PDE for Finance, Spring 2006 — Homework 2
Due 3/20/06

1. In the lecture, we showed that the solution of

$$\begin{cases} w_t = w_{xx} & \text{for } t>0 \text{ and } x>0 \\ w(x,t=0)=0 & \\ w(x=0,t)=\phi(t) & \end{cases}$$

can be expressed as

$$w(x,t) = \int_0^t \frac{\partial}{\partial y} G(x,0,t-s)\,\phi(s)\,ds \tag{1}$$

where $G(x,y,s)$ is the probability that a random walker (i.e., $dy=\sqrt{2}dW$), starting at x at time 0, reaches y at time s without first hitting the boundary at 0. The following line of reasoning provides a different way of looking at this solution:

(a) Express, in terms of G, the probability that the random walker, starting at x at time 0, hits the boundary before time t. Differentiate in t to obtain the probability that it hits the boundary at time t (This is known as the first passage time density).

(b) Use the forward Kolmogorov equation and integration by parts to show that the first passage time density is $\frac{\partial}{\partial y} G(x,0,t)$.

(c) Deduce the formula (1).

2. For the process $dy = \mu dt + dW$ with an absorbing boundary at $y=0$,

(a) suppose the process starts at $x>0$ at time 0, let $G(x,y,t)$ be the probability that the random walker is at position y at time t without first hitting the boundary. Show that

$$G(x,y,t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{|y-x-\mu t|^2}{2t}} - \frac{1}{\sqrt{2\pi t}} e^{-2\mu x} e^{-\frac{|y+x-\mu t|^2}{2t}}$$

i.e., to verify that this G solves the relevant forward Kolmogorov equation with appropriate boundary and initial conditions.

(b) Show that the first passage time density is

$$\frac{1}{2}\frac{\partial}{\partial y} G(x,0,t) = \frac{x}{t\sqrt{2\pi t}} e^{-\frac{|x+\mu t|^2}{2t}}$$

3. Consider the heat equation $u_t - u_{xx} = 0$ in one space dimension, with discontinuous initial data

$$u(x,0) = \begin{cases} 0 & \text{if } x<0 \\ 1 & \text{if } x>0 \end{cases}$$

(a) Show that

$$u(x,t) = N\left(\frac{x}{\sqrt{2t}}\right)$$

where

$$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{y^2}{2}}\,dy$$

i.e., the cumulative normal distribution.

(b) What is $\max_x u_x(x,t)$ as a function of time t? Where is it achieved? What is $\min_x u_x(x,t)$? Sketch the graph of u_x as a function of x at a given time $t>0$.

(c) Show that

$$v(x,t) = \int_{-\infty}^{x} u(z,t)\,dz$$

solves

$$\begin{cases} v_t - v_{xx} = 0 \\ v(x,0) = \max\{x,0\}. \end{cases}$$

Discuss the qualitative behavior of $v(x,t)$ as a function of x for a given t : how rapidly does v tend to 0 as $x \to -\infty$? What is the behavior of v as $x \to \infty$? What is the value of $v(0,t)$? Sketch the graph of $v(x,t)$ as a function of x for given $t > 0$.

4. Give "solution formulas" for the following initial-boundary-value problems for the heat equation

$$w_t - w_{xx} = 0 \text{ for } t > 0, \quad \text{and } x > 0$$

with the following initial and boundary conditions:

(a) $w_1(x=0,t) = 0$ and $w_1(x,t=0) = 1$. Express the solution in terms of the cumulative normal distribution $N(\cdot)$.

(b) $w_2(x=0,t) = 0$ and $w_2(x,t=0) = (x-K)_+$ with $K > 0$. Express your solution in terms of the function $v(x,t)$ defined in Problem 3(c)

(c) $w_3(x=0,t) = 0$ and $w_3(x,t=0) = (x-K)_+$ with $K < 0$

(d) $w_4(x=0,t) = 1$ and $w_4(x,t=0) = 0$.

Interpret each as the expected payoff of a suitable barrier-type option, whose underlying is described by $dy = \sqrt{2}dW$ with initial condition $y(0) = x$ and an absorbing barrier at 0.