YOUR NAME:

Midterm, October 18, 2011, with solutions
Linear Algebra I

Cross out what is not meant to be part of your final answer.

1. Let u_1, u_2 and u_3 be three linearly independent elements of a vector space V and let W be the subspace spanned by these three elements.

   Consider the three elements $u_1 + u_2 + u_3, u_2 + u_3$ and u_3. Do these three elements always span the same space W?

   Solution: Consider

   $$a_1(u_1+u_2+u_3, u_2+u_3)+a_2(u_2+u_3)+a_3u_3 = a_1u_1+(a_1+a_2)u_2+(a_1+a_2+a_3)u_3 = 0.$$

   Then, $a_1 = 0, a_1 + a_2 = 0$ **and** $a_1 + a_2 + a_3 = 0$ **and therefore** $a_1 = a_2 = a_3 = 0$.

2. Let $P(R)$ be the space of all polynomials with real coefficients. Consider the mapping $T : P(R) \to P(R)$ where

   $$T(p(x)) \to \int_0^x p(t)dt.$$

   Show that this is a linear transformation and that it is one-to-one but not onto.

   Solution: We need only to check that $T(cp_1 + p_2) = cT(p_1) + T(p_2)$; **this is easy. (See p. 65 of text book.)**

   If $T(p_1) = T(p_2)$, **then** $\int_0^x (p_1 - p_2)dt = 0, \forall x$ **and then** $p_1(t) = p_2(t)$. **Therefore,** T **is one-to-one.**

   T **is not onto since there is no** $p \in P(R)$ **such that** $T(p) = 1$.

3. Find a linear transformation $T : R^2 \to R^2$ such that range(T) = null(T). Here R^2 is the space of vectors with two real components.

   Also show that there does not exist any $T : R^3 \to R^3$, which satisfies range(T) =null(T).

   Solution: The simplest example is probably $T(x, y) = (0, x)$. **For** R^3 **no such transformation can exist since by Theorem 2.5 the nullity and the rank would then have to be** 1.5.

4. A skew-symmetric matrix A is a square matrix such that $a_{ij} = -a_{ji}, \ \forall i, j$. Let us assume that all the matrix elements are real numbers.

   (a) Show that these matrices form a vector space and determine the dimension of the space of all $n \times n$ skew-symmetric matrices.

   Solution: Check that $cA_1 + A_2$ is skew-symmetric for any scalar c and any pair of skew-symmetric matrices A_1 and A_2. The dimension of the space equals the number of elements above the diagonal, i.e., equals $n(n-1)/2$. A basis elements can be chosen to vanish except for position i, j and j, i where $i \neq j$. We can choose $a_{ij} = -a_{ji} = 1$.

   (b) Show that $x^T Ax = 0, \ \forall x$. Here A is skew-symmetric, x a column vector with n components, i.e., a matrix of order $n \times 1$, x^T its transpose and $x^T Ax$ the product of three matrices.

   Solution: The product equals

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j.$$

   Note that $a_{ii} = 0$ and pair up terms and use that $a_{ij} + a_{ji} = 0$.

   (c) Assume that $x^T Ax = 0, \ \forall x$. Can we then conclude that A is skew-symmetric?

   Solution: Note that this formula is assumed to hold for all x. Use the summation formula above and choose $x_i = x_j = 1$ and all other components of x equal to 0. Then $a_{ij} + a_{ji} = 0$, i.e., A is skew-symmetric.

5. (a) Consider the set of polynomials $\{L_k(x)\}_0^n$ where

$$L_k(x) = \prod_{i=0, i \neq k}^{n} \left(\frac{x - x_i}{x_k - x_i}\right),$$

   and $x_0, x_1, \ldots, x_n$, $n+1$ distinct real numbers.

   Show that this set spans the space of all polynomials of degree n or less.

   Solution: These are the polynomials that are featured in the Lagrange interpolation formula

$$p_n(x) = \sum_{k=0}^{n} f(x_k) L_k(x)$$

and which satisfies $p_n(x_k) = f(x_k)$. Now note that if we choose $f(x) = p(x)$, where $p(x)$ is an arbitrary polynomial of degree n or less, then the interpolation formula returns $p(x)$. Therefore, any polynomial can be written as a linear combination of the $L_k(x)$.

We can also argue that if $p_n(x_k) = 0$ for all k then it vanishes everywhere and the $L_k(x)$ must form a basis.

(b) Consider the two sets of polynomials $H_k(x)$ and $K_k(x)$, $0 \leq k \leq n$, where

$$H_k(x) = L_k(x)^2(1 - 2L_k(x_k)(x - x_k)), \quad K_k(x) = L_k(x)^2(x - x_k).$$

Show that these polynomials span the space of all polynomials of degree $2n + 1$ or less.

Solution: Most unfortunately, there is misprint. The factor $L_k(x_k)$ in the formula for $H_k(x)$ should be replaced by $L'_k(x_k)$, the derivative of L_k at the point in question. Then, we show that $H_k(x_i)$ vanishes for $i \neq k$, $H_k(x_k) = 1$, and $H'_k(x_i) = 0, \forall i$. Similarily, $K_k(x_i) = 0, \forall i, K'_k(x_i) = 0, i \neq k$, and $K'_k(x_k) = 1$. We the aid if these functions we can match values of $f(x_k)$ and $f'(x_k)$ and the $n+1$ points; this gives the solution of the Hermite interpolation problem. We can then argue very much in the same way as in the first part of this question.

(Hint: It might be helpful to start out by considering the special case of $n = 1$.)

6. Let $Ax = b$ be a linear system of algebraic equations with the same number of equations as unknowns.

(a) Briefly describe Gaussian elimination with partial pivoting.

Solution: See the handout. Note that partial pivoting involves always choosing the pivot element as the one which is largest in absolute value of all the candidates.

(b) Suppose that A is not invertible and that we can compute using exact arithmetic. How does the fact that A is not invertible manifest itself when using Gaussian elimination with partial pivoting?

Solution: We will either get $u_{nn} = 0$, where U is the upper triangular matrix, or we will encounter a situation

where all potential pivots, in the relevant column of the transformed matrix, on and below the diagonal vanish.

(c) Show that we can solve an upper triangular system of linear equations with n unknowns in about n^2 arithmetic operations.

Solution: It is easy to see that we will use every element in the upper triangular matrix exactly once. Therefore, the number of operations is proportional to n^2.

(d) How can we tell if an upper triangular matrix is invertible or not?

Solution: U **is invertible if and only if all diagonal elements differ from** 0**; try to solve the system and it becomes obvious.**