CHRIS RORRES

Completing Book II of Archimedes's *On Floating Bodies*

Archimedes (c. 287 B.C. to 212/211 B.C.) lived in the Greek city-state of Syracuse, Sicily, up to the time that it was conquered by the Romans, a conquest that led to his death. Of his works that survive, the second of his two books of *On Floating Bodies*[1] is considered his most mature work, commonly described as a *tour de force* [6, 15]. This book contains a detailed investigation of the stable equilibrium positions of floating right paraboloids[2] of various shapes and relative densities, but restricted to the case when the base of the paraboloid lies either entirely above or entirely below the fluid surface.

This paper summarizes the results of research in which I completed Archimedes's investigation to include also the more complex cases when the base of the floating paraboloid is partially submerged. Modern scientific computing and computer graphics enabled me to construct a three-dimensional surface that summarizes all possible equilibrium positions (both stable and unstable) for all possible shapes and relative densities. This equilibrium surface contains folds and cusps that explain certain catastrophic phenomena—for example, the sudden tumbling of a melting iceberg or the toppling of a tall structure due to liquefaction of the ground beneath it—that have long been observed but not previously explained fully.

Books I and II

Book I of *On Floating Bodies* begins with a derivation of Archimedes's Law of Buoyancy from more fundamental principles and finishes with a simple, elegant geometric proof that a floating segment of a homogeneous solid sphere is always in stable equilibrium when its base is parallel to the surface of the fluid, either above the fluid surface or below it. Book I introduced the concept of fluid pressure and initiated the science of hydrostatics. It took almost eighteen centuries before this work on the nature of fluids was continued by such scientists as Simon Stevin (Dutch, 1548-1620), Galileo Galilei (Italian, 1564-1642), Evangelista Torricelli (Italian, 1608-1647), Blaise

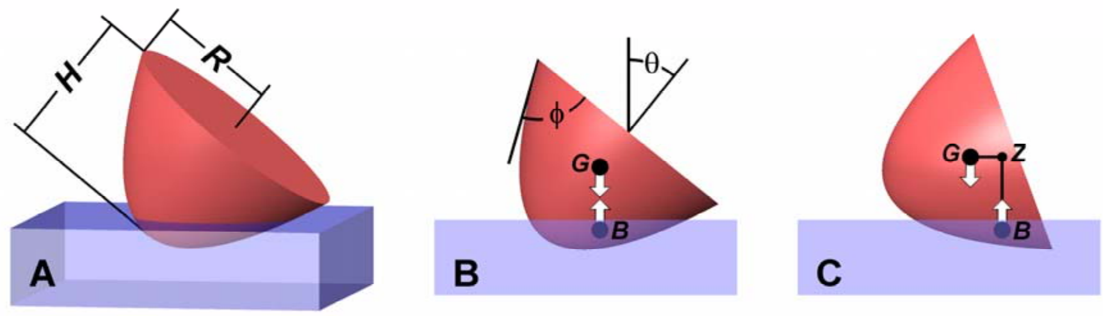


Figure 1. (**A** and **B**) Two views of a floating right paraboloid with ϕ= 67º, s = 0.08 and θ=38.95º in equilibrium according to Archimedes's Proposition 8. (**C**) The paraboloid has been rotated clockwise from this equilibrium position while keeping the weight of the displaced fluid equal to the weight of the paraboloid.

Pascal (French, 1623-1662), and Isaac Newton (English, 1642-1727). In the interim, Book I served mainly as the basis for determining the density of objects, such as gemstones and precious-metal artifacts, by comparing their weights in air and in water.

In Book II Archimedes extended his stability analysis of floating bodies from a segment of a sphere to a right paraboloid. However, Book II contained many sophisticated ideas and complex geometric constructions and did not have the appeal of Book I. Only after Greek geometry was augmented with algebra, trigonometry, and analytical geometry and the field of mechanics reached the maturity to handle the concepts of equilibrium and stability that Archimedes introduced was Book II seriously studied. It then became the standard starting point for scientists and naval architects examining the stability of ships and other floating bodies[3].

To describe the results Archimedes obtained in Book II let us first precisely define his object of study:

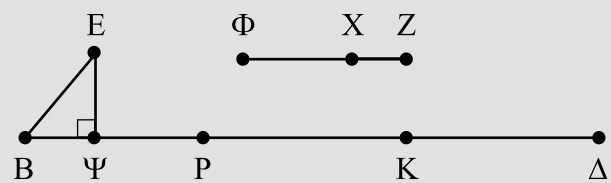
> **Definition:** A *paraboloid* is a homogeneous solid convex object bounded by a surface obtained by rotating a parabola about its axis of symmetry and by a plane that is not parallel to the parabola's axis of symmetry. If the plane is perpendicular to the axis of symmetry it is called a *right paraboloid*, otherwise it is called an *oblique paraboloid*. The planar portion of the surface, which is either circular or elliptical, is called the *base* of the paraboloid.

Let R be the radius of the base of a right paraboloid and let H be its height (Fig. 1A). Define its *base angle* ϕ as the angle between 0º and 90º for which $\tan\phi = 2H/R$. In a profile view of the paraboloid it is the angle between its base and the tangent line to the parabolic cross section at the base (Fig. 1B). This base angle determines the shape of the parabola. Next, let ρ_{body} be the mass-density of the paraboloid and let ρ_{fluid} be the mass-density of the fluid in which it is floating within a uniform gravitational field. Following Archimedes, let us neglect the density of the air above the fluid[4] and define

Proposition 8 of Book II of Archimedes's *On Floating Bodies*

The following statements and diagram appear in the proof of Proposition 8:

1. BΔ is equal to the axis.
2. BK is twice KΔ.
3. KP is equal to the line-up-to-the-axis.
4. The weight of the body is to that of the fluid [of equal volume] as the square of side ΦZ is to that of side BΔ.
5. ΦX is twice XZ.
6. ΦX is equal to PΨ.
7. The square of side ΨE is half of the rectangle of sides KP and BΨ.



These statements describe a compass-and-straightedge construction beginning with:

1. The "axis" of the paraboloid. This is a line segment of length H, the height of the paraboloid.
2. The "line-up-to-the-axis". This is the semilatus rectum of the paraboloid, which is a line segment of length $R^2/2H$, where R is the radius of the base of the paraboloid. Alternatively, we could begin with a line segment of length R and construct the line-up-to-the-axis.
3. The magnitude s. This is the ratio of the weight of the paraboloid to that of an equal volume of fluid. Floating the paraboloid in the fluid vertically with the base up, we have that $s = (h/H)^2$, where h is the height of the submerged portion of the paraboloid. The line segment ΦZ in the diagram has length h.

Archimedes shows that the paraboloid's angle of inclination (the angle its axis makes with the surface of the fluid) is angle EBΨ. The complementary angle BEΨ is my tilt angle θ. Using algebraic notation, where AB appearing in an equation represents the length of the corresponding line segment, the seven statements above become

1. $\mathrm{B\Delta} = H$
2. $\mathrm{BK} = 2(\mathrm{K\Delta})$
3. $\mathrm{KP} = R^2/2H$
4. $s = (\mathrm{\Phi Z}/\mathrm{B\Delta})^2$
5. $\mathrm{\Phi X} = 2(\mathrm{XZ})$
6. $\mathrm{\Phi X} = \mathrm{P\Psi}$
7. $\mathrm{\Psi E}^2 = (\mathrm{KP})(\mathrm{B\Psi})/2$

From Eqs. 1-7 and the fact that the base angle ϕ of the paraboloid satisfies $\tan\phi = 2H/R$, we obtain

$$\tan^2\theta = \left(\frac{\mathrm{B\Psi}}{\mathrm{\Psi E}}\right)^2 = \frac{\mathrm{B\Psi}^2}{\frac{1}{2}(\mathrm{KP})(\mathrm{B\Psi})} = \frac{2\mathrm{B\Psi}}{\mathrm{KP}} = \frac{2(\mathrm{BK}-\mathrm{P\Psi}-\mathrm{KP})}{\mathrm{KP}} = \frac{2(\mathrm{BK}-\mathrm{\Phi X}-\mathrm{KP})}{\mathrm{KP}}$$

$$= \frac{2\left(\frac{2}{3}\mathrm{B\Delta}-\frac{2}{3}\mathrm{\Phi Z}-\mathrm{KP}\right)}{\mathrm{KP}} = \frac{2\left(\frac{2}{3}\mathrm{B\Delta}-\frac{2}{3}\sqrt{s}\,\mathrm{B\Delta}-\mathrm{KP}\right)}{\mathrm{KP}} = \frac{2\left(\frac{2}{3}H-\frac{2}{3}\sqrt{s}H-\frac{R^2}{2H}\right)}{\frac{R^2}{2H}}$$

$$= 2\left(\frac{4}{3}\frac{H^2}{R^2}\left(1-\sqrt{s}\right)-1\right) = \frac{2}{3}\left(1-\sqrt{s}\right)\tan^2\phi-2$$

the *relative density* (or specific gravity) of the paraboloid as $s = \rho_{body}/\rho_{fluid}$, which is a number in the interval [0, 1] for a floating paraboloid. Finally, let θ be the *tilt angle* (or *heel angle*), by which is meant the angle of inclination in the interval [0º, 180º] of the axis of the paraboloid from the vertical with 0º corresponding to the base above the fluid level (Fig. 1B). As with Archimedes, let us confine the rotation of the paraboloid so that its axis always lies in a fixed vertical plane.

Below is an example of one of the ten propositions in Book II, in which I first give a very literal translation of the Greek text [22] and then a very liberal modern translation. In the literal translation the ‘axis’ is a line segment whose length is the height H of the paraboloid and the ‘line-up-to-the-axis’ is the semilatus rectum of the paraboloid, which is a line segment of length $R^2/2H$. The last sentence in my translation actually consists of seven excerpts from the beginning of the proof of Proposition 8 where a geometric construction is described.

Archimedes’s Proposition 8. *Literal Translation:*

> A right segment of a right-angled conoid, when its axis is greater than one-and-a-half times the line-up-to-the-axis, but small enough so that its ratio to the line-up-to-the-axis is less than fifteen to four, and when further its weight has to that of the fluid [of equal volume] a ratio less than that which the square of the amount by which the axis exceeds one-and-a-half times the line-up-to-the-axis bears to the square of the axis, will, when so placed in the fluid that the base does not touch the surface of the fluid, not return to the vertical position and not remain in the inclined position except when its axis makes with the surface of the fluid a certain angle to be described. [This angle is EBΨ in the diagram (Fig. 2) in which] (1) BΔ is equal to the axis; (2) BK is twice KΔ; (3) KP is equal to the line-up-to-the-axis; (4) the weight of the body is to that of the fluid [of equal volume] as the square of side ΦZ is to that of side BΔ; (5) ΦX is twice XZ; (6) ΦX is equal to PΨ; and (7) the square of side ΨE is half of the rectangle of sides KP and BΨ.

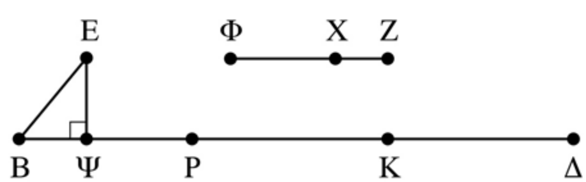


Figure 2. Diagram for the statement of Proposition 8, scaled for the paraboloid in Figure 1.

Archimedes’s Proposition 8. *Modern Translation:*

> A right paraboloid whose base angle ϕ satisfies $3 < \tan^2\phi < 15/2$ and whose relative density s satisfies $s < (1 - 3\cot^2\phi)^2$ has precisely one stable equilibrium position with its base completely above the fluid surface. The corresponding tilt angle is $\theta = \tan^{-1}\sqrt{\frac{2}{3}\left(1-\sqrt{s}\right)\tan^2\phi - 2}$.

Archimedes's objective in Proposition 8 was to describe a geometric construction using compass and straightedge that begins with three lines segments describing the shape and relative density of the paraboloid (the axis, the line-up-to-the-axis, and the line segment ΦZ whose length is $\sqrt{s}H$) and ends with a diagram in which the tilt angle is revealed. My objective in the modern translation, however, was to summarize the geometric construction in a single analytical expression in which the equilibrium tilt angle θ is expressed as an explicit function of s and ϕ. My modern translation incorporates centuries of algebraic, trigonometric, and analytical developments and considerably alters how the Greeks would have grasped Archimedes's results. It also shows the limitations of Greek geometry in formulating and describing complicated physical phenomena.

Archimedes's other propositions in Book II complete his study of the stable equilibrium tilt angles when the base is either completely above or completely below the fluid surface for appropriate values of the base angle and the relative density. The main geometric tools he used were the formulas for the volumes and centroids of right and oblique paraboloids, formulas that he himself derived in other works[5]. The mechanical tools he used—again, tools that he himself first formulated—were his Law of Buoyancy for a floating body, his Law of the Lever, and the equilibrium condition that the center of gravity of the floating body must lie on the same vertical line as its center of buoyancy. (Because a paraboloid is a homogeneous convex body, its center of buoyancy coincides with the center of gravity of its submerged portion.)

Righting and Energy Arms

The numerical techniques I used required the evaluation of the moment acting on an unbalanced floating paraboloid. In Figure 1C a right paraboloid is floating in a fluid with the weight of the displaced fluid equal to the weight of the right paraboloid. However, it is not in equilibrium because the center of gravity ***G*** of the body is not on the same vertical line as its center of buoyancy ***B***. Rather, the weight of the paraboloid and the buoyancy force form a couple that will cause the paraboloid to rotate in a counterclockwise direction toward the equilibrium position shown in Figure 1B. The value of the couple, called the *righting moment*, is the weight of the paraboloid times the horizontal displacement ***GZ*** between ***G*** and ***B***, taken as positive if ***B*** is to the right of ***G***. This horizontal displacement is called the *righting arm* and its use is preferred by naval architects to the righting moment. If a wave causes a ship to heel, the righting-arm expressed as a function of the heel angle affects the dynamics of how the ship will return to its vertical equilibrium orientation. One of the standard specifications of a ship is a graph of its righting arm for a wide range of heel angles.

If the base is completely above or below the fluid surface, it is possible to determine an exact expression for the righting arm of a floating right paraboloid using the exact formulas for the volume and centroid of an oblique paraboloid. For example, if the base is above the fluid surface then

(1) $$\frac{\text{Righting Arm}}{H} = \frac{\sin\theta}{\tan^2\phi}\left[2 - \frac{2}{3}\left(1-\sqrt{s}\right)\tan^2\phi + \tan^2\theta\right].$$

Setting this equal to zero determines all equilibrium tilt angles with the base above the fluid surface and, in particular, returns the expression for the tilt angle determined by Archimedes's Proposition 8 above. When the base is completely submerged, symmetry principles can be used to obtain an analogous expression[6].

While the righting arm provides the necessary information for the stability analysis of a floating body, its potential energy also provides some insight. Taking the fluid surface as the level of zero potential energy, the potential energy of the paraboloid/fluid system is the sum of the potential energy of the paraboloid and the potential energy of the displaced fluid. The potential energy of the paraboloid is its weight multiplied by the height of its center of gravity ***G*** above the fluid surface. Likewise, the potential energy of the displaced fluid is its weight (the same as the weight of the paraboloid) multiplied by the distance of its center of gravity ***B*** below the fluid surface. The total potential energy is then the weight of the paraboloid multiplied by the vertical distance between ***B*** and ***G***. For a homogeneous convex paraboloid, ***G*** will always lie above ***B*** if the relative density is less than one and so the potential energy will always be positive.

By analogy with the term 'righting arm' I shall call the vertical displacement from ***B*** to ***G*** (***BZ*** in Figure 1C) the *energy arm* of the floating paraboloid. The fundamental relationship between force and energy shows that when the righting arm and energy arm are expressed as functions of the tilt angle θ then

$$\frac{d(\text{energy arm})}{d\theta} = \text{righting arm}\,. \tag{2}$$

In order to work with dimensionless units, let us divide both the righting arm and the energy arm by the height H of the paraboloid. Thus one unit of the normalized energy arm is the energy needed to raise the paraboloid in air a distance equal to its height.

Figure 3 is an example of the normalized righting arm and the normalized energy arm as a function of the tilt angle for a right paraboloid with base angle 74.330º and relative density 0.510. When its base is above the fluid surface ($0° \le \theta \le 28.2°$) I used Eq. (1) and when the base is below the fluid surface ($151.0° \le \theta \le 180°$) a similar exact expression was used.

When the base is cut by the fluid surface I used numerical integration to determine the volume and first moments of the unsubmerged portion of the paraboloid, from which the center of buoyancy and resulting righting-arm and righting-arm curves were determined[7]. The six roots of this righting-arm curve, or, equivalently, the six stationary points of the energy-arm curve, determine the six equilibrium positions of the corresponding paraboloid.

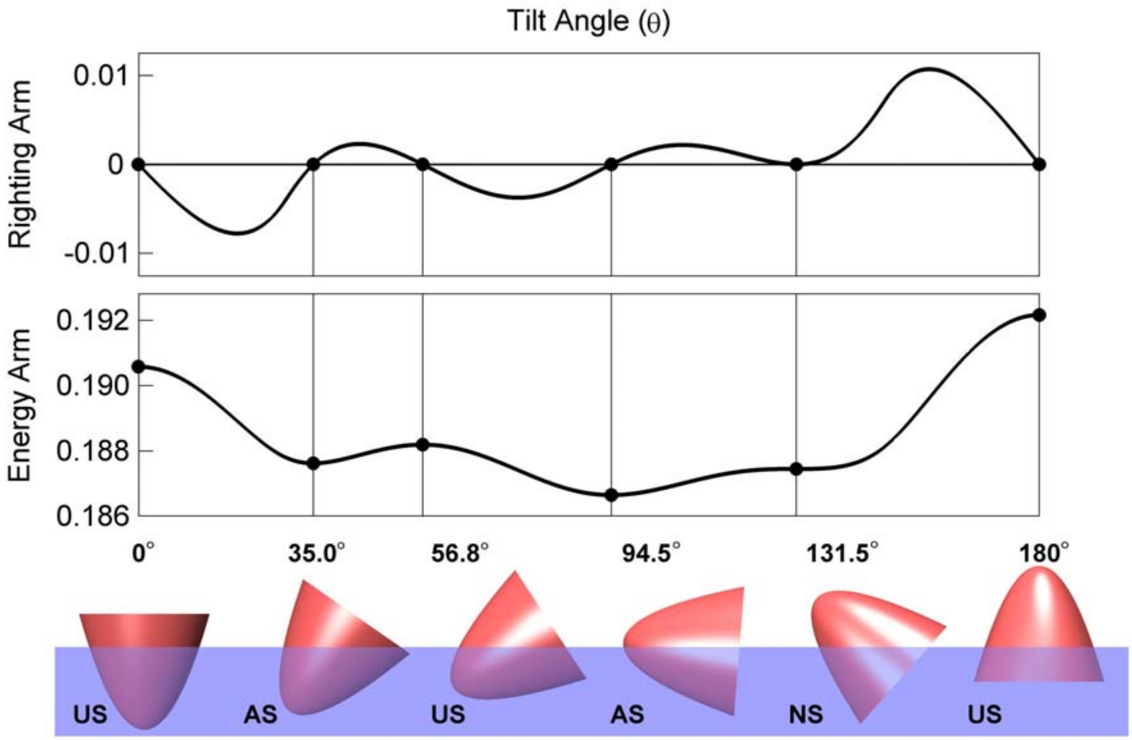


Figure 3. (Top) The normalized righting-arm curve of a paraboloid with base angle 74.330º and relative density 0.510. **(Middle)** The normalized energy-arm curve of the paraboloid. **(Bottom)** The six equilibrium configurations of the paraboloid determined by the roots of the righting-arm curve (or the stationary values of the energy-arm curve) together with their stability classifications (AS, US, or NS). The base is partially submerged if $28.2º \leq \theta \leq 151.0º$.

Because a positive righting arm produces a counterclockwise rotation and a negative righting arm produces a clockwise rotation, the way in which the algebraic sign changes through a root determines the stability classification of the corresponding equilibrium configuration. In particular, a root is *asymptotically stable* (AS), *neutrally stable to first order* (NS), or *unstable* (US) if the slope of the righting curve at the root is positive, zero, or negative, respectively[8]. None of the six equilibrium positions for the particular paraboloid described in Figure 3 were present in Archimedes's studies since they are either unstable or correspond to the base being cut by the fluid surface.

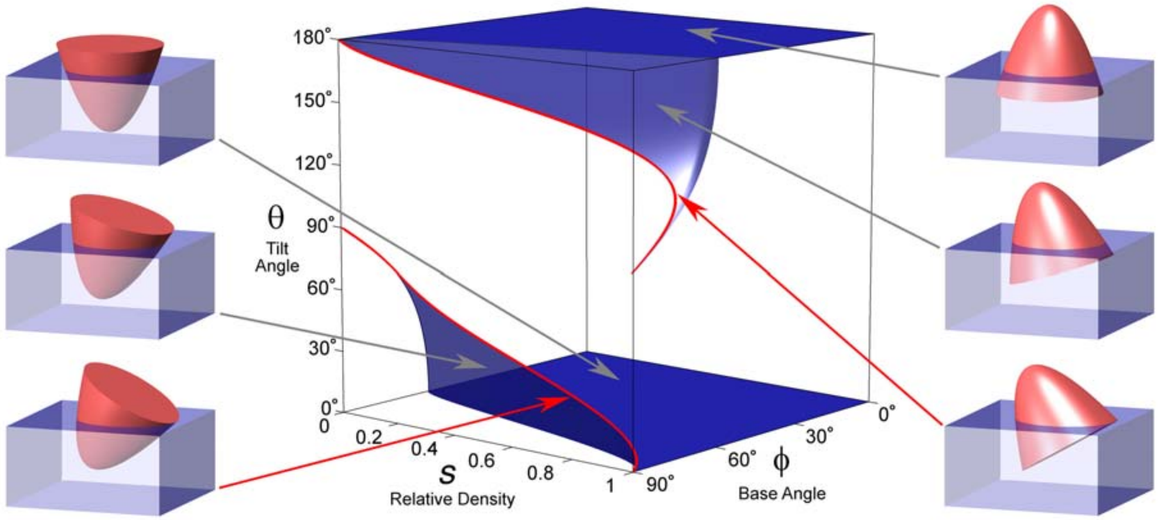


Figure 4. Archimedes's results in graphical form. Each point on the surface identifies an AS equilibrium configuration of the paraboloid in which the base is not cut by the fluid surface. Typical configurations are shown for different parts of the surface. The red curves identify those limiting configurations in which the base touches the fluid at one point.

Archimedes's Results

Let us next summarize Archimedes's results in Book II in graphical form. In Figure 4 I have plotted a surface in (ϕ, s, θ)-space in the region $[0°, 90°] \times [0, 1] \times [0°, 180°]$ in which each point identifies a combination of base angle, relative density, and tilt angle for an AS equilibrium configuration of a paraboloid whose base in not cut by the fluid surface. The bottom portion of this equilibrium surface is associated with the base lying above the fluid surface and the top portion with the base lying below the fluid surface. Because of certain symmetry considerations[6] the top portion of the equilibrium surface is a rotation of its bottom portion about the line $s = 1/2$ and $\theta = 90°$.

The curved piece of the bottom portion of the equilibrium surface, as partially determined by Archimedes's Proposition 8 above, has the explicit equation

$$\theta = \tan^{-1}\sqrt{\frac{2}{3}\left(1-\sqrt{s}\right)\tan^2\phi - 2} \tag{3}$$

restricted to the appropriate domain in ϕ and s. This curved surface is delineated below by its intersection with the plane $\theta = 0°$ and this delineation identifies those configurations in which the paraboloid starts tilting from a vertical AS configuration. The curved surface is delineated above by the bottom red curve in Figure 4, which marks those configurations when the base of the paraboloid touches the fluid surface at precisely one point. In Proposition 10 of *On Floating Bodies II,* Archimedes developed a complicated geometric construction to determine these configurations. His geometric construction is so ingenious as to warrant Cicero's assessment of him as being "endowed with greater genius that one would imagine it possible for a human being to possess" [17].

In modern analytical notation Archimedes's geometric construction for the bottom red curve is given by the following equations:

$$(4) \qquad s = \left(\frac{6+\tan^2\theta}{6+5\tan^2\theta} \right)^4, \quad \phi = \tan^{-1}\left(\frac{6+5\tan^2\theta}{4\tan\theta} \right), \qquad 0° \le \theta \le 90°,$$

where at $\theta = 90°$ the limiting values $s = 1/625$ and $\phi = 90°$ are taken.

For the upper surface corresponding equations can be obtained by replacing s by $1 - s$ and θ by $180° - \theta$ in Eqs. (3) and (4).

It should again be emphasized that Figure 4 and its analytical descriptions in Eqs. (3) and (4) are quite alien to Greek mathematics. As in the literal translation of Proposition 8, Archimedes could only express his complete results in convoluted sentences and complicated geometric constructions.

Complete Equilibrium Surface

My own research involved completing the equilibrium surface in Figure 4 by appending those points corresponding to AS configurations in which the base is cut by the fluid surface and also all points corresponding to US and NS configurations. The result is shown in Figure 5.

The construction of Figure 5 required determining all of the roots of the righting arm curves for a large number of base angles and relative densities using numerical techniques. The base angle ϕ turned out to be a single-valued function of s in [0,1] and θ in (0°, 90°). I used this fact, together with the rotational symmetry, to explicitly plot the surface. That is, rather that compute and plot θ as a multiple-valued function of ϕ and s, I computed and plotted ϕ as a single-valued function of s and θ for all s in [0, 1] and all θ other than 0°, 90°, and 180°. For those three exceptional values of θ I used the facts that (1) the entire planes $\theta = 0°$ and $\theta = 180°$ are part of the equilibrium surface, indicating that the right paraboloid is always in equilibrium when its axis of symmetry is vertical, and (2) the cross section of the equilibrium surface at $\theta = 90°$ consists of the three line segments $\{s=1/2, \theta=90°\}$, where the paraboloid is on its side half in and half out of the fluid; $\{\phi=0°, \theta=90°\}$, where the paraboloid has collapsed to a circular disk; and $\{\phi=90°, \theta=90°\}$, where the paraboloid has collapsed to a line segment.

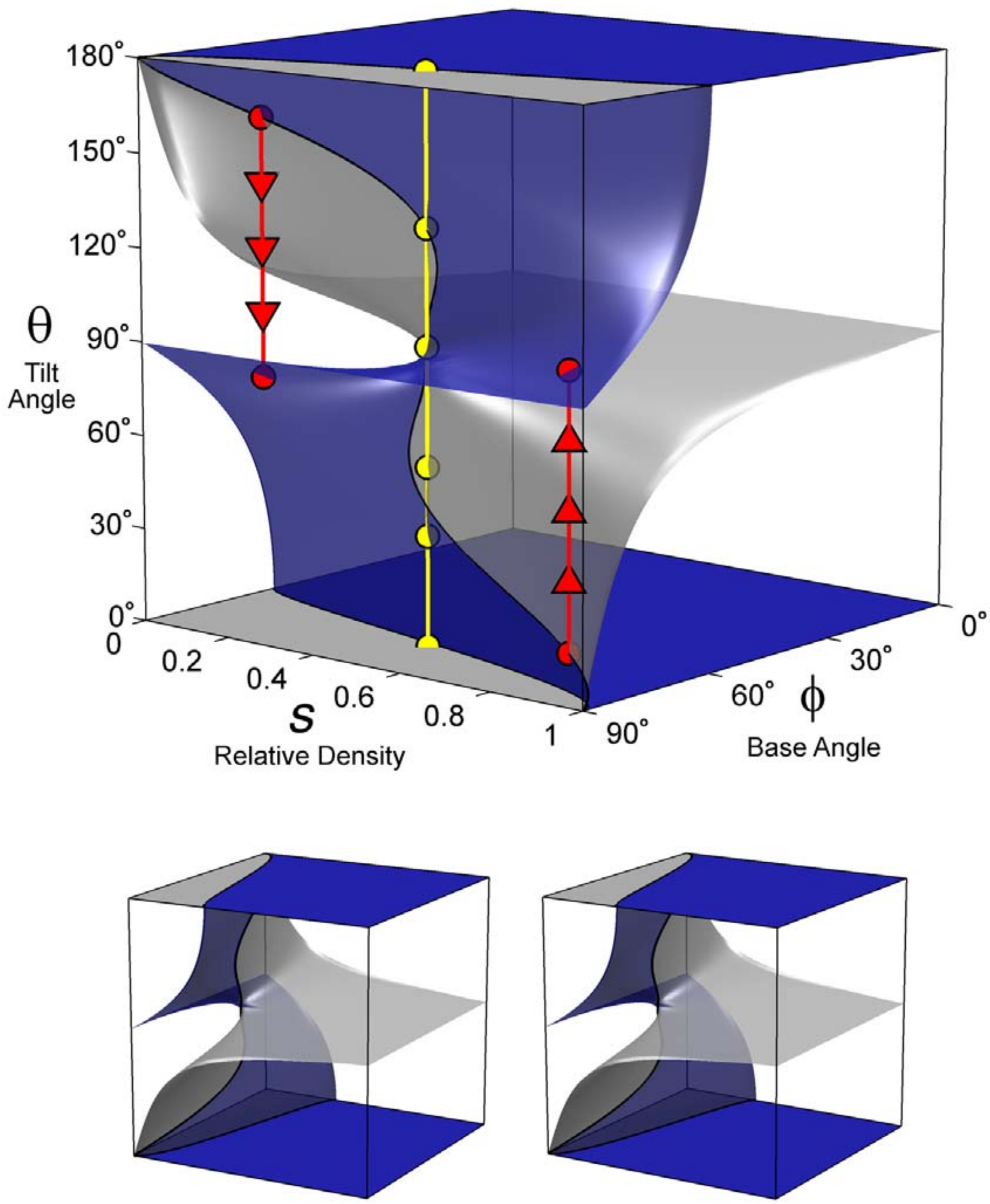


Figure 5. (Top) The complete equilibrium surface of a floating paraboloid with the equilibrium tilt angles θ plotted against the base angle ϕ and the relative density *s*. The AS points are in blue and the US points are in gray. The NS points lie on the winding curve separating the two regions. The yellow vertical line cuts through the six equilibrium points in Figure 3. The red vertical line to the right is the jump (*b* to *c*) in Figure 8 associated with a tumbling iceberg and the red vertical line to the left is the jump (*c* to *d*) in Figure 9 associated with a toppling structure. (**Bottom**) Stereoview of the equilibrium surface.

The curved portion of the equilibrium surface resembles three-fourths of a turn of a helical surface, which is, appropriately enough, also the shape of an Archimedes screw. However, the axis of the helical surface is distorted. It is about this distorted axis, near the vertical line $\{\phi = 74°, s = 1/2\}$, that one finds up to seven distinct values of the tilt angle for fixed values of ϕ and s and a variety of complicated equilibria transitions.

The equilibrium surface is colored with the AS points in blue, the US points in gray, and the NS points in black. The NS points lie on one continuous curve that separates the equilibrium surface into AS and US pieces. On the plane $\theta=0°$ the curve of NS points has the equation $s = \left(1 - 3\cot^2\phi\right)^2$, which Archimedes had previously identified as the limiting condition for a vertical stable equilibrium (cf., Proposition 8). By symmetry, a similar curve lies on the plane $\theta=180°$.

To determine the stability of each nonvertical equilibrium, I determined the algebraic sign of the slope of the corresponding righting curve at the corresponding root. When the fluid level does not cut the base, Archimedes's results are applicable. When the base is cut by the fluid level, the algebraic sign can be determined by computing the ratio of the two principal moments of inertia of the cross section of the intersection of the paraboloid with the fluid surface [12]. The cross section in this case is a right segment of an ellipse and I determined its principal moments of inertia using exact formulas.

For base angles of less than 60° the right paraboloid has the same floating characteristics as the spherical segment that Archimedes studied in Book I: namely, for any relative density it floats stably at the vertical tilt angles 0° and 180° and unstably at a tilt angle close to 90°. I shall refer to this as *plate-like* behavior, in contrast to the *rod-like* behavior when the base angle of the paraboloid is close to 90°. In the latter case the paraboloid floats unstably at 0° and 180° for most densities and floats stably at a tilt angle close to 90°, when it is lying on its side. Because plate-like and rod-like paraboloids float in totally different ways, the transition between the two shapes produces a complicated equilibrium surface with correspondingly complicated floating behaviors.

The NS points on the curved portion of the equilibrium surface lie along the edge of a helical fold that leads to catastrophic transitions between two equilibria as the base angle and/or the relative density of the floating paraboloid changes. These NS points identify saddle-node bifurcations where an US point and an AS point meet and annihilate each other, forming a fold catastrophe. If the parameters of a floating paraboloid change in such a way as to pass over a fold, the equilibrium configuration will jump catastrophically from the NS point on the fold to an AS point lying on the vertical line through the NS point. The NS points on the curved portion of the equilibrium surface and the corresponding fold catastrophes arise only when the base of the paraboloid is partially submerged and so did not enter into Archimedes's consideration.

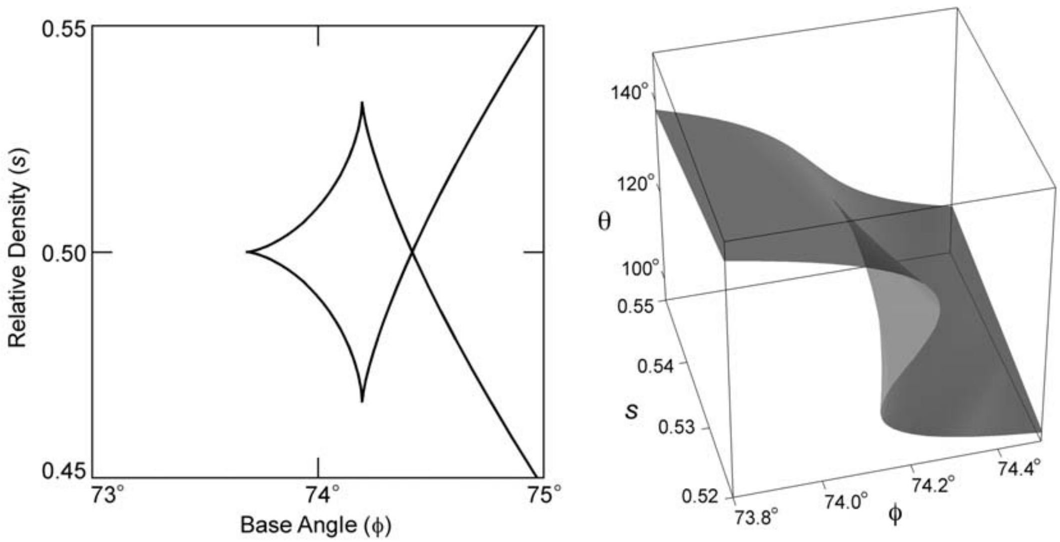


Figure 6 (Left) The projection of a portion of the curve of fold catastrophes onto the ϕ*s*-plane. Its three cusps identify three cusp catastrophes at tilt angles of 60.0°, 90°, and 120.0°. **(Right)** An oblique view of the topmost cusp catastrophe at θ=120.0°.

Cusp Catastrophes, Bifurcations, and Hysteresis Loops

Figure 6(left) is a projection of a portion of the curve of fold catastrophes onto the ϕ*s*-plane. Although this curve is smooth in three-dimensional space, its two-dimensional projection has three cusps. These cusps identify three cusp catastrophes at (ϕ, *s*, θ)-values of (74.19°, 0.467, 60.0°), (73.68°, 1/2, 90°), and (74.19°, 0.533, 120.0°). These are points where the equilibrium surface folds over and locally changes from a single-value function of θ to a triple-valued function. Figure 6(right) is an oblique view of the topmost cusp catastrophe illustrating this folding behavior. Within the diamond-shaped region in Figure 6(left) outlined on the left by the three cusps, the equilibrium surface has seven tilt-angle values, including two US values of 0° and 180°.

Figure 7 contains twelve slices of the equilibrium surface for fixed values of the tilt angle, base angle, and relative density. These slices exhibit the complicated geometric nature of the equilibrium surface, which leads to complicated changes in the equilibrium position of the paraboloid as its base angle or relative density changes. Figures 7A to 7D illustrate the fact that ϕ is a single-valued function of *s* for all θ between 0º and 180º other than 90º. Figures 7E to 7H exhibit pitchfork bifurcations at θ equal to 0º and 180º and show the bifurcations associated with the passing of the slice through the cusp catastrophe at a base angle of 73.682° $\left(=\tan^{-1}\sqrt{35/3}\right)$ between Figures 7E and 7F. In Figure 7G the slice passes through the two cusp catastrophes at tilt angles of 60.0º and 120.0º producing two more pitchfork bifurcations.

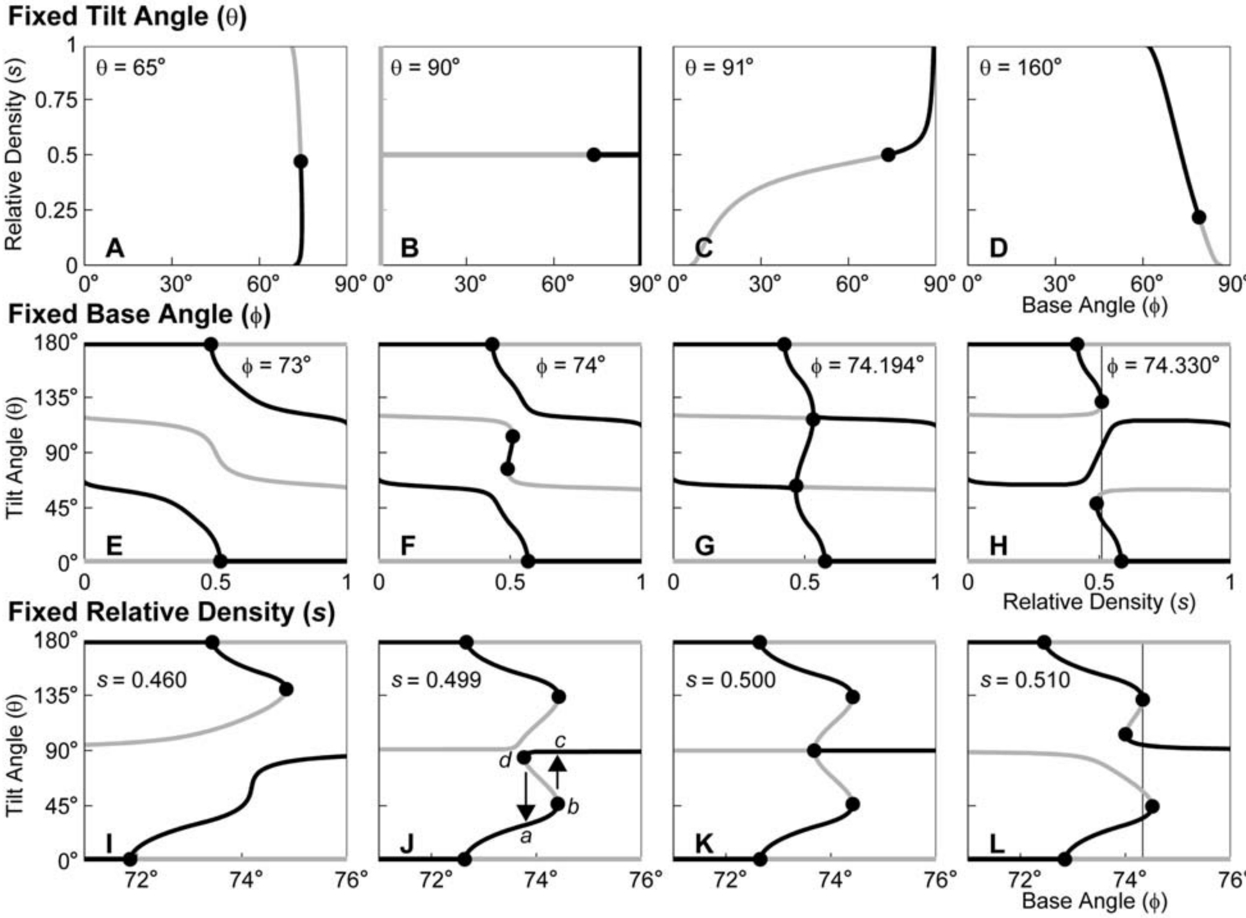


Figure 7. Slices of the equilibrium surface. The black curves are AS points, the gray curves are US points, and the black dots are NS points. A hysteresis loop (*a* through *d*) is shown in (J) and the common vertical line in (H) and (L) cuts through the six equilibrium tilt angles described in Figure 3.

Figures 7I to 7L show passages through the three cusp catastrophes using slices of constant relative density. Within the slice at $s = 1/2$ (Fig. 7K) the cross-section of the cusp at $\theta = 90^o$ appears as a subcritical pitchfork bifurcation [13].

Small hysteresis loops appear in the *s*-slices for *s* between 0.467 and 0.500 associated with the cusp at $\theta = 60.0^o$ and for *s* between 0.500 and 0.533 associated with the cusp $\theta = 120.0^o$. Figure 7J highlights the loop for $s = 0.499$. The paraboloid flips catastrophically about this loop between the two orientations (*a*) and (*c*) in a periodic manner as the base angle oscillates between the values 73.8º and 74.4º, a change of only 0.6º. As the base angle goes through one oscillation the tilt angle continuously increases from 29.0º to 43.3º (*a* to *b*), then jumps to 89.6º (*b* to *c*), then decreases continuously to 86.6º (*c* to *d*), and finally returns catastrophically to 29.0º (*d* to *a*).

The vertical lines in Figures 7H and 7L are at s = 0.510 and ϕ = 74.330º, respectively, and pass through the six tilt angles shown in Figure 3. A slight increase in either the relative density or the base angle from these values cause the structurally unstable NS point at θ =131.5º to be annihilated, while a slight decrease causes it to split into an AS-US pair.

Tumbling of icebergs due to melting

Icebergs are notoriously unstable and may tumble over for no apparent reason [2, 3, 20]. Jules Verne gave an explanation of this phenomenon in his 1870 novel *20,000 Leagues Under the Sea.* After a tumbling iceberg strikes the Nautilus, Captain Nemo explains, “An enormous block of ice; a mountain turned over. When icebergs are undermined by warmer waters or by repeated collisions, their center of gravity rises, with the result that they overturn completely” [21].

Figure 8 quantifies this phenomenon for a paraboloidal iceberg with uniform relative density of 0.9 melting in such a way that its base angle slowly increases (i.e., it gets narrower[9]). The cross-section of the equilibrium surface at this relative density shows that for base angles less that 82.54º the iceberg can float stably in a vertical orientation with its base above water (*a*). As its base angle slowly melts from 82.54º to 82.65º, its tilt angle slowly increases from 0º to 12.3º (*a* to *b*), and then suffers a catastrophic jump to 98.1º when the base angle increases past 82.65º (*b* to *c*).

The paraboloidal iceberg will tumble, rather than gradually roll over, only if its relative density is greater than 0.467 (Figures 8I and 8J). The tumbling then takes place almost immediately after the base cuts the fluid surface.

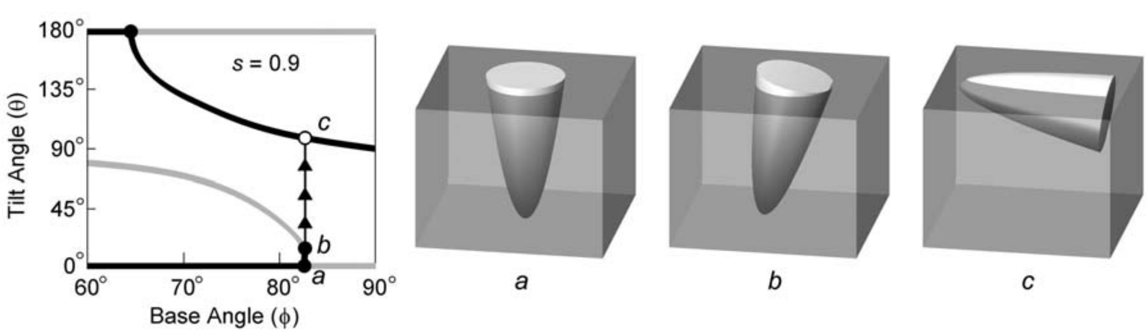


Figure 8. A paraboloidal iceberg of relative density 0.9 tumbles over as it melts and its base angle passes through 82.65º (*b* to *c*).

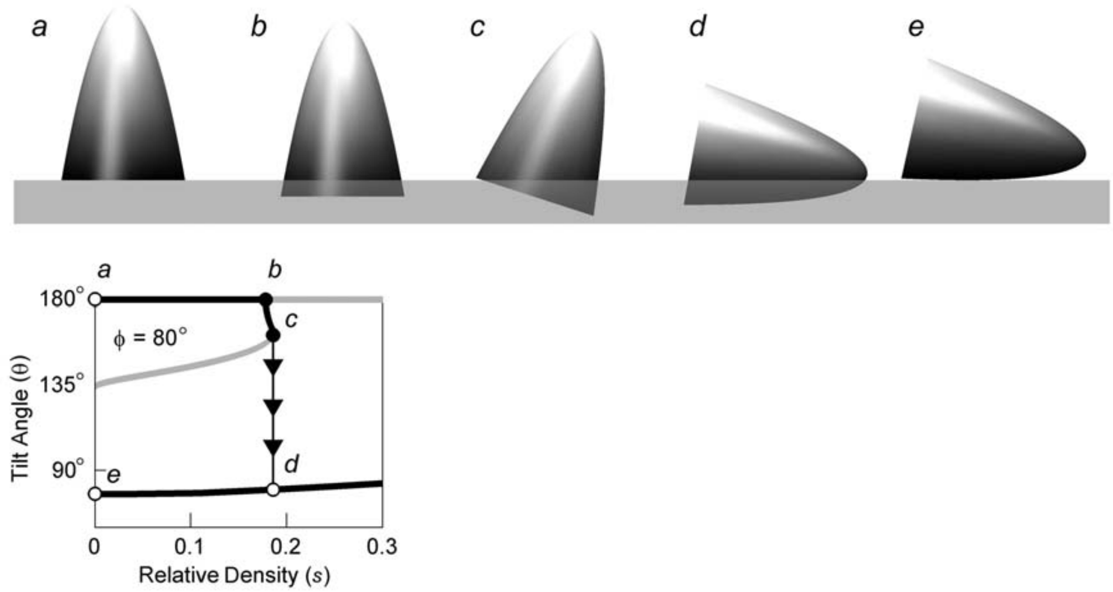


Figure 9. A paraboloidal structure with base angle 80º topples as the soil under it liquefies and the relative density of the structure passes through 0.186 (*c* to *d*).

Toppling of structures due to soil liquefaction

During an earthquake loose, water-saturated soil can behave like a viscous fluid, a phenomenon known as soil liquefaction. Structures originally supported by the soil begin to float on it when it liquefies and can then sink and topple as the density of the liquefied soil decreases.

Figure 9 illustrates this phenomenon for a paraboloidal structure with a base angle of 80º initially standing vertically on solid ground at a tilt angle of 180º (*a*). Let us consider solid ground as a liquid with infinite density, so that the relative density of the structure is zero. As the ground liquefies its density slowly decreases from infinity through large finite values and the relative density of the structure increases from zero through small finite values. The cross-section of the equilibrium surface at a base angle of 80º shows that as the relative density of the structure increases from 0 to 0.177 the structure slowly sinks into the ground in a vertical position (*a* to *b*), then starts to slowly tilt until it reaches a tilt angle of 162.6º at a relative density of 0.187 (*b* to *c*), at which point its base is barely above ground. If the relative density increases further, the structure topples catastrophically to a tilt angle of 79.9º (*c* to *d*). This toppling is irreversible. If the soil returns to its solid state, the structure, if still in one piece, ends up at a tilt angle of 77.2º (*d* to *e*).

As with the iceberg, the paraboloidal structure cannot topple until its base is partially exposed above the soil level. Additionally, this toppling can only occur if the base angle of the structure is greater that 74.194º (cf., Fig. 7G). For smaller base angles the paraboloidal structure gradually sinks and tilts into the soil without toppling as the soil's density decreases.

Conclusion

One need only glance at Archimedes's Proposition 8 above to see that *On Floating Bodies* is several orders of magnitude more sophisticated than anything else found in ancient mathematics. It ranks with Newton's *Principia Mathematica* as a work in which basic physical laws are both formulated and accompanied by superb applications.

However, Archimedes's investigation of floating paraboloids had to await the computer age for its continuation, just as did his famous Cattle Problem [24]. This latter problem has an integer solution with more than 200,000 digits that needed modern computers to determine. Likewise, I needed advanced computing and graphics systems to determine all possible equilibrium positions of Archimedes's floating paraboloids and to represent them in a single diagram.

No doubt Archimedes would have been interested in seeing the results in this paper, but one could ask how much of the mathematics developed in the last two millennia he would need to learn to understand them. At the very least he would have to learn about three-dimensional Cartesian coordinate systems, although he should have no trouble with this concept considering how close he came to defining a polar-coordinate system in his description of the spiral that bears his name. Unhooking him from the straitjacket of compass-and-straightedge construction to explain how the relationship among three variables can be represented by the points on a surface might take a little longer. He could then see how the equilibrium surface in Figure 5 presents a global picture of the behavior of his floating paraboloids and how the twists and turns of that surface lead to catastrophic[10] transitions. He could also then appreciate some of the advances made in mathematics in the last 23 centuries, although my guess is that he would have expected more considering the enormous advances that he alone made in his lifetime.

Acknowledgments

My sincere thanks to Professor C. Henry Edwards of the University of Georgia and Prof. Dr. Horst Nowacki of the Technical University of Berlin for their advice and support.

Web site

A web site maintained by the author containing QuickTime movies animating the figures is available at http://mail.vet.upenn.edu/~rorres/. The author also maintains an extensive website on Archimedes at
http://www.math.nyu.edu/~crorres/Archimedes/contents.html.

NOTES

1. A Greek manuscript dating from about the ninth century and containing both books of *On Floating Bodies* was translated into Latin by the Flemish Dominican William of Moerbeke in 1269, along with other works of Archimedes from other manuscripts. The tracks of the Greek manuscript were lost in the fourteenth century, but Moerbeke's holograph remains intact in the Vatican library (*Codex Ottobonianus Latinus* 1850) [5, 8]. Moerbeke's Latin translation was the source of all versions of *On Floating Bodies* from his time until the twentieth century. Moerbeke's translation of both books of *On Floating Bodies* was first printed in 1565, independently by Curtius Troianus in Venice and by Federigo Commandino in Bologna [4]. A palimpsest from the tenth century, discovered and edited by J. L. Heiberg in 1906, contains the only extant Greek text [16, 25]. The texts by Dijksterhuis [8] and Heath [14] are the only translations/paraphrases presently available in English.

2. Also called *parabolic conoids* or *orthoconoids*.

3. Some classic works concerned with how things float are: **Christiaan Huygens** (Dutch, 1629-1695), *De iis quae liquido supernatant;* **Pierre Bouguer** (French, 1698-1758), *Traité du Navire, de sa Construction, et de ses Mouvements*; **Leonhard Euler** (Swiss, 1707-1783), *Scientia navalis*; **Jean Le Rond d'Alembert** (French, 1717-1783), *Traité de l'équilibre et du Mouvement des Fluide*; **Fredrik Henrik af Chapman** (Swedish, 1721-1808), *Architectura Navalis Mercatoria*; **George Atwood** (English, 1745-1807); *The Construction and Analysis of Geometrical Propositions Determining the Positions Assumed by Homogeneal Bodies Which Float Freely, and at Rest, on the Fluid's Surface; also Determining the Stability of Ships and of Other Floating Bodies*; **Pierre Dupin** (French, 1784-1873), *Applications de géométrie et de mécanique*; **August Yulevich Davidov** (Russian, 1823-1885); *The Theory of Equilibrium of Bodies Immersed in a Liquid* [in Russian]. More recent works include [7, 9-12, 18, 19].

4. If ρ_{air} is the mass-density of the air, then, because the paraboloid is a homogeneous convex body, the buoyancy effect of the air can be accounted for by defining the relative density as $s = (\rho_{body} - \rho_{air})/(\rho_{fluid} - \rho_{air})$. Actually, Archimedes's description of *s* as the ratio of the weight of the body to the weight of an equal volume of fluid results in this expression if the weighing is done in air, but it is doubtful that he was aware of the buoyancy effects of air.

5. Archimedes's proof for the volume of a right or oblique paraboloid is contained in Propositions 21-22 of *On Conoids and Spheroids*. He gave a 'mechanical' proof of the location of the centroid of a right paraboloid in

Proposition 5 of *The Method*. He used the correct expression for the centroid of an oblique paraboloid in *On Floating Bodies II*, but no proof survives [8, 14].

6. Symmetry considerations show that if θ is an equilibrium tilt angle when the relative density of a floating body of revolution is *s*, then $180°–\theta$ is an equilibrium tilt angle for the body when its relative density is 1–*s*. Thus only tilt angles in the range [0°, 90°] need be explicitly computed. Although Archimedes does not mention this fact, it is clear that he was aware of it for his paraboloids since his proofs when the base is below the fluid surface are the same, *mutatis mutandis*, as his proofs when the base is above the fluid surface.

7. The integrals determining the volume and centroids of the unsubmerged portion can be found in closed form using symbolic algebra programs, but they are page-long monstrosities and numerical integration yields results much quicker and with more accuracy. Additionally, numerical techniques were used to determine when the weight of the displaced fluid is equal to the weight of the paraboloid and to find the roots of the righting arm curve. The symbolic calculations were performed with Maple™ and Mathematica™ and the numerical calculations and graphs were performed with MatLab™.

8. Points NS to first order may be AS or US when higher-order terms are considered. In particular, the NS points when $\theta=0°$ and 180º are actually AS and the rest are US. These NS points are also classified as *nonhyperbolic*, *degenerate*, and *structurally unstable* [1, 13].

9. Unlike Verne's iceberg, the center of gravity of the paraboloidal iceberg remains fixed relative to its size at a distance of one-third of its height along its axis from its base.

10. In Greek: CATASTROPHE = ΚΑΤΑΣΤΡΟΦΗ = a downward turn

REFERENCES

1. D. M. Arrowsmith and C. M. Place, *An Introduction to Dynamical Systems,* Cambridge University Press, Cambridge, 1990, p. 79.

2. R. C. Bailey, "Implications of iceberg dynamics for iceberg stability estimation," *Cold Regions Science and Technology* 22 (1994), 197-203.

3. D. W. Bass, "Stability of icebergs," *Annals of Glaciology* 1 (1980), 43-47.

4. M. Clagett, "The impact of Archimedes on medieval science," *Isis* 50 (1959), 419-429.

5. M. Clagett, *Archimedes in the Middle Ages,* Volume I, The Arabo-Latin Tradition, The University of Wisconsin Press, Madison, 1964, pp. 3-8.

6. M. Clagett, *Biographical Dictionary of Mathematicians, (*article on Archimedes) Charles Scribner's Sons, New York, 1991, Vol. 1, p. 95.

7. R. Delbourgo, “The floating plank,” *American Journal of Physics* 55 (1987), 799-802.

8. E. J. Dijksterhuis*, Archimedes*, (In Dutch: Chapters I-V, I. P. Noordhoff, Groningen, 1938: Chapters VI-XV, *Euclides*, VX-XVII, 1938-44). English translation by C. DIKSHOORN, Princeton University Press, Princeton, NJ, 1987.

9. B. R. Duffy, “A bifurcation problem in hydrostatics,” *American Journal of Physics* 61 (1993), 264-269.

10. P. Erdös, G. Schibler, and R. C. Herndon, “Floating equilibrium of symmetrical objects and the breaking of symmetry. Part 1: Prisms,” *American Journal of Physics* 60 (1992), 335-345.

11. P. Erdös, G. Schibler, and R. C. Herndon, “Floating equilibrium of symmetrical objects and the breaking of symmetry. Part 2: The cube, the octahedron, and the tetrahedron,” *American Journal of Physics* 60 (1992), 345-356.

12. E. N. Gilbert, “How things float,” *The American Mathematical Monthly* 98 (1991), 201-216.

13. J. Hale and H. Koçak, *Dynamics and Bifurcations,* Springer-Verlag, New York, 1991, p. 19.

14. T. L. Heath, T. L.*, The Works of Archimedes,* Dover Publications, New York, 2002. Unabridged republication of *The Works of Archimedes* (1897) and *The Method of Archimedes* (1912), both published by The Cambridge University Press.

15. T. L. Heath*, A History of Greek Mathematics,* Dover Publications, New York, 1981, Vol. II, p. 95*;* Unabridged republication from Clarendon Press, Oxford, 1921*.*

16. J. L. Heiberg, *Archimedis Opera Omnia*, with corrections by E.S. Stamatis, B. G. Teubner, Stuttgart, 1972.

17. C. W. Keyes, *Cicero: De Re Publica,* Book I, Section 22, Loeb Classical

Library, Harvard University Press, Cambridge, 1928.

18. H. Nowacki, *Archimedes and Ship Stability*, Max Planck Institute for the History of Science, Berlin, 2002.

19. H. Nowacki and L. D. Ferreiro, *Historical Roots of the Theory of Hydrostatic Stability of Ships*, Max Planck Institute for the History of Science, Berlin, 2003.

20. J. F. Nye and J. R. Potter, "The use of catastrophe theory to analyze the stability and toppling of icebergs," *Annals of Glaciology* 1 (1980), 49-54.

21. J. Verne, *20,000 Leagues Under the Sea*, (translated into English by M. T. Brunetti), Penguin Putnam Inc., New York, 1969, p. 368.

22. E. S. Stamatis, *The Complete Archimedes*, (In Greek: Ε. Σ. ΣΤΑΜΑΤΗΣ, *ΑΡΧΙΜΗΔΟΥΣ ΑΠΑΝΤΑ)* Volume B', Athens, 1973.

23. S. Stein, "Archimedes and his Floating Paraboloids," in *Mathematical Adventures for Students and Amateurs*, Mathematical Association of America, Washington, D.C., 2004.

24. H. C. Williams, R. A. German, and C. R. Zarnke, "Solution of the Cattle Problem of Archimedes," *Mathematics of Computation* 19 (1965), 671-674.

25. N. Wilson, *The Archimedes Palimpsest,* Christie's Auction Catalog, New York, 1998.

CHRIS RORRES

University of Pennsylvania
New Bolton Center
382 West Street Road
Kennett Square, PA 19348
USA

wwwsite: w3.vet.upenn.edu/faculty/rorres

e-mail: rorres@vet.upenn.edu

Chris Rorres did his doctorate at the Courant Institute with Joseph B. Keller on scattering theory, then for 33 years taught at Drexel University in Philadelphia. Now emeritus at Drexel, he is working at the University of Pennsylvania's School of Veterinary Medicine, modeling spread of diseases in food animals (such as avian flu in poultry). All the while, he has been fascinated with the life and work of Archimedes. These studies led to appearances on two BBC documentaries, and his Web site has received more than 250,000 hits just in the last 12 months. Here he is seen with the Archimedes-inspired finials he designed for his fence at home. The theorems they evoke—that on the area and volume of a sphere, and that on the volume of a right paraboloid—were among Archimedes's favorites.

This is a reformatted article that orginally appeared in

THE MATHEMATICAL INTELLIGENCER
Volume 26, Number 3, Summer 2004, Pages 32-42