# 1 The Black-Scholes Formula for a European Call or Put

Recall:

$$V(f) = e^{-r(T-t)} \mathbb{E}_{RN} [f(S_T)]$$

where the expectation is taken with respect to the risk-neutral measure.

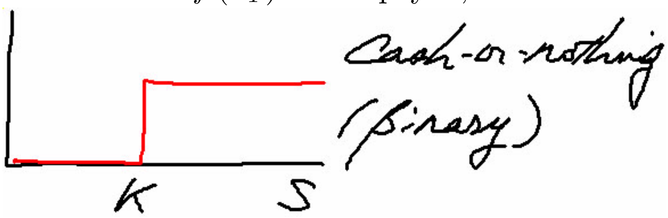
In a risk-neutral world, the stock price dynamics is

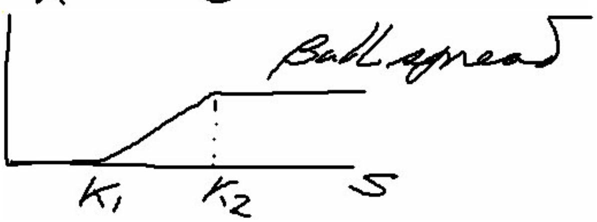
$$S_T = S_t e^{\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma\sqrt{T-t}Z}, \qquad Z \sim \mathcal{N}(0,1)$$

or equivalently

$$\log\left(\frac{S_T}{S_t}\right) \sim \mathcal{N}\left[\left(r - \frac{1}{2}\sigma^2\right)(T-t), \sigma^2 (T-t)\right]$$

Note that $f(S_T)$ is the payoff, a know function of $S_T$, e.g.,




## 1.1 Evaluation of European Options

Evaluation of a European Call/Put at $t = 0$. Let us quote the results first:

$$\begin{aligned} c[S_0, T, K] &= S_0 N(d_1) - K e^{-rT} N(d_2), \\ p[S_0, T, K] &= K e^{-rT} N(-d_2) - S_0 N(-d_1) \end{aligned}$$

where

$$\begin{aligned} N(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy \\ d_1 &= \frac{1}{\sqrt{\sigma^2 T}} \log\left[\frac{S_0 e^{\left(r + \frac{1}{2}\sigma^2\right)T}}{K}\right] \\ d_2 &= \frac{1}{\sqrt{\sigma^2 T}} \log\left[\frac{S_0 e^{\left(r - \frac{1}{2}\sigma^2\right)T}}{K}\right] \end{aligned}$$

Note that

$$d_2 = d_1 - \sqrt{\sigma^2 T}$$

First, let us evaluate the expectation of the following function

$$f(x) \equiv \begin{cases} e^{ax}, & x \geq k \\ 0, & \text{otherwise} \end{cases}$$

where $X$ is a Gaussian-distributed random variable with mean $m$ and variance $\sigma'^2$:

$$\begin{aligned} \mathbb{E}[f(x)] &= \int_{-\infty}^{\infty} f(x) \frac{1}{\sqrt{2\pi\sigma'^2}} e^{-\frac{(x-m)^2}{2\sigma'^2}} dx \\ &= \frac{1}{\sqrt{2\pi\sigma'^2}} \int_k^{\infty} e^{ax} e^{-\frac{(x-m)^2}{2\sigma'^2}} dx \end{aligned}$$

Complete the square:

$$ax - \frac{(x-m)^2}{2\sigma'^2} = am + \frac{1}{2}a^2\sigma'^2 - \frac{[x - (m + a\sigma'^2)]^2}{2\sigma'^2}$$

therefore,

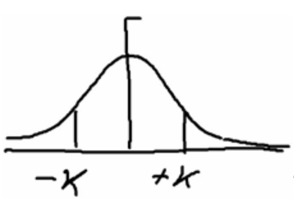
$$\mathbb{E}[f(x)] = e^{am + \frac{1}{2}a^2\sigma'^2} \frac{1}{\sigma'\sqrt{2\pi}} \int_k^{\infty} e^{-\frac{\left[x-\left(m+a\sigma'^2\right)\right]^2}{2\sigma'^2}} dx$$

Changing variable,

$$y \equiv \frac{x - (m + a\sigma'^2)}{\sigma'},$$

yields

$$\begin{aligned} \mathbb{E}[f(x)] &= e^{am + \frac{1}{2}a^2\sigma'^2} \int_{\kappa = \frac{k - \left(m + a\sigma'^2\right)}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-y^2}{2}} dy \\ &= e^{am + \frac{1}{2}a^2\sigma'^2} \int_{-\infty}^{-\kappa} \frac{1}{\sqrt{2\pi}} e^{\frac{-y^2}{2}} dy \qquad \text{(even-symmetry of a Gaussian)} \\ &= e^{am + \frac{1}{2}a^2\sigma'^2} N\left(-\frac{k - (m + a\sigma'^2)}{\sigma'}\right) \end{aligned}$$




Therefore, we have

$$\mathbb{E}[f(x)] = e^{am + \frac{1}{2}a^2\sigma'^2} N(d), \qquad d \equiv \frac{-k + m + a\sigma'^2}{\sigma'} \tag{1}$$

#### 1.1.1 European Call

Applying Eq. (1) to a European call:

$$V(f) = e^{rT} \int_{-\infty}^{\infty} (S_0 e^x - K)_+ \frac{1}{\sqrt{2\pi\sigma^2 T}} e^{-\frac{\left(x - \left(r - \frac{1}{2}\sigma^2\right)T\right)^2}{2\sigma^2 T}} dx$$

Note that

$$S_0 e^x - K \geq 0 \quad \Longrightarrow \quad x > \log \frac{K}{S_0}$$

1. For the first term in the payoff, i.e., $S_0 e^x$, we use the result above with

   $$a = 1, \quad k = \log \frac{K}{S_0}, \ m = \left(r - \frac{1}{2}\sigma^2\right) T, \ \ \sigma'^2 = \sigma^2 T$$

   therefore,

   $$\begin{aligned} & e^{-rT} \int_{k}^{\infty} S_0 e^x \frac{1}{\sqrt{2\pi\sigma^2 T}} e^{-\frac{\left(x - \left(r - \frac{1}{2}\sigma^2\right)T\right)^2}{2\sigma^2 T}} dx \\ = \ & S_0 e^{-rT} e^{\left(r - \frac{1}{2}\sigma^2\right)T + \frac{1}{2}\sigma^2 T} N(d_1) \\ = \ & S_0 N(d_1) \end{aligned}$$

   where

   $$\begin{aligned} d_1 &= \frac{-\log \frac{K}{S_0} + \left(r - \frac{1}{2}\sigma^2\right) T + \sigma^2 T}{\sqrt{\sigma^2 T}} \\ &= \frac{1}{\sqrt{\sigma^2 T}} \log \left[ \frac{S_0 e^{\left(r + \frac{1}{2}\sigma^2\right)T}}{K} \right] \end{aligned}$$

2. For the $2^{nd}$ term (i.e., -K), choose

   $$a = 0,$$

   then,

   $$e^{-rT} \int_{k}^{\infty} K \frac{1}{\sqrt{2\pi\sigma^2 T}} e^{-\frac{\left(x - \left(r - \frac{1}{2}\sigma^2\right)T\right)^2}{2\sigma^2 T}} dx = K e^{-rT} N(d_2)$$

   where

   $$\begin{aligned} d_2 &= \frac{-\log \frac{K}{S_0} + \left(r - \frac{1}{2}\sigma^2\right) T}{\sqrt{\sigma^2 T}} \\ &= \frac{1}{\sqrt{\sigma^2 T}} \log \left[ \frac{S_0 e^{\left(r - \frac{1}{2}\sigma^2\right)T}}{K} \right] \end{aligned}$$

Therefore,

$$c(S_0, T, K) = S_0 N(d_1) - K e^{-rT} N(d_2)$$

### 1.1.2 European Put

How to evaluate a put? Use the put-call parity

$$p - c = Ke^{-rT} - S_0$$

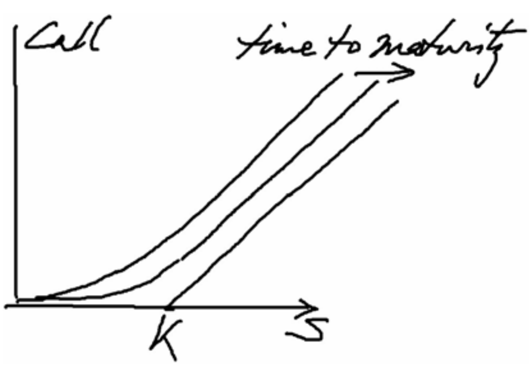
Therefore,

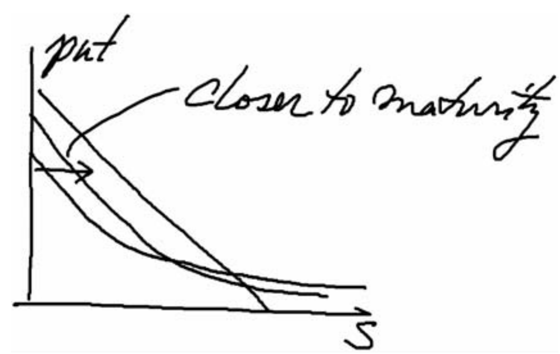
$$\begin{aligned} p(S_0, T, K) &= c(S_0, T, K) + Ke^{-rT} - S_0 \\ &= S_0 N(d_1) - S_0 + Ke^{-rT} - Ke^{-rT} N(d_2) \\ &= -S_0 (1 - N(d_1)) + Ke^{-rT} (1 - N(d_2)) \\ &= -S_0 N(-d_1) + Ke^{-rT} N(-d_2) \end{aligned}$$

Hence

$$p(S_0, T, K) = Ke^{-rT} N(-d_2) - S_0 N(-d_1)$$

Note that the notation $T$ can be understood as the maturity of the contract counting from the day when the option is setup or it can also be understood as the time-to-maturity — which is, sometimes, emphasized through the notation $T - t$ with $T$ being reserved for maturity.




Note that

1. These prices are good as long as the lognormal stock price dynamics is a good model for our market;

2. Parameters in the formula:

$$\begin{aligned} &S_0 \quad \text{— the present value of a stock,} \\ &K \quad \text{— Strike,} \\ &r \quad \text{— risk-free interest rate,} \\ &T \quad \text{— maturity or time-to-maturity} \\ &\sigma \quad \text{— volatility} \end{aligned}$$

   what is the value of $\sigma$?

(a) Historical volatility;

(b) Implied volatility — cf. Volatility smile, volatility skew.

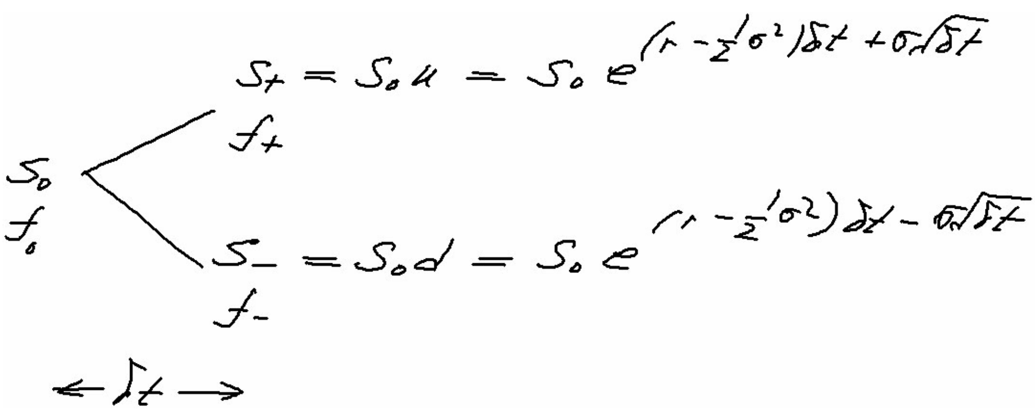

Issues:

1. How good is the lognormal dynamics?

2. How to hedge away some of the problem?

# 2 Hedging

## 2.1 Hedging in a Binomial World



$$f_0 = f(S_0, t+\delta t) + \left[ -rf(S_0, t+\delta t) + rS_0 \frac{\partial f}{\partial S_0} + \frac{1}{2}\sigma^2 S_0^2 \frac{\partial^2 f}{\partial S_0^2} \right] \delta t + o(\delta t)$$

where all the derivatives are evaluated at $t + \delta t$.

Suppose we misparameterized $\sigma$ :

$$\sigma' = \sigma + \delta\sigma$$

where $\delta\sigma$ is the error.

Then, the incorrect price for our contingent claim:

$$f_0' = f(S_0, t+\delta t) + \frac{\partial f}{\partial \sigma}\delta\sigma + \left[-rf(S_0, t+\delta t) + rS_0\frac{\partial f}{\partial S_0} + \frac{1}{2}\sigma^2 S_0^2 \frac{\partial^2 f}{\partial S_0^2}\right]\delta t + o(\delta t)$$

For the purpose of illustrating the idea of hedging, we have assumed $\delta\sigma$ and $\delta t$ are of the same order, otherwise, there are further expansions of those derivatives with respect to $\delta\sigma$. Here, we neglect higher order terms, e.g. $O(\delta t \delta\sigma)$.

Therefore, the mispriced amount is

$$\begin{aligned} \delta f_0 &\equiv f_0' - f_0 \\ &\approx \frac{\partial f}{\partial \sigma}\delta\sigma \end{aligned}$$

However, if we have another contingent claim on the same stock to form a portfolio:

$$f + xg, \quad x \text{ — the number of unit of } g\text{-option.}$$

Then the total mispricing will be

$$\delta(f + xg) \approx \left(\frac{\partial f}{\partial \sigma} + x\frac{\partial g}{\partial \sigma}\right)\delta\sigma$$

if

$$\begin{aligned} \frac{\partial f}{\partial \sigma} + x\frac{\partial g}{\partial \sigma} &= 0 \\ \text{i.e.,} \quad x &= \frac{-\frac{\partial f}{\partial \sigma}}{\frac{\partial g}{\partial \sigma}} \end{aligned}$$

then we can hedge away potential mispricing due to incorrect volatility parameterization to $O(\delta\sigma)$ — a Vega hedging.

Terminology:

$$\begin{aligned} Vega &: \\ \mathcal{V}_f &\equiv \frac{\partial f}{\partial \sigma}, \quad \mathcal{V}_g \equiv \frac{\partial g}{\partial \sigma} \end{aligned}$$

i.e.,

$$x = -\frac{\mathcal{V}_f}{\mathcal{V}_g}$$

Question: Have we hedged away all risks yet? Let's analyze this issue further.

Recall that the replicating portfolio in a risk-neutral valuation is

$$-f_0 + \underbrace{\Delta_f S_0 + B_f}_{\text{replicating prtflo}} = 0$$

in a correctly parameterized world. i.e.,

$$-f_+ + \Delta_f S_+ + B_f e^{r\delta t} = -f_- + \Delta_f S_- + B_f e^{r\delta t}$$

$$\therefore \quad \Delta_f = \frac{f_+ - f_-}{S_+ - S_-}$$

is the amount of stock needed to hedge away the risk

Due to $\sigma$-misparameterization:

$$-f_0' + \Delta_f' S_0 + B_f' = 0 \qquad \text{at} \quad t = 0$$

therefore,

$$\Delta_f' = \frac{f_+' - f_-'}{S_+ - S_-}$$

where $f_+'$, $f_-'$ and $B_f'$ are computed using $\sigma'$. So in a time-step $\delta t$, our risk is

$$\delta\Pi_f = \left(-f_+ + \Delta_f' S_+ + B_f' e^{r\delta t}\right) - \left(-f_- + \Delta_f' S_- + B_f' e^{r\delta t}\right)$$

$\Uparrow$

N.B. in the real world, our $f$ has to pay $f_+$ rather than $f_+'$

$$\therefore \quad \delta\Pi_f = -\left(f_+ - f_-\right) + \Delta_f'\left(S_+ - S_-\right)$$

$$\because \quad \Delta_f = \Delta_f(\sigma)$$

$$\therefore \quad \Delta_f' \approx \Delta_f + \frac{\partial \Delta_f}{\partial \sigma}\delta\sigma + o(\delta\sigma)$$

therefore,

$$\delta\Pi_f = \underbrace{-\left(f_+ - f_-\right) + \Delta_f\left(S_+ - S_-\right)}_{\substack{=0 \\ \because \text{ perfectly hedged} \\ \text{with correct } \sigma}} + \frac{\partial \Delta_f}{\partial \sigma}\delta\sigma\left(S_+ - S_-\right)$$

$$\therefore \quad \delta\Pi_f = \frac{\partial \Delta_f}{\partial \sigma}\left(S_+ - S_-\right)\delta\sigma$$

which, in general, is not zero. However,

$$S_+ - S_- \approx O\left(\sigma\sqrt{\delta t}\right)$$

$$\therefore \quad \delta\Pi_f = O\left(\left(\frac{\partial \Delta_f}{\partial \sigma}\sigma\sqrt{\delta t}\right)\delta\sigma\right)$$

which contains risks — Either we are content to live with these risks (they could be small or large, depending on the combination of $\frac{\partial \Delta_f}{\partial \sigma}\sigma\delta\sigma$) or we can try to hedge further — Let's see how theoretically this can be done. First note that, even for our portfolio

$$f + xg$$

we have

$$\delta\Pi_{f+xg} = \delta\Pi_f + \delta\Pi_g = \left(\frac{\partial\Delta_f}{\partial\sigma} + x\frac{\partial\Delta_g}{\partial\sigma}\right)(S_+ - S_-)\,\delta\sigma$$

which means there are still risks, i.e. our portfolio $f + xg$ is not completely $\Delta$-neutral.

Now suppose we have another contingent claim on the same stock to form a new portfolio:

$$f + xg + yh$$

we can choose $x, y$, such that

$$\text{Vega-Neutral:} \quad \frac{\partial f}{\partial\sigma} + x\frac{\partial g}{\partial\sigma} + y\frac{\partial h}{\partial\sigma} = 0 \quad \text{i.e.} \quad \mathcal{V}_f + x\mathcal{V}_g + y\mathcal{V}_h = 0 \tag{2a}$$

$$\text{and } \Delta\text{-Neutral:} \quad \frac{\partial\Delta_f}{\partial\sigma} + x\frac{\partial\Delta_g}{\partial\sigma} + y\frac{\partial\Delta_h}{\partial\sigma} = 0 \tag{2b}$$

then we have hedged away potential mispricing and risks due to misparameterization of $\sigma$.

**Importance of being nonlinear:** Question: can we use the stock as our third option for hedging, i.e.,

$$h(S_T) = S_T$$

i.e., the stock itself for our option $h(S)$? Note that

$$\begin{aligned}
\mathcal{V}_h &= \frac{\partial S}{\partial\sigma} = 0 \text{ — a stock has vanishing } Vega \\
\text{and} \quad \Delta_h &= \frac{\partial S_0}{\partial S} = 1 \quad \left[\text{or } \Delta_h = \frac{S_+ - S_-}{S_+ - S_-} = 1\right] \\
\textit{i.e.}, \quad \text{a stock has } \Delta &= 1 \\
&\Longrightarrow \quad \text{stock } S \text{ is a linear derivative}
\end{aligned}$$

Can we use the stock itself for our $h$?

Since Eqs. (2) now become

$$\begin{aligned}
\frac{\partial f}{\partial\sigma} + x\frac{\partial g}{\partial\sigma} + y\cdot 0 &= 0 \\
\text{and} \quad \frac{\partial\Delta_f}{\partial\sigma} + x\frac{\partial\Delta_g}{\partial\sigma} + y\cdot 0 &= 0
\end{aligned}$$

leading to no solution for $x$, and $y$, in general.

The story is just to give you some sense of how issues of hedging arise and how hedging can be done. This simple example illustrates the need for **nonlinear** derivatives for hedging purposes.

**Conclusion:**

Even if a stock price dynamics is not 100% accurate, as long as it is sufficiently close to the true dynamics — meaning both model specification and model parameterization — then we can use a well-balanced (hedged) portfolio to eliminate most of risks.

## 2.2 Hedging (General Formulation) — Greeks

### 2.2.1 Greeks

Portfolio value:

$$\Pi = \Pi(t, S, \sigma, r)$$

where $t$ is time-to-maturity. Then

$$\begin{aligned} \delta\Pi &= \frac{\partial \Pi}{\partial t}\delta t + \frac{\partial \Pi}{\partial S}\delta S + \frac{\partial \Pi}{\partial \sigma}\delta \sigma + \frac{\partial \Pi}{\partial r}\delta r \\ &\quad + \frac{1}{2}\frac{\partial^2 \Pi}{\partial S^2}(\delta S)^2 + \cdots \end{aligned}$$

where $\delta t$ indicates the changing of time. What are the Greeks? They are no more than

$$\begin{aligned} Theta &: \quad \Theta = \frac{\partial \Pi}{\partial t}, \\ Delta &: \quad \Delta = \frac{\partial \Pi}{\partial S}, \\ Vega &: \quad \mathcal{V} = \frac{\partial \Pi}{\partial \sigma}, \\ rho &: \quad \rho = \frac{\partial \Pi}{\partial r} \\ Gamma &: \quad \Gamma = \frac{\partial^2 \Pi}{\partial S^2} \end{aligned}$$

Note that a portfolio contains, e.g., stocks, calls, puts, etc. each of which has its own corresponding $\Delta, \Gamma$,etc. For example, for a stock,

$$\begin{aligned} \Delta_S &= 1 \\ \Gamma_S &= 0 \\ \mathcal{V}_S &= 0 \end{aligned}$$

### 2.2.2 Greeks for a European Call/Put:

$$\begin{aligned} \Delta_c &= \frac{\partial}{\partial S_0} c(S_0, T, K) = N(d_1) \qquad \text{(How to evaluate? HW)} \\ \Delta_p &= \frac{\partial}{\partial S_0} p(S_0, T, K) = -N(-d_1) = N(d_1) - 1 \end{aligned}$$

the second line of which can be seen directly from the put-call parity.

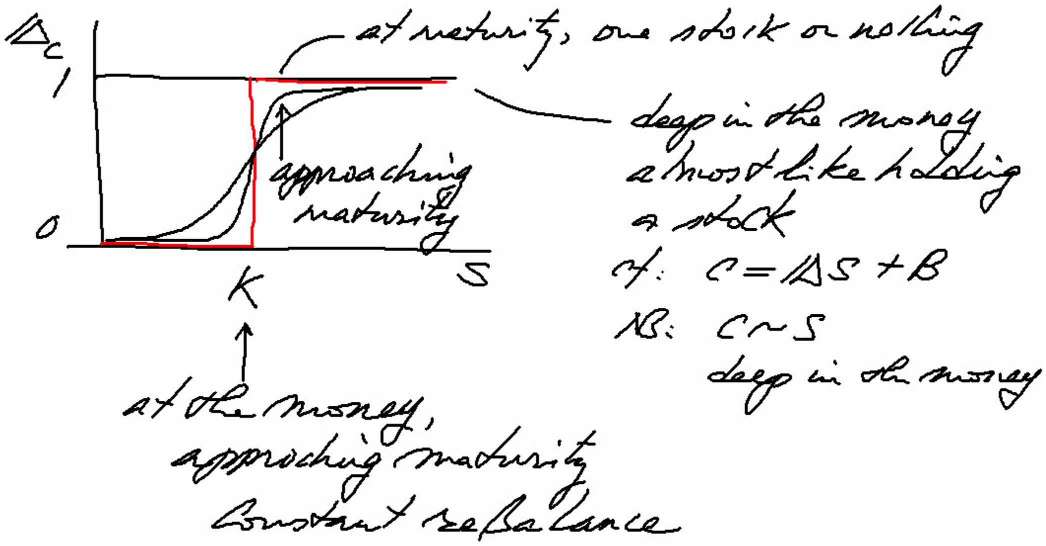
Note that the hedging portfolio in a risk-neutral way would be

$$\Delta_c S_0 - c$$

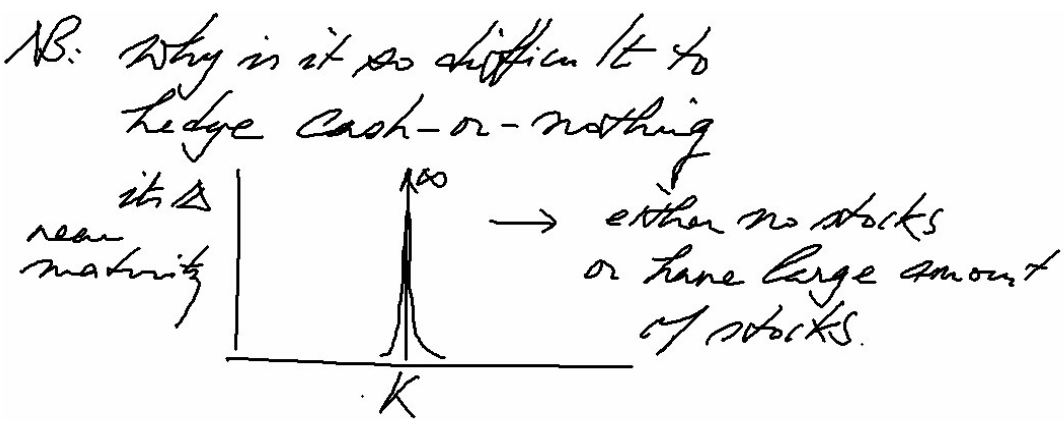
With changing $\Delta_c$, one has to rebalance.

$\Delta := \dfrac{\partial}{\partial S_0}$ **— sensitivity to the change of the stock price.**

Call: $1 \geq N(d_1) \geq 0$ Put: $-1 \leq N(d_1) - 1 \leq 0$



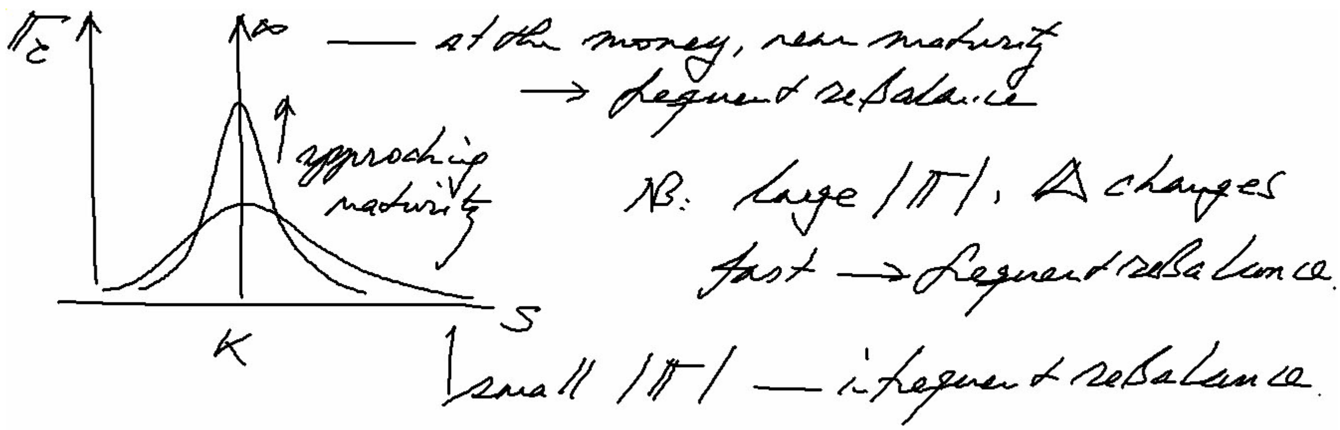

$\mathbb{Q}$: Why is it so difficult to hedge a cash-or-nothing?




$\Gamma := \dfrac{\partial^2}{\partial S_0^2}$ **— Δ-sensitivity to the change of** $S_0$

Call: $\Gamma_c = \dfrac{1}{S_0\sqrt{2\pi\sigma^2 T}} \exp\left[-\dfrac{d_1^2}{2}\right] > 0$ Put: $\Gamma_p = \Gamma_c$

$\Gamma_p = \Gamma_c$ can be seen directly from the put-call parity.

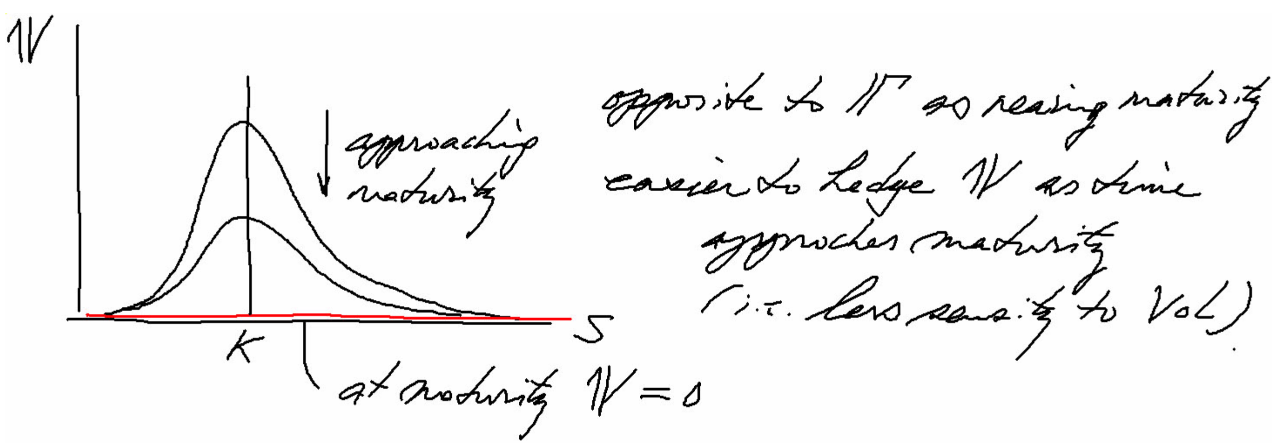



$\mathcal{V} := \dfrac{\partial}{\partial \sigma}$ — **sensitivity to the volatility change**

$$\text{Call:} \qquad \mathcal{V}_c = S_0 \sqrt{\frac{T}{2\pi}} \exp\left[-\frac{d_1^2}{2}\right] > 0, \qquad\qquad \text{Put:} \quad \mathcal{V}_p = \mathcal{V}_c$$

again, the relation $\mathcal{V}_p = \mathcal{V}_c$ can be seen from the put-call parity.

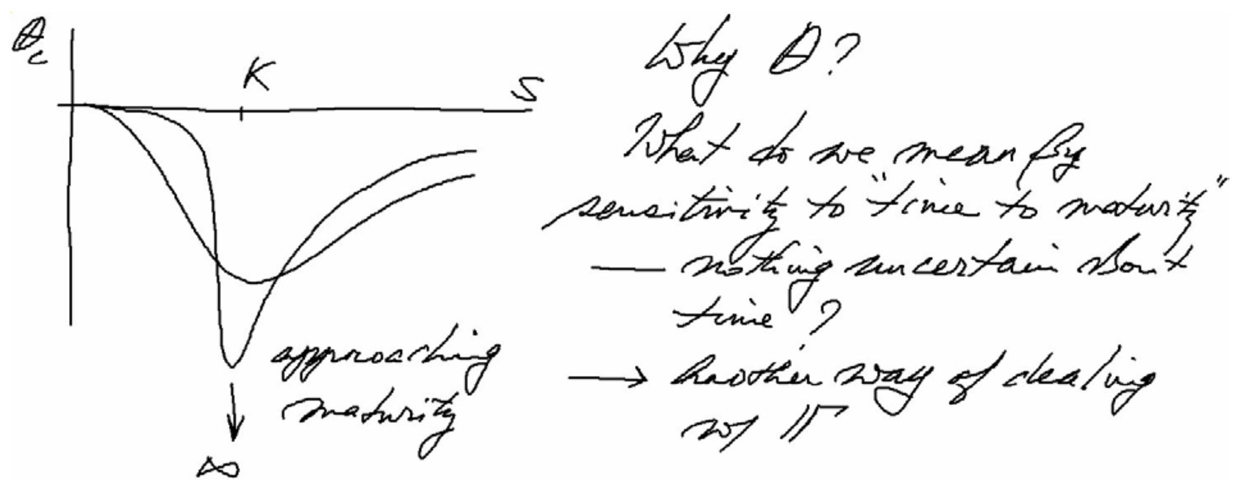
Note that the Black-Scholes formula assumes a constant $\sigma$. But, if volatility changes, then rebalance is needed.



$\Theta := \frac{\partial}{\partial t}$ — **sensitivity to "time-to-maturity"**

$$\text{Call} \quad : \quad \Theta_c = -\frac{S_0\sigma}{2\sqrt{2\pi T}}\exp\left[-\frac{d_1^2}{2}\right] - rKe^{-rT}N(d_2) < 0$$

$$\text{Put} \quad : \quad \Theta_p = -\frac{S_0\sigma}{2\sqrt{2\pi T}}\exp\left[-\frac{d_1^2}{2}\right] + rKe^{-rT}N(d_2) = \{ \begin{matrix} > 0 \\ < 0 \end{matrix}$$




What do we mean by sensitivity to "time-to-maturity" since there is nothing uncertain about time after all? Actually this is another way of looking at Gamma $\Gamma$.

Recall the Black-Scholes PDE:

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + rS\frac{\partial f}{\partial S} - rf = 0$$

i.e.,

$$\Theta + \frac{1}{2}\sigma^2 S^2 \Gamma + rS\Delta = rf \tag{3}$$

Note that

1. If we need nothing to enter a contract (a portfolio with many options), then

   $$f(0) = 0$$

   If we want to maintain that way, i.e., constant rebalance to ensure $f = 0$, then, Eq. (3) yields

   $$\Delta\text{-neutral, } \Theta\text{-neutral} \quad \Longrightarrow \quad \Gamma\text{-neutral}$$

2. For a $\Delta$-neutral portfolio, we have

$$\Theta + \frac{1}{2}\sigma^2 S^2 \Gamma = rf$$

then

$$\begin{array}{rcl} \text{large, positive } \Theta & \Longrightarrow & \text{large, negative } \Gamma \\ \text{large, negative } \Theta & \Longrightarrow & \text{large, positive } \Gamma \end{array}$$

$\rho := \frac{\partial}{\partial r}$ — **sensitivity to the change of interest rate**

$$\begin{array}{lcl} \text{Call} & : & \rho_c = TKe^{-rT} N(d_2) > 0 \\ \text{Put} & : & \rho_p = -TKe^{-rT} N(-d_2) = TKe^{-rT}\left[N(d_2) - 1\right] < 0 \end{array}$$


### 2.2.3 General Hedging:

**Δ-Hedge:** e.g.,

$n_1$ units of an option $f_1$ on the same stock.

and the portfolio is

$$\Pi_1 = n_1 f_1 + n_s S_0 + B$$

where $B$ stands for the amount of bond. $\Delta$-neutral is

$$\begin{aligned} \frac{\partial}{\partial S_0}\Pi_1 &= 0 \\ \textit{i.e.,} \qquad n_1\Delta_1 + n_s &= 0 \end{aligned}$$

which is insensitive to the change of the stock price. Since,

$$\begin{aligned} n_1 f_1 + n_s S_0 + B &= V_1 \\ n_1 \Delta_1 + n_s &= 0 \end{aligned}$$

which are two equations with two unknowns, $n_1, n_s$, there is no more freedom to hedge other Greeks.

**$\mathcal{V}$-Hedge:** However, if we have two options $f_1$, $f_2$ (on the same stock) with $n_1$ and $n_2$ units, respectively, then the value of the portfolio is

$$\Pi_2 = n_1 f_1 + n_2 f_2 + n_s S_0 + B$$

its value is

$$n_1 f_1 + n_2 f_2 + n_s S_0 + B = V_2, \tag{4}$$

with $\Delta$-neutral position, i.e.,

$$\begin{aligned} \frac{\partial \Pi_2}{\partial S_0} &= 0 \\ \Longrightarrow \quad & n_1 \Delta_1 + n_2 \Delta_2 + n_s = 0 \end{aligned} \tag{5}$$

Now we have more freedom for hedging. For example, we can demand $Vega$-neutral, i.e., $\frac{\partial}{\partial \sigma} \Pi_2 = 0 \quad \Longrightarrow$

$$n_1 \mathcal{V}_1 + n_2 \mathcal{V}_2 = 0 \tag{6}$$

Eqs. $(4)$, $(5)$ and $(6)$ can be solved for the three unknowns, $n_1, n_2, n_3$ — thus, achieving $\Delta$-neutral and $Vega$-neutral, reducing the exposure to the changes or mis-specification of volatility, etc.

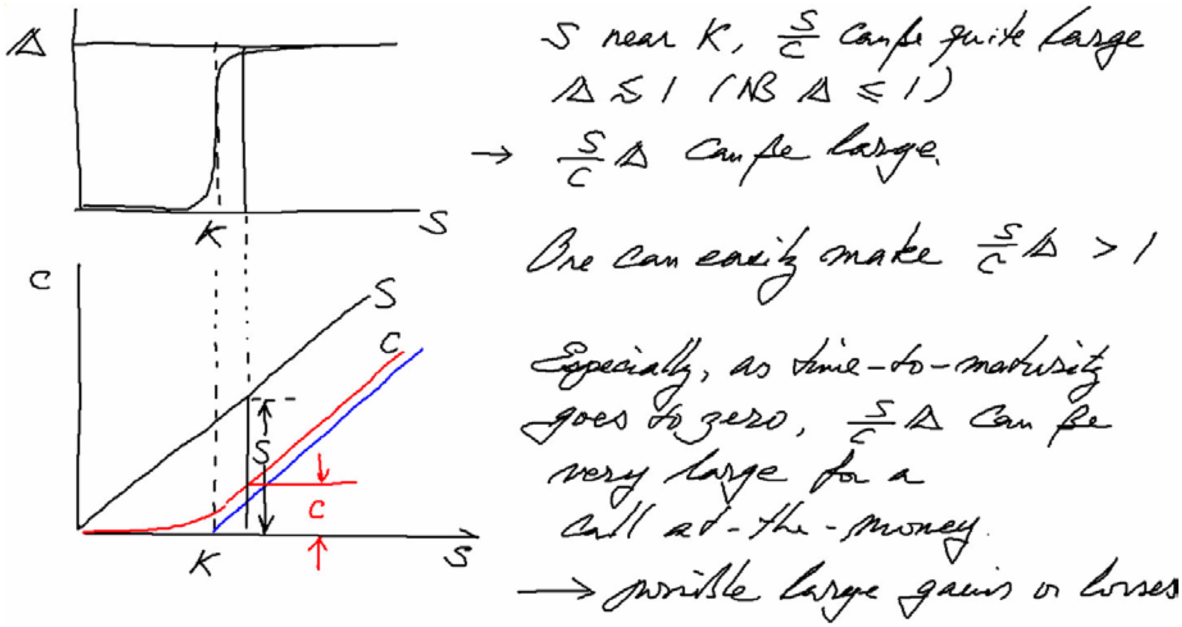
Note that

1. By increasing types of options, we can hedge away other kinds of risks described by other Greeks.
2. Dynamic balancing: Often times $\Gamma$ and $Vega$ are monitored but not zeroed out. $\Delta$ is zeroed out daily by rebalancing shares.
3. There is a difficulty of $\Gamma$-neutral and $\mathcal{V}$-neutral — which require **nonlinear** derivatives that are traded at competitive prices.

### 2.2.4 Speculation using Greeks

Consider only a European call

$$\begin{aligned} \delta c &= \frac{\partial c}{\partial S} \delta S = \Delta \delta S \\ \frac{\delta c}{c} &= \frac{\Delta}{c} \delta S = \frac{S\Delta}{c} \frac{\delta S}{S} \end{aligned}$$




If

$$\frac{S}{c}\Delta \gg 1$$

then, a percentage change of stock price $\delta S/S$ can lead to an appreciably large percentage change of option price.

One can also bet on volatility:

$$\begin{aligned} \delta\Pi &= \frac{\delta\Pi}{\delta\sigma}\delta\sigma = \mathcal{V}\delta\sigma \\ \frac{\delta\Pi}{\Pi} &= \frac{\mathcal{V}}{\Pi}\sigma\left(\frac{\delta\sigma}{\sigma}\right) \end{aligned}$$

If a portfolio is constructed in such a way that

$$\frac{\mathcal{V}}{\Pi}\sigma \gg 1$$

then, it is possible to use this portfolio for speculation with high leverage, i.e., a change of volatility is magnified by a factor $\frac{\mathcal{V}}{\Pi}\sigma$ in the portfolio price.