int_0^T y(s)\,ds\right)\right]$ is equal to $u(x, 0; 0)$.

6. Consider the stochastic volatility model with continuous dividend yield D_0 :

$$\begin{aligned} dX_t &= (r - D_0)\,X_t dt + f(Y_t)\,X_t dW_t \\ dY_t &= \frac{1}{\varepsilon}(m - Y_t)\,dt + \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}\left(\rho dW_t + \sqrt{1-\rho^2}dZ_t\right) \end{aligned}$$

where $\varepsilon \ll 1$, $\sigma_t = f(Y_t) > 0$, W_t and Z_t are independent Brownian motions.

(a) Generalize the PDE for a European option price to include the dividend in this stochastic volatility environment.

(b) Letting the price

$$P(t, x) = P_0 + \sqrt{\varepsilon}P_1 + \cdots$$

use the perturbation theory to derive the PDE that governs P_0 (make sure you give all detailed arguments).

(c) Show that the first correction to the price can still be expressed as

$$\tilde{P}_1(t, x) = -(T - t)\left(V_2 x^2 \frac{\partial^2}{\partial x^2}P_0 + V_3 x^3 \frac{\partial^3}{\partial x^3}P_0\right)$$

where P_0 is the $0th$ order solution and $\tilde{P}_1 = \sqrt{\varepsilon}P_1$. (make sure you give all detailed arguments).

(d) Describe how to use the implied volatility surface to extract the parameters V_2 and V_3 for European puts.