CS Peskin 5/8/2019

Elastic energy of a surface

Let

(1) $$\underline{X}(q_1, q_2)$$

be the parametric description of a surface with q_1, q_2 as material coordinates. Let G be the 2×2 matrix with elements

(2) $$G_{ij} = \frac{\partial \underline{X}}{\partial q_i} \cdot \frac{\partial \underline{X}}{\partial q_j} = \frac{\partial X_\alpha}{\partial q_i} \frac{\partial X_\alpha}{\partial q_j}$$

We use Latin indices for subscripts that take on the values 1,2, and Greek indices for subscripts that take on the values 1,2,3, and repeated indices in a given term are summed over the values that they may take on.

The matrix G is obviously symmetric, and if we assume that $\partial \underline{X}/\partial q_1$ and $\partial \underline{X}/\partial q_2$ are linearly independent, then G is also positive definite. To see this, note that

(3) $$u_i G_{ij} u_j = \left\| u_i \frac{\partial \underline{X}}{\partial \xi_i} \right\|^2 \geq 0$$

with equality only if

(4) $$u_i \frac{\partial \underline{X}}{\partial \xi_i} = 0$$

and (4) implies that $u_1 = u_2 = 0$, since $\partial \underline{X}/\partial \xi_1$ and $\partial \underline{X}/\partial \xi_2$ are linearly independent.

Let a curve on the surface be given in the form

(5) $$\xi_i = \bar{\xi}_i(t) \qquad i = 1,2$$

where t is a metrical parameter. The trajectory of this curve in space is given by

(6) $$\overline{\underline{X}}(t) = \underline{X}(\bar{\xi}_1(t), \bar{\xi}_2(t))$$

so

(7) $$d\overline{\underline{X}} = \frac{\partial \underline{X}}{\partial \xi_i} \frac{d\bar{\xi}_i}{dt} \, dt$$

and

(8) $$\frac{d\bar{\underline{X}}}{dt} \cdot \frac{d\bar{\underline{X}}}{dt} = \frac{\partial \underline{X}}{\partial \bar{q}_i} \frac{d\bar{q}_i}{dt} \cdot \frac{\partial \underline{X}}{\partial \bar{q}_j} \frac{d\bar{q}_j}{dt}$$

$$= \frac{d\bar{q}_i}{dt} G_{ij} \frac{d\bar{q}_j}{dt}$$

Let

(9) $$u_i = \frac{d\bar{q}_i}{dt}$$

Then (8) becomes

(10) $$\frac{d\bar{\underline{X}}}{dt} \cdot \frac{d\bar{\underline{X}}}{dt} = u_i G_{ij} u_j$$

Now we consider two different configurations of the surface, denoted $\underline{X}^0$ and $\underline{X}$, and the corresponding curve on each of them. We define λ as the stretch ratio

(11) $$\lambda = \frac{\sqrt{u_i G_{ij} u_j}}{\sqrt{u_i G^0_{ij} u_j}}$$

Now consider a pair of curves that intersect orthogonally on the surface $\underline{X}^0$. If we choose t = arclength on both curves, then this implies

$$(12) \qquad u_i^{(k)} G_{ij}^0 u_j^{(l)} = \delta_{kl}$$

at the point of intersection, for $k, l = 1, 2$.

Equation (12) can also be written

$$(13) \qquad \left(u^{(k)}\right)^T G^0 \left(u^{(l)}\right) = \delta_{kl}$$

Let

$$(14) \qquad \left(G^0\right)^{1/2} u^{(k)} = v^{(k)}, \qquad k = 1, 2$$

Then

$$(15) \qquad \left(v^{(k)}\right)^T \left(v^{(l)}\right) = \delta_{kl}$$

and

$$(16) \qquad v^{(k)} \left(v^{(k)}\right)^T = I$$

Then

$$(17) \qquad \lambda_1^2 + \lambda_2^2 = u_i^{(k)} G_{ij} u_j^{(k)} = (u^{(k)})^T G\, u^{(k)}$$

$$= \text{trace}\left((u^{(k)})^T G\, u^{(k)}\right)$$

$$= \text{trace}\left((v^{(k)})^T (G^0)^{-1/2} G (G^0)^{-1/2} v^{(k)}\right)$$

$$= \text{trace}\left((G^0)^{-1/2} G (G^0)^{-1/2}\; v^{(k)} (v^{(k)})^T\right)$$

$$= \text{trace}\left((G^0)^{-1/2} G (G^0)^{-1/2}\right)$$

$$= \text{trace}\left(G (G^0)^{-1}\right) = \text{trace}\left((G^0)^{-1} G\right)$$

Note that the direction of the pair of curves has dropped out of the foregoing.

Thus, for *any* two curves that cross orthogonally in the *reference* configuration, the sum of the squares of the stretches is the same at the point of intersection, regardless of the orientation of the pair of curves.

Note, however, that two such curves are not necessarily orthogonal where they cross in the *current* configuration.

Now we look for directions u that maximize or minimize λ. It is equivalent to maximize or minimize

(18) $$\log(\lambda^2) = \log(u^T G u) - \log(u^T G^0 u)$$

This leads to the condition

(19) $$\frac{(du)^T G u}{u^T G u} = \frac{(du)^T G^0 u}{u^T G^0 u}$$

which is supposed to hold for all du, and this is only possible if

(20) $$\frac{G u}{u^T G u} = \frac{G^0 u}{u^T G^0 u}$$

Equation (20) can also be written as

(21) $$(G - \mu G^0) u = 0$$

where

(22) $$\mu = \lambda^2 = \frac{u^T G u}{u^T G^0 u}$$

Equation (21) can be put in the standard form of an eigenvalue problem by multiplying both sides on the left by $(G^0)^{-1/2}$ and also making the change of variables

(23) $$u = (G^0)^{-1/2} v$$

This gives

(24) $$\left((G^0)^{-1/2} G (G^0)^{-1/2} - \mu I \right) v = 0$$

Since $(G^0)^{-1/2} G (G^0)^{-1/2}$ is symmetric and positive definite, the eigenproblem (24) has two real and positive eigenvalues that we call

(25) $$\mu^{(1)}, \quad \mu^{(2)}$$

with corresponding eigenvectors

(26) $$v^{(1)}, \quad v^{(2)}$$

If $\mu^{(1)} \neq \mu^{(2)}$, then $v^{(1)}$ and $v^{(2)}$ are uniquely determined except that their lengths are arbitrary, and moreover

$v^{(1)}$ and $v^{(2)}$ are orthogonal. We can choose their lengths equal to 1, and then we have

(27) $$(v^{(k)})^T v^{(l)} = \delta_{kl} \quad , \quad k,l = 1,2$$

If $\mu^{(1)} = \mu^{(2)} = \mu$, then

(28) $$(G^0)^{-1/2} G (G^0)^{-1/2} = \mu I$$

and therefore

(29) $$G = \mu G^0$$

In this situation, the stretch is independent of the direction of the curve.

Now we return to the typical case, in which $\mu^{(1)} \neq \mu^{(2)}$. When equation (27) is rewritten in terms of u, it reads

(30) $$(u^{(k)})^T G^0 (u^{(l)}) = \delta_{kl} \quad , \quad k,l = 1,2$$

and this is precisely the condition that the two curves are orthogonal where they cross in the reference ~~and~~ configuration:

But

(31) $$G\,u^{(l)} = \mu^{(l)}\, G^0\, u^{(l)} \quad , \quad l = 1, 2$$

(with no sum on l), see (21). Thus, equation (30) can be rewritten in terms of G as follows

(32) $$(u^{(k)})^T\, G\, u^{(l)} = \frac{\delta_{kl}}{\mu^{(l)}} = \frac{\delta_{kl}}{\mu^{(k)}}$$

(with no sum on l or k). (Note that $\mu^{(l)}$ and $\mu^{(k)}$ cannot be zero, and indeed are positive, since G and G^0 are positive definite.) With $k \neq l$, equation (32) shows that the two curves cross orthogonally in the current configuration of the surface as well as in the reference configuration!

We now consider the area element on each of the surfaces $\underline{X}(q_1, q_2)$ and $\underline{X}^0(q_1, q_2)$. The element of area is given by $\|d\underline{A}\|$ where

(33) $$d\underline{A} = \frac{\partial \underline{X}}{\partial q_1} \times \frac{\partial \underline{X}}{\partial q_2} \, dq_1 \, dq_2$$

and we have

(34) $$\left\| \frac{\partial \underline{X}}{\partial q_1} \times \frac{\partial \underline{X}}{\partial q_2} \right\|^2 =$$

$$\varepsilon_{\alpha\beta\gamma} \frac{\partial X_\beta}{\partial q_1} \frac{\partial X_\gamma}{\partial q_2} \varepsilon_{\alpha\beta'\gamma'} \frac{\partial X_{\beta'}}{\partial q_1} \frac{\partial X_{\gamma'}}{\partial q_2} =$$

$$\left(\delta_{\beta\beta'}\delta_{\gamma\gamma'} - \delta_{\beta\gamma'}\delta_{\gamma\beta'}\right) \frac{\partial X_\beta}{\partial q_1} \frac{\partial X_\gamma}{\partial q_2} \frac{\partial X_{\beta'}}{\partial q_1} \frac{\partial X_{\gamma'}}{\partial q_2} =$$

$$\left\| \frac{\partial \underline{X}}{\partial q_1} \right\|^2 \left\| \frac{\partial \underline{X}}{\partial q_2} \right\|^2 - \left(\frac{\partial \underline{X}}{\partial q_1} \cdot \frac{\partial \underline{X}}{\partial q_2} \right)^2 =$$

$$G_{11} G_{22} - G_{12}^2 = \det(G)$$

and of course (34) is also valid with $\underset{\sim}{X}$ replaced by $\underset{\sim}{X}^0$ and with G replaced by G^0.

Therefore,

(35) $$\frac{\| \underset{\sim}{A} \|}{\| \underset{\sim}{A}^0 \|} = \sqrt{\frac{\det(G)}{\det(G^0)}}$$

$$= \sqrt{\det\left((G^0)^{-1/2}\, G\, (G^0)^{-1/2}\right)}$$

$$= \sqrt{\mu^{(1)}\mu^{(2)}} = \lambda_1 \lambda_2$$

Thus, the ratio of the areas is the product of the principal stretches.

The stretch ratio of an arbitrary curve can be determined from the principal stretches as follows. From (11),

(36) $$\lambda = \frac{\sqrt{u^T G u}}{\sqrt{u^T G^0 u}}$$

Now let

(37) $$u = (G^0)^{-1/2} v = (G^0)^{-1/2}\left((\cos\theta) v^{(1)} + (\sin\theta) v^{(2)}\right)$$

where $v^{(1)}$ and $v^{(2)}$ are the orthonormal eigenvectors of $(G^0)^{-1/2} G (G^0)^{-1/2}$ with eigenvalues $\mu^{(1)}$ and $\mu^{(2)}$, respectively. Then

(38) $$u^T G^0 u = \left((\cos\theta) v^{(1)} + (\sin\theta) v^{(2)}\right)^T \left((\cos\theta) v^{(1)} + (\sin\theta) v^{(2)}\right)$$
$$= \cos^2\theta + \sin^2\theta = 1$$

and

(39) $$u^T G u = \left((\cos\theta) v^{(1)} + (\sin\theta) v^{(2)} \right)^T (G^0)^{-1/2} G (G^0)^{-1/2} \left((\cos\theta) v^{(1)} + (\sin\theta) v^{(2)} \right)$$

$$= \left((\cos\theta) v^{(1)} + (\sin\theta) v^{(2)} \right)^T \left(\mu^{(1)} (\cos\theta) v^{(1)} + \mu^{(2)} (\sin\theta) v^{(2)} \right)$$

$$= \mu^{(1)} \cos^2\theta + \mu^{(2)} \sin^2\theta$$

$$= \lambda_1^2 \cos^2\theta + \lambda_2^2 \sin^2\theta$$

Substituting (38-39) into (36) gives

(40) $$\lambda = \sqrt{\lambda_1^2 \cos^2\theta + \lambda_2^2 \sin^2\theta}$$

This formula can be rewritten in terms of u by noting that

(41) $$u^T G^0 u^{(k)} =$$

$$\left((\cos\theta)\, v^{(1)} + (\sin\theta)\, v^{(2)}\right)^T G_0^{-1/2}$$

$$G^0 G_0^{-1/2} v^{(k)}$$

$$= (\cos\theta)\left(v^{(1)}\right)^T v^{(k)} + (\sin\theta)\left(v^{(2)}\right)^T v^{(k)}$$

$$= \begin{cases} \cos\theta, & k = 1 \\ \sin\theta, & k = 2 \end{cases}$$

Therefore (40) can be written

(42) $$\lambda = \sqrt{\lambda_1^2 \left(u^T G^0 u^{(1)}\right)^2 + \lambda_2^2 \left(u^T G^0 u^{(2)}\right)^2}$$

In this formula, note that t = arclength along all three of the curves in the reference configuration (i.e., the two curves in the direction of the principal stretches, and the third curve that passes through the point of intersection making angle θ with respect to one of those ~~~~ curves.

Now we consider the effect of a change of variables in the material coordinates:

(43) $$\xi_1(\xi_1', \xi_2'), \quad \xi_2(\xi_1', \xi_2')$$

Let

(44) $$\underline{X}'(\xi_1', \xi_2') = \underline{X}(\xi_1(\xi_1', \xi_2'), \xi_2(\xi_1', \xi_2'))$$

Then

(45) $$\frac{\partial \underline{X}'}{\partial \xi_i'} = \frac{\partial \underline{X}}{\partial \xi_k} \frac{\partial \xi_k}{\partial \xi_i'}$$

so

(46) $$G_{ij}' = \frac{\partial \underline{X}}{\partial \xi_k} \frac{\partial \xi_k}{\partial \xi_i'} \cdot \frac{\partial \underline{X}}{\partial \xi_l} \frac{\partial \xi_l}{\partial \xi_j'}$$

$$= G_{kl} \frac{\partial \xi_k}{\partial \xi_i'} \frac{\partial \xi_l}{\partial \xi_j'}$$

and

(47) $$G' = \left(\frac{\partial \xi}{\partial \xi'}\right)^T G \left(\frac{\partial \xi}{\partial \xi'}\right)$$

where we have introduced the 2x2 matrix $\partial \xi / \partial \xi'$ with elements

$$(48) \qquad \left(\frac{\partial \xi}{\partial \xi'}\right)_{ij} = \frac{\partial \xi_i}{\partial \xi'_j}$$

Of course, the foregoing considerations are also applicable to the reference configuration, so we have

$$(49) \qquad (G^0)' = \left(\frac{\partial \xi}{\partial \xi'}\right)^T G^0 \left(\frac{\partial \xi}{\partial \xi'}\right)$$

Now suppose that the transformation (43) is a rotation, so that

$$(50) \qquad \left(\frac{\partial \xi}{\partial \xi'}\right)^T \left(\frac{\partial \xi}{\partial \xi'}\right) = \frac{\partial \xi}{\partial \xi'} \left(\frac{\partial \xi}{\partial \xi'}\right)^T = I$$

In this special case, we claim that

$$(51) \qquad \left((G^0)'\right)^{1/2} = \left(\frac{\partial \xi}{\partial \xi'}\right)^T (G^0)^{1/2} \left(\frac{\partial \xi}{\partial \xi'}\right)$$

To check this, we square both sides, and get

(52) $$(G^0)' = \left(\frac{\partial q}{\partial q'}\right)^T (G^0)^{1/2} \left(\frac{\partial q}{\partial q'}\right) \left(\frac{\partial q}{\partial q'}\right)^T (G^0)^{1/2} \left(\frac{\partial q}{\partial q'}\right)$$

$$= \left(\frac{\partial q}{\partial q'}\right)^T (G^0)^{1/2} (G^0)^{1/2} \frac{\partial q}{\partial q'}$$

$$= \left(\frac{\partial q}{\partial q'}\right)^T G^0 \left(\frac{\partial q}{\partial q'}\right) = (G^0)'$$

as required (see (49)).

Again with $(\partial q/\partial q')$ as a rotation matrix, we also have

(53) $$((G^0)')^{-1/2} = \left(\left(\frac{\partial q}{\partial q'}\right)^T (G^0)^{1/2} \left(\frac{\partial q}{\partial q'}\right)\right)^{-1}$$

$$= \left(\frac{\partial q}{\partial q'}\right)^{-1} (G^0)^{-1/2} \left(\frac{\partial q}{\partial q'}\right)^{-T} = \left(\frac{\partial q}{\partial q'}\right)^T (G^0)^{-1/2} \frac{\partial q}{\partial q'}$$

Therefore

(54) $$\left((G^0)'\right)^{-1/2} G' \left((G^0)'\right)^{-1/2} =$$

$$\left(\frac{\partial q}{\partial q'}\right)^T (G^0)^{-1/2} \cancel{\left(\frac{\partial q}{\partial q'}\right)\left(\frac{\partial q}{\partial q'}\right)^T} G \cancel{\left(\frac{\partial q}{\partial q'}\right)\left(\frac{\partial q}{\partial q'}\right)^T} (G^0)^{-1/2} \left(\frac{\partial q}{\partial q'}\right) =$$

$$\left(\frac{\partial q}{\partial q'}\right)^T (G^0)^{-1/2} G (G^0)^{-1/2} \left(\frac{\partial q}{\partial q'}\right)$$

Now we consider the stretching energy per unit area in the reference configuration* that is involved in deforming the surface from the reference configuration $\underline{X}^0(q_1, q_2)$ to the current configuration $\underline{X}(q_1, q_2)$.

*i.e., the energy density with respect to reference-configuration area.

Since the energy density must be invariant under rotations of the material coordinates for an isotropic material, we assume it is of the form

(55) $$\mathcal{E}^0(\lambda_1^2+\lambda_2^2\,,\,\lambda_1\lambda_2)$$

The superscript ^{0} means that this is energy density with respect to ~~refer~~ reference-configuration area. Thus, the total energy of the deformed surface is given by

(56) $$\int_Q \mathcal{E}^0(\lambda_1^2+\lambda_2^2,\lambda_1\lambda_2)\sqrt{\det(G^0)}\;dq_1\,dq_2$$

$$=\int_Q \frac{\mathcal{E}^0(\lambda_1^2+\lambda_2^2,\lambda_1\lambda_2)}{\lambda_1\lambda_2}\,\lambda_1\lambda_2\sqrt{\det(G^0)}\;dq_1\,dq_2$$

$$=\int_Q \mathcal{E}(\lambda_1^2+\lambda_2^2,\lambda_1\lambda_2)\sqrt{\det(G)}\;dq_1\,dq_2$$

Here Q is part of the g_1, g_2 plane that is used in defining the surface, and

(57) $$\mathcal{E}(\lambda_1^2+\lambda_2^2, \lambda_1\lambda_2) = \frac{\mathcal{E}^0(\lambda_1^2+\lambda_2^2, \lambda_1\lambda_2)}{\lambda_1\lambda_2}$$

so that $\mathcal{E}$ is the stretching energy density of the deformation with respect to current-configuration area.

For example, if $\mathcal{E}^0 = \sigma\lambda_1\lambda_2$, then $\mathcal{E} = \sigma$, so the total energy is just σ times the current-configuration area of the surface. This case is called surface tension.

Another interesting case is obtained by setting

(58) $$\mathcal{E}^0 = \left(\frac{\sigma}{2}\right)(\lambda_1^2+\lambda_2^2)$$

Then

$$\mathcal{E} = \frac{\sigma}{2}\,\frac{\lambda_1^2 + \lambda_2^2}{\lambda_1 \lambda_2} \tag{59}$$

$$= \sigma + \frac{\sigma}{2}\left(\frac{\lambda_1^2 + \lambda_2^2}{\lambda_1 \lambda_2} - 2\right)$$

$$= \sigma + \frac{\sigma}{2}\,\frac{(\lambda_1 - \lambda_2)^2}{\lambda_1 \lambda_2}$$

$$= \sigma + \frac{\sigma}{2}\left(\frac{\lambda_1 - \lambda_2}{\sqrt{\lambda_1 \lambda_2}}\right)^2$$

$$= \sigma + \frac{\sigma}{2}\left(\sqrt{\frac{\lambda_1}{\lambda_2}} - \sqrt{\frac{\lambda_3}{\lambda_1}}\right)^2$$

Thus the energy per unit current-configuration area is a constant plus a term that depends on the ratio of the principal stretches and is minimized when that ratio is one.

Note that we do not need to find λ_1 and λ_2 separately to evaluate $\lambda_1^2+\lambda_2^2$ and $\lambda_1\lambda_2$, since we have the following formulae

(60) $$\lambda_1^2+\lambda_2^2 = \operatorname{trace}\left(G(G^0)^{-1}\right)$$

(61) $$\lambda_1\lambda_2 = \sqrt{\frac{\det(G)}{\det(G^0)}}$$

Now consider the case in which a triangle in the reference configuration is deformed into a different triangle in the current configuration, with barycentric coordinates as material coordinates. Let

(62) $$\underline{A}^0, \underline{B}^0, \underline{C}^0$$

be the vertices of the triangle in the reference configuration, and let

(63) $$\underline{A}, \underline{B}, \underline{C}$$

be the corresponding vertices in the current configuration. Then

(64) $$\underline{X}^0(\xi_1,\xi_2) = \xi_1 \underline{A}^0 + \xi_2 \underline{B}^0 + (1-\xi_1-\xi_2)\underline{C}^0$$

(65) $$\underline{X}(\xi_1,\xi_2) = \xi_1 \underline{A} + \xi_2 \underline{B} + (1-\xi_1-\xi_2)\underline{C}$$

Let

(66) $$\underline{a}^0 = \underline{A}^0 - \underline{C}^0 \quad , \quad \underline{b}^0 = \underline{B}^0 - \underline{C}^0$$

(67) $$\underline{a} = \underline{A} - \underline{C} \quad , \quad \underline{b} = \underline{B} - \underline{C}$$

Then

(68) $$G^0 = \begin{pmatrix} \|\underline{a}^0\|^2 & (\underline{a}^0 \cdot \underline{b}^0) \\ (\underline{a}^0 \cdot \underline{b}^0) & \|\underline{b}^0\|^2 \end{pmatrix}$$

(69) $$G = \begin{pmatrix} \|\underline{a}\|^2 & \underline{a} \cdot \underline{b} \\ \underline{a} \cdot \underline{b} & \|\underline{b}\|^2 \end{pmatrix}$$

(70) $$\det(G^0) = \|\underline{a}^0\|^2 \|\underline{b}^0\|^2 - (\underline{a}^0 \cdot \underline{b}^0)^2$$

(71) $$\det(G) = \|\underline{a}\|^2 \|\underline{b}\|^2 - (\underline{a} \cdot \underline{b})^2$$

(72) $$(G^0)^{-1} = \frac{1}{\det(G^0)} \begin{pmatrix} \|\underline{b}^0\|^2 & -(\underline{a}^0 \cdot \underline{b}^0) \\ -(\underline{a}^0 \cdot \underline{b}^0) & \|\underline{a}^0\|^2 \end{pmatrix}$$

(73) $$G G_0^{-1} =$$

$$\frac{1}{\det(G^0)} \begin{pmatrix} \|\underline{a}\|^2 & \underline{a}\cdot\underline{b} \\ \underline{a}\cdot\underline{b} & \|\underline{b}\|^2 \end{pmatrix} \begin{pmatrix} \|\underline{b}^0\|^2 & -(\underline{a}^0\cdot\underline{b}^0) \\ -(\underline{a}^0\cdot\underline{b}^0) & \|\underline{a}^0\|^2 \end{pmatrix}$$

(74) $$\mathrm{trace}(G G_0^{-1}) =$$

$$\frac{\|\underline{a}\|^2 \|\underline{b}^0\|^2 + \|\underline{b}\|^2 \|\underline{a}^0\|^2 - 2(\underline{a}\cdot\underline{b})(\underline{a}^0\cdot\underline{b}^0)}{\|\underline{a}^0\|^2 \|\underline{b}^0\|^2 - (\underline{a}^0\cdot\underline{b}^0)^2}$$

Let

(75) $$\underline{c}^0 = \underline{b}^0 - \underline{a}^0, \quad \underline{c} = \underline{b} - \underline{a}$$

Then

(76) $$\|\underline{c}^0\|^2 = \|\underline{b}^0\|^2 - 2(\underline{a}^0\cdot\underline{b}^0) + \|\underline{a}^0\|^2$$

(77) $$\|\underline{c}\|^2 = \|\underline{b}\|^2 - 2(\underline{a}\cdot\underline{b}) + \|\underline{a}\|^2$$

Therefore

$$(78) \qquad \underline{a}^0 \cdot \underline{b}^0 = \frac{1}{2}\left(\|\underline{a}^0\|^2 + \|\underline{b}^0\|^2 - \|\underline{c}^0\|^2\right)$$

$$(79) \qquad (\underline{a}^0 \cdot \underline{b}^0)^2 = \frac{1}{4}\left(\|\underline{a}^0\|^4 + \|\underline{b}^0\|^4 + \|\underline{c}^0\|^4\right)$$

$$+ \frac{1}{2}\left(\|\underline{a}^0\|^2 \|\underline{b}^0\|^2 - \|\underline{a}^0\|^2 \|\underline{c}^0\|^2 - \|\underline{b}^0\|^2 \|\underline{c}^0\|^2\right)$$

$$(80) \qquad \|\underline{a}^0\|^2 \|\underline{b}^0\|^2 - (\underline{a}^0 \cdot \underline{b}^0)^2$$

$$= -\frac{1}{4}\left(\|\underline{a}^0\|^4 + \|\underline{b}^0\|^4 + \|\underline{c}^0\|^4\right)$$

$$+ \frac{1}{2}\left(\|\underline{a}^0\|^2 \|\underline{b}^0\|^2 + \|\underline{a}^0\|^2 \|\underline{c}^0\|^2 + \|\underline{b}^0\|^2 \|\underline{c}^0\|^2\right)$$

Note that

(81) $$\left(\|\underline{a}^0\| + \|\underline{b}^0\| + \|\underline{c}^0\|\right)\left(-\|\underline{a}^0\| + \|\underline{b}^0\| + \|\underline{c}^0\|\right)$$

$$\left(\|\underline{a}^0\| - \|\underline{b}^0\| + \|\underline{c}^0\|\right)\left(\|\underline{a}^0\| + \|\underline{b}^0\| - \|\underline{c}^0\|\right) =$$

$$\left(-\|\underline{a}^0\|^2 + \left(\|\underline{b}^0\| + \|\underline{c}^0\|\right)^2\right)$$

$$\left(\|\underline{a}^0\|^2 - \left(\|\underline{b}^0\| - \|\underline{c}^0\|\right)^2\right) =$$

$$-\|\underline{a}^0\|^4 + \|\underline{a}^0\|^2\left(\|\underline{b}^0\|^2 + \cancel{2\|\underline{b}^0\|\|\underline{c}^0\|} + \|\underline{c}^0\|^2\right)$$

$$+ \|\underline{a}^0\|^2\left(\|\underline{b}^0\|^2 - \cancel{2\|\underline{b}^0\|\|\underline{c}^0\|} + \|\underline{c}^0\|^2\right)$$

$$- \left(\|\underline{b}^0\|^2 - \|\underline{c}^0\|^2\right)^2$$

$$= -\|\underline{a}^0\|^4 - \|\underline{b}^0\|^4 - \|\underline{c}^0\|^4$$

$$+ 2\left(\|\underline{a}^0\|^2\|\underline{b}^0\|^2 + \|\underline{a}^0\|^2\|\underline{c}^0\|^2 + \|\underline{b}^0\|^2\|\underline{c}^0\|^2\right)$$

Thus, equation (80) can also be written

$$(82)\quad \|\underline{a}^0\|^2\,\|\underline{b}^0\|^2-\left(\underline{a}^0\cdot\underline{b}^0\right) =$$

$$\frac{1}{4}\left(\|\underline{a}^0\|+\|\underline{b}^0\|+\|\underline{c}^0\|\right)\left(-\|\underline{a}^0\|+\|\underline{b}^0\|+\|\underline{c}^0\|\right)$$

$$\left(\|\underline{a}^0\|-\|\underline{b}^0\|+\|\underline{c}^0\|\right)\left(\|\underline{a}^0\|+\|\underline{b}^0\|-\|\underline{c}^0\|\right)$$

This is Heron's formula. Note that all of the factors are positive. The first one is a sum of positive quantities, and the last three are positive by the triangle inequality.

Remark: Although Heron's formula is very beautiful, the left-hand side of (82) is easier to differentiate with respect to the coordinates of the vertices of the triangle, since it is a polynomial in the vertices, whereas Heron's formula involves the lengths of the sides, which have square roots in their formulae.

The numerator on the right-hand side of (74) can be evaluated as follows:

(83) $$\|\underline{a}\|^2\|\underline{b}^0\|^2 + \|\underline{b}\|^2\|\underline{a}^0\|^2 - 2(\underline{a}\cdot\underline{b})(\underline{a}^0\cdot\underline{b}^0) =$$

$$\|\underline{a}\|^2\|\underline{b}^0\|^2 + \|\underline{b}\|^2\|\underline{a}^0\|^2$$

$$-\frac{1}{2}\left(\|\underline{a}\|^2 + \|\underline{b}\|^2 - \|\underline{c}\|^2\right)\left(\|\underline{a}^0\|^2 + \|\underline{b}^0\|^2 - \|\underline{c}^0\|^2\right)$$

$$= -\frac{1}{2}\left(\|\underline{a}\|^2\|\underline{a}^0\|^2 + \|\underline{b}\|^2\|\underline{b}^0\|^2 + \|\underline{c}\|^2\|\underline{c}^0\|^2\right)$$

$$+\frac{1}{2}\Big(\|\underline{a}\|^2\|\underline{b}^0\|^2 + \|\underline{b}\|^2\|\underline{a}^0\|^2 + \|\underline{a}\|^2\|\underline{c}^0\|^2 + \|\underline{c}\|^2\|\underline{a}^0\|^2$$

$$+ \|\underline{b}\|^2\|\underline{c}^0\|^2 + \|\underline{c}\|^2\|\underline{b}^0\|^2\Big)$$

$$= \frac{1}{2}\left(\|\underline{a}\|^2 + \|\underline{b}\|^2 + \|\underline{c}\|^2\right)\left(\|\underline{a}^0\|^2 + \|\underline{b}^0\|^2 + \|\underline{c}^0\|^2\right)$$

$$-\left(\|\underline{a}\|^2\|\underline{a}^0\|^2 + \|\underline{b}\|^2\|\underline{b}^0\|^2 + \|\underline{c}\|^2\|\underline{c}^0\|^2\right)$$

Now combining (82) & (83), and rewriting (83) slightly, we get

$$(84)\quad \lambda_1^2+\lambda_2^2 = \operatorname{trace}\left(G(G^0)^{-1}\right) =$$

$$\frac{\frac{1}{2}\Big(\|\underline{a}\|^2\left(-\|\underline{a}^0\|^2+\|\underline{b}^0\|^2+\|\underline{c}^0\|^2\right) + \|\underline{b}\|^2\left(\|\underline{a}^0\|^2-\|\underline{b}^0\|^2+\|\underline{c}^0\|^2\right) + \|\underline{c}\|^2\left(\|\underline{a}^0\|^2+\|\underline{b}^0\|^2-\|\underline{c}^0\|^2\right)\Big)}{\frac{1}{4}\left(\|\underline{a}^0\|+\|\underline{b}^0\|+\|\underline{c}^0\|\right)\left(-\|\underline{a}^0\|+\|\underline{b}^0\|+\|\underline{c}^0\|\right)\left(\|\underline{a}^0\|-\|\underline{b}^0\|+\|\underline{c}^0\|\right)\left(\|\underline{a}^0\|+\|\underline{b}^0\|-\|\underline{c}^0\|\right)}$$

In the special case that the reference triangle is equilateral, with $\|\underline{a}^0\| = \|\underline{b}^0\| = \|\underline{c}^0\| = l_0$, this reduces to

$$(85)\quad \lambda_1^2+\lambda_2^2 = \frac{2}{3}\left(\|\underline{a}\|^2+\|\underline{b}\|^2+\|\underline{c}\|^2\right)/l_0^2$$

and finally,

$$(86)\quad \lambda_1 \lambda_2 = \sqrt{\frac{\|\underline{a}\|^2 \|\underline{b}\|^2 - (\underline{a}\cdot\underline{b})^2}{\|\underline{a}^0\|^2 \|\underline{b}^0\|^2 - (\underline{a}^0\cdot\underline{b}^0)^2}}$$

$$= \sqrt{\frac{\begin{gathered}(\|\underline{a}\| + \|\underline{b}\| + \|\underline{c}\|)(-\|\underline{a}\| + \|\underline{b}\| + \|\underline{c}\|)\\ (\|\underline{a}\| - \|\underline{b}\| + \|\underline{c}\|)(\|\underline{a}\| + \|\underline{b}\| - \|\underline{c}\|)\end{gathered}}{\begin{gathered}(\|\underline{a}^0\| + \|\underline{b}^0\| + \|\underline{c}^0\|)(-\|\underline{a}^0\| + \|\underline{b}\|^0 + \|\underline{c}\|^0)\\ (\|\underline{a}^0\| - \|\underline{b}^0\| + \|\underline{c}^0\|)(\|\underline{a}^0\| + \|\underline{b}\|^0 - \|\underline{c}\|^0)\end{gathered}}}$$