PDE for Finance, Spring 2006 — Homework 1
Due 2/27/06

1. Consider the lognormal process

$$dy = \mu y dy + \sigma y dW$$

starting at $y(0) = x$. Assume $\mu \neq \frac{1}{2}\sigma^2$. In the lecture, we used the PDE for the mean exit time from the interval $[a, b]$,

$$\mu x u_x + \frac{1}{2}\sigma^2 x^2 u_{xx} = -1 \quad \text{for } 0 < a < x < b \tag{1}$$

with boundary conditions $u(a,t) = u(b,t) = 0$, to derive an explicit formula for u.

(a) Show that the general solution of (1), without taking any boundary conditions into account, is

$$u = \frac{1}{\frac{1}{2}\sigma^2 - \mu} \log x + c_1 + c_2 x^\gamma$$

with $\gamma = 1 - 2\mu/\sigma^2$. Here c_1 and c_2 are arbitrary constants.

(b) Argue as in Prof. Kohn's notes to show that the mean exit time from the interval $[a, b]$ is finite.

(c) Let p_a be the probability that the process exits at a, and $p_b = 1 - p_a$ the probability that it exits at b. Give an equation for p_a in terms of the barriers a, b and the initial value x.

(d) What if $\mu = \frac{1}{2}\sigma^2$ in problems $(a), (b), (c)$ above?

2. For a diffusion $dy = f(y)\,dt + g(y)\,dW$ starting at x at time 0, show that the second moment of the arrival time τ to the boundary, i.e., $h(x) = \mathbb{E}\left[\tau^2\right]$ satisfies

$$f h_x + \frac{1}{2} g^2 h_{xx} = -2v(x)$$

with $h = 0$ at $x = a, b$, where $v(x)$ is the mean arrival time to the boundary, i.e. $v(x) = \mathbb{E}[\tau]$

3. Consider a diffusion $dy = f(y)\,dt + g(y)\,dW$, with initial condition $y(0) = x$. Suppose $u(x,t)$ is the solution of the PDE

$$\begin{cases} f u_x + \frac{1}{2} g^2 u_{xx} - r(x)\, u = 0 & \text{for } a < x < b \\ u(a,t) = u(b,t) = 1 & \end{cases}$$

for some function $r(x)$. Assuming $\mathbb{E}[\tau] < \infty$, where τ is the exit time from $[a, b]$, show that

$$u(x) = \mathbb{E}_{y(0)=x}\left[e^{-\int_0^\tau r(y(s))ds}\right].$$

4. For Asian option, if we define

$$A_t = \frac{1}{t}\int_0^t S_\tau d\tau$$

assuming S satisfies the risk-neutral lognormal process, show that the option value satisfies

$$\partial_t V + \frac{1}{2}\sigma^2 S^2 \partial_{SS}^2 V + rS\partial_S V + \frac{1}{t}(S - A)\,\partial_A V - rV = 0.$$