Calculus II: Solutions for "Final Exam: December 19, 2008"

1. Integration by parts. Let $u = e^x$ and $dv = \cos x\ dx$:

$$\int e^x \cos x\ dx = e^x \sin x - \int e^x \sin x\ dx. \tag{0.1}$$

Next, let $u = e^x$ and $dv = \sin x\ dx$:

$$\int e^x \sin x\ dx = -e^x \cos x + \int e^x \cos x\ dx. \tag{0.2}$$

Combining (**??**) and (**??**):

$$\int e^x \cos x\ dx = \tfrac{1}{2} e^x (\sin x + \cos x) + C.$$

2. Trigonometric substitution. Let $x = 2\sin\theta$:

$$\begin{aligned}
\int \frac{x^2}{(4-x^2)^{\frac{3}{2}}}\ dx &= \int \frac{4\sin^2\theta}{(4\cos^2\theta)^{\frac{3}{2}}}(2\cos\theta)\ d\theta \\
&= \int \frac{\sin^2\theta}{\cos^2\theta}\ d\theta \\
&= \int \frac{1}{\cos^2\theta}\ d\theta - \int \frac{\cos^2\theta}{\cos^2\theta}\ d\theta \\
&= \tan\theta - \theta + C \\
&= \frac{x}{\sqrt{4-x^2}} - \arcsin\left(\frac{x}{2}\right) + C.
\end{aligned}$$

3. Integration by completing the square.

$$\int \frac{2x}{x^2+2x+5}\ dx = \int \frac{2x}{(x+1)^2+4}\ dx.$$

Let $u = x + 1$:

$$\begin{aligned}
\int \frac{2x}{(x+1)^2+4}\ dx &= \int \frac{2u-2}{u^2+4}\ du \\
&= \int \frac{2u}{u^2+4}\ du - \int \frac{2}{u^2+4}\ du \\
&= \ln(u^2+4) - \arctan\left(\frac{u}{2}\right) + C \\
&= \ln(x^2+2x+5) - \arctan\left(\frac{x+1}{2}\right) + C.
\end{aligned}$$

4. Solid of revolution.

$$\begin{aligned}\int_0^2 \pi(2y)^2 - \pi(y^2)^2\ dy &= \int_0^2 4\pi y^2 - \pi y^4\ dy \\ &= \pi\left[\frac{4}{3}y^3 - \frac{1}{5}y^5\right]\Bigg|_0^2 \\ &= \frac{64}{15}\pi.\end{aligned}$$

5. Differential equation.

$$\begin{aligned}\int y^2\ dy &= \int \cos x\ dx \\ \frac{1}{3}y^3 &= \sin x + C \\ y &= (3\sin x + C)^{\frac{1}{3}}.\end{aligned}$$

Then,

$$1 = y(0) = (3\sin 0 + C)^{\frac{1}{3}} = C^{\frac{1}{3}}$$

which means

$$y = (3\sin x + 1)^{\frac{1}{3}}.$$

6. Series: Tests for convergence.

(a) Comparison Test.

$$\frac{n^2+1}{n^4+2} = \frac{n^2}{n^4+2} + \frac{1}{n^4+2} < \frac{n^2}{n^4} + \frac{1}{n^4}.$$

Both series indicated by the right-hand-side are p-series with $p > 1$, so each series is convergent and the original series is convergent.

(b) Ratio Test.

$$\left|\frac{2^{n+1}}{(n+1)!}\cdot\frac{n!}{2^n}\right| = \left|\frac{2}{n+1}\right| \to 0 \quad \text{as } n \to \infty.$$

The limit is less than 1, so the series is absolutely convergent (and therefore convergent).

7. Power series expansion.

$$\frac{x}{4+x^2} = \frac{\frac{x}{4}}{1-\left(\frac{-x^2}{4}\right)} = \frac{x}{4}\sum_{n=0}^{\infty}\left(\frac{-x^2}{4}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{4^{n+1}}x^{2n+1}.$$

The series is geometric, so it converges when $|\frac{-x^2}{4}| < 1 \Leftrightarrow |x| < 4$. The radius of convergence is 4 and the interval of convergence is $(-4, 4)$.

8. Maclaurin series expansion.

$$\begin{aligned}\int \frac{\sin x}{x}\ dx &= \int \frac{1}{x}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right]\ dx \\ &= \int \left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots\right]\ dx \\ &= C + x - \frac{x^3}{3(3!)} + \frac{x^5}{5(5!)} - \frac{x^7}{7(7!)} + \cdots \\ &= C + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+1)!}\ x^{2n+1}.\end{aligned}$$

9. Taylor series expansion.

$$T_6(x) = \frac{1}{1!}\ x + \frac{3}{3!}\ x^3 + \frac{25}{5!}\ x^5$$

10. Polar coodinates.

(a) Conversion to Cartesian coordinates.

$$\begin{aligned} x = &\ r\cos\theta = (1+2\cos\theta)\cos\theta = \left(1 + 2\left(-\frac{\sqrt{2}}{2}\right)\right)\left(-\frac{\sqrt{2}}{2}\right) \\ y = &\ r\sin\theta = (1+2\cos\theta)\sin\theta = \left(1 + 2\left(-\frac{\sqrt{2}}{2}\right)\right)\left(\frac{\sqrt{2}}{2}\right)\end{aligned}$$

(b) Tangents to polar curves.

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{(-2\sin\theta)\sin\theta + (1+2\cos\theta)\cos\theta}{(-2\sin\theta)\cos\theta - (1+2\cos\theta)\sin\theta} = -\frac{\sqrt{2}}{4-\sqrt{2}}.$$

11. Parametric curves.

$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1-3t^2}{2t}; \\ \frac{d^2y}{dx^2} &= \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{-1-3t^2}{4t^3}.\end{aligned}$$