PDE for Finance, Spring 2006 — Homework 4
Due 05/01/06

1. Suppose the stock price X_t and the process Y_t underlying the stochastic volatility are described by

$$\begin{aligned} dX_t &= \mu X_t dt + \sigma_t X_t dW_t \\ \sigma_t &= f(Y_t) \\ dY_t &= \frac{1}{\varepsilon}\mu_Y(t, Y_t)\, dt + \frac{1}{\sqrt{\varepsilon}}\sigma_Y(t, Y_t)\, d\tilde{Z}_t \end{aligned}$$

where f is a positive function and $\tilde{Z}_t = \rho W_t + \sqrt{1-\rho^2} Z_t$, with W_t and Z_t being two independent Brownian motions. Derive the PDE that governs the price of a European option with the payoff function $h(x)$ and maturity T.

2. For stochastic volatility models,

$$dX_t = \mu X_t dt + \sigma_t X_t dW$$

and $\sigma_t = f(Y_t)$ and Y_t is described by a diffusion process. If we assume σ_t and W_t are independent, under certain conditions, one can show that the implied volatility curve $I(K)$ (for fixed stock price x, time t and maturity T) is a locally convex function around $K_m = xe^{r(T-t)}$. We will demonstrate a special case of this using the following procedure. Under the assumption that

$$\bar{\sigma}^2 \equiv \frac{1}{T-t}\int_t^T f^2(Y_s)\, ds$$

is a Bernoulli random variable, i.e.,

$$\bar{\sigma}^2 = \begin{cases} \sigma_1^2 & \text{with probability } p \\ \sigma_2^2 & \text{with probability } 1-p \end{cases}$$

(a) Show that, from the Hull-White pricing formula, we can determine the implied volatility from

$$C_{BS}(K; I(p, K)) = pC_{BS}(K; \sigma_1) + (1-p)\, C_{BS}(K; \sigma_2) \quad (1)$$

where $C_{BS}(K;\sigma)$ is the standard Black-Scholes pricing formula for a European call with strike K and volatility σ and $I(p, K)$ is the implied volatility.

(b) Define $g(p)$ by

$$g(p) \equiv p\frac{\partial C_{BS}}{\partial K}(\sigma_1) + (1-p)\frac{\partial C_{BS}}{\partial K}(\sigma_2) - \frac{\partial C_{BS}}{\partial K}(I(p, K)),$$

show that

$$sign\left(\frac{\partial I}{\partial K}\right) = sign(g(p))$$

and $g(0) = g(1) = 0$ (Note that $\partial C_{BS}/\partial\sigma > 0$)

(c) From Eq. (1), show that

$$C_{BS}(\sigma_1) - C_{BS}(\sigma_2) = \frac{\partial C_{BS}}{\partial \sigma}(I(p, K))\frac{\partial I}{\partial p}$$

and further show that

$$\frac{d^2 g}{dp^2} = 2\frac{(C_{BS}(\sigma_1) - C_{BS}(\sigma_2))^2}{\left.\frac{\partial C_{BS}}{\partial \sigma}\right|_{\sigma=I}}\frac{\log\left(xe^{r(T-t)}/K\right)}{(T-t)\, I^3}$$

(d) By noticing $I > 0$, show that

$$sign\left(\frac{d^2 g}{dp^2}\right) = sign\left(\log\left(\frac{xe^{r(T-t)}}{K}\right)\right),$$

and further using (b) above show that the implied volatility $I(K)$ is locally convex around $K_m = xe^{r(T-t)}$, which is the forward price of the stock.

3. Let us generalize the two-state Markov chain. Suppose that, instead of merely jumping between two states, the process Y_t jumps after exponentially holding times to random variables, uniformly distributed between -1 and $+1$. We assume that (1) the jump sizes and holing times are independent, so Y_t is a pure jump Markov process in $[-1,+1]$, (2) the mean holding time is $1/\alpha$ (which means that the number of jumps N_t before time t is a Poisson process with intensity α, i.e.,

$$P\{N_t = k\} = \frac{(\alpha t)^k}{k!} e^{-\alpha t}$$

for integers $k > 0$.

(a) For any bounded function g on $(-1,1)$, show that

$$\mathbb{E}[g(Y_t)] = g(y)e^{-\alpha t} + \left(\int g(z)\,p(z)\,dz\right)\alpha t e^{-\alpha t} + \mathcal{O}(t^2)$$

where $\mathbb{E}[g(Y_t)] = \mathbb{E}[g(Y_t)\,|N_t = 0]\,P\{N_t = 0\} + \mathbb{E}[g(Y_t)\,|N_t \geq 1]\,P\{N_t \geq 1\}$, and $p(y)$ is the density function for the uniformly distributed jumps, i.e., $p(y) = \frac{1}{2}\mathbf{1}_{(-1,1)}(y)$.

(b) By taking the limit,

$$\lim_{t\to 0^+} \frac{\mathbb{E}[g(Y_t)] - g(y)}{t}$$

show that the infinitesimal generator for this process is

$$\mathcal{L}g(y) = \alpha \int [g(z) - g(y)]\,p(z)\,dz$$

(c) Find the invariant distribution p^* for the process Y_t.

(d) Defining

$$\langle g \rangle \equiv \int g(z)\,p^*(z)\,dz$$

show that

$$\mathbb{E}[g(Y_0)\,h(Y_t)] = \langle g \rangle \langle h \rangle + e^{-\alpha t}[\langle gh \rangle - \langle g \rangle \langle h \rangle]$$

for any continuous bounded functions g and h. Therefore, as $t \to \infty$, Y_t decorrelates from the initial Y_0 at the exponential rate α.

(e) Find the solution u that satisfies

$$\mathcal{L}u(y) = 0.$$