PDE for Finance, Spring 2006 — Homework 3
Due 4/10/06

1. Consider the linear heat equation $u_t - u_{xx} = 0$ on the interval $0 < x < 1$, with boundary condition $u = 0$ at $x = 0; 1$ and initial condition $u = 1$.

   (a) Interpret u as the value of a suitable double-barrier option.

   (b) Express $u(t, x)$ as a Fourier sine series.

   (c) At time $t = 1/100$, how many terms of the series are required to give $u(t, x)$ within one percent accuracy?

2. Consider the SDE $dy = f(y)dt + g(y)dW$. Let $G(x, y, t)$ be the fundamental solution of the forward Kolmogorov PDE, i.e. the probability that a walker starting at x at time 0 is at y at time t. Show that if the infinitesimal generator is *self-adjoint*, i.e.

$$-\left(fu\right)_x + \frac{1}{2}\left(g^2 u\right)_{xx} = fu_x + \frac{1}{2}g^2 u_{xx},$$

   then the fundamental solution is symmetric, i.e. $G(x, y, t) = G(y, x, t)$.

3. Consider the stochastic differential equation $dy = f(y, s)ds + g(y, s)dW$, and the associated backward and forward Kolmogorov equations

$$u_t + f\left(x, t\right)u_x + \frac{1}{2}g^2\left(x, t\right)u_{xx} = 0 \qquad \text{for } t < T, \text{ with } u = \Phi \text{ at } t = T$$

   and

$$\rho_s + \left(f\left(z, s\right)\rho\right)_z - \frac{1}{2}\left(g^2\left(z, s\right)\rho\right)_{zz} = 0 \qquad \text{for } s > 0 \text{ with } \rho\left(z\right) = \rho_0\left(z\right) \text{ at } s = 0.$$

   Recall that $u(x, t)$ is the expected value (starting from x at time t) of payoff $\Phi(y(T))$, whereas $\rho(z, s)$ is the probability distribution of the diffusing state $y(s)$ (if the initial distribution is ρ_0).

   (a) The solution of the backward equation has the following property: if $m = \min_z \Phi(z)$ and $M = \max_z \Phi(z)$ then $m \le u(x, t) \le M$ for all $t < T$. Give two distinct justications: one using the maximum principle for the PDE, the other using the probabilistic interpretation.

   (b) The solution of the forward equation does not in general have the same property; in particular, $\max_z \rho(z, s)$ can be larger than the maximum of ρ_0. Explain why not, by considering the example $dy = -yds$. (Intuition: $y(s)$ moves toward the origin; in fact, $y(s) = e^{-s}y_0$. Viewing $y(s)$ as the position of a moving particle, we see that particles tend to collect at the origin no matter where they start. So $\rho\left(z, s\right)$ should be increasingly concentrated at $z = 0$.) Show that the solution in this case is $\rho\left(z, s\right) = e^s \rho_0\left(e^s z\right)$. This counterexample has $g = 0$; can you also give a counterexample using $dy = -yds + \varepsilon dW$?

4. The solution of the forward Kolmogorov equation is a probability density, so we expect it to be nonnegative (assuming the initial condition $\rho_0\left(z\right)$ is everywhere nonnegative). There is a question whether the PDE has this property. Let's show that it does.

   (a) Consider the initial-boundary-value problem

$$w = a\left(x, t\right)w_{xx} + b\left(x, t\right)w_x + c\left(x, t\right)w$$

   with $x \in (0, 1)$ and $t =\in \left(0, T\right)$. We assume as usual that $a(x, t) > 0$. Suppose furthermore that $c < 0$ for all x and t. Show that if $0 \le w \le M$ at the initial time and the spatial boundary then $0 \le w \le M$ for all x and t. (Hint: a positive maximum cannot be achieved in the interior or at the final boundary. Neither can a negative minimum.)

(b) Now consider the same PDE but with $\max_{x,t} c(x;t)$ positive. Suppose the initial and boundary data are nonnegative. Show that the solution w is nonnegative for all x and t. (Hint: apply part (a) not to w but rather to $\tilde{w} = e^{-Ct}w$ with a suitable choice of C).

(c) Consider the solution of the forward Kolmogorov equation in the interval, with $\rho = 0$ at the boundary. (It represents the probability of arriving at z at time s without hitting the boundary first.) Show using part (b) that $\rho(z, s) \geq 0$ for all s and z.

[Comment: statements analogous to (a)-(c) are valid for the initial-value problem as well, when we solve for all $x \in \mathbb{R}$ rather than for x in a bounded domain. The justification takes a little extra work however, and it requires some hypothesis on the growth of the solution at ∞.]

5. Consider the solution of

$$u_t + au_{xx} = 0 \qquad \text{for } t < T, \quad \text{with } u = \Phi \quad \text{at } t = T$$

where a is a positive constant. Recall that in the stochastic interpretation, a is $\frac{1}{2}g^2$ where g represents volatility. Let's use the maximum principle to understand qualitatively how the solution depends on volatility.

(a) Show that if $\Phi_{xx} \geq 0$ for all x then $u_{xx} \geq 0$ for all x and t. (Hint: differentiate the PDE.)

(b) Suppose $\bar{u}$ solves the analogous equation with a replaced by $\bar{a} > a$, using the same final-time data Φ. We continue to assume that $\Phi_{xx} \geq 0$. Show that $\bar{u} \geq u$ for all x and t. (Hint: $w = \bar{u} - u$ solves $w_t + \bar{a}w_{xx} = f$ with $f = (a - \bar{a})\, u_{xx} \leq 0$)

6. Consider the standard finite difference scheme

$$\frac{u\left((m+1)\Delta t, n\Delta x\right) - u\left(m\Delta t, n\Delta x\right)}{\Delta t} = \frac{u\left(m\Delta t, (n+1)\Delta x\right) - 2u\left(m\Delta t, n\Delta x\right) + u\left(m\Delta t, (n-1)\Delta x\right)}{(\Delta x)^2} \tag{1}$$

for solving $u_t - u_{xx} = 0$. The stability restriction $\Delta t < \frac{1}{2}\Delta x^2$ leaves a lot of freedom in the choice of Δx and Δt. Show that

$$\Delta t = \frac{1}{6}\Delta x^2$$

is special, in the sense that the numerical scheme (1) has errors of order Δx^4 rather than Δx^2. In other words: when u is the exact solution of the PDE, the left and right sides of (1) differ by a term of order Δx^4.