Solutions to Final Examination

V63.0121
Calculus I

December 19, 2008

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1. (10 Points) *Compute the following derivatives.* Please leave your answers unsimplified.

(i) $\dfrac{d}{dx}\left(\ln\left(x\sin x+1\right)\right)$

Solution. We need the chain rule and the product rule:

$$\frac{d}{dx}\left(\ln\left(x\sin x+1\right)\right)=\frac{1}{x\sin x+1}\cdot\frac{d}{dx}\left(x\sin x+1\right)=\frac{x\cos(x)+\sin(x)}{x\sin(x)+1}$$

▲

(ii) $\dfrac{d}{dx}\left(\arctan\left(\dfrac{x+1}{e^x+1}\right)\right)$

Solution. Again by the chain rule,

$$\frac{d}{dx}\left(\arctan\left(\frac{x+1}{e^x+1}\right)\right)=\frac{1}{1+\left(\frac{x+1}{e^x+1}\right)^2}\cdot\frac{d}{dx}\left(\frac{x+1}{e^x+1}\right)=\frac{1}{1+\left(\frac{x+1}{e^x+1}\right)^2}\left\{\frac{(1+e^x)-e^x(x+1)}{(1+e^x)^2}\right\}$$

▲

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2. (10 Points) *Find the y-intercept of the line that is tangent to the ellipse* $4x^2 + 9y^2 = 900$ *at the point* $(12, 6)$. Put your answer in the box.

Hint. Implicit differentiation may help here.

Solution. Implicitly differentiating gives us

$$8x + 18y\frac{dy}{dx} = 0 \implies \frac{dy}{dx} = -\frac{4x}{9y}$$

At the point $(12, 6)$. we have

$$\left.\frac{dy}{dx}\right|_{(12,6)} = -\frac{4 \cdot 12}{9 \cdot 6} = -\frac{8}{9}$$

The equation of the line through $(6, 12)$ with slope $-\frac{8}{9}$ is

$$y - 6 = -\tfrac{8}{9}(x - 12) \implies y = -\tfrac{8}{9}x + \tfrac{8}{9} \cdot 12 + 6$$

The y-intercept is

$$\frac{8}{9} \cdot 12 + 6 = \frac{96}{9} + \frac{54}{9} = \frac{150}{9} = \frac{50}{3} = 16\tfrac{2}{3}$$

▲

y-intercept: $\boxed{16\tfrac{2}{3}}$

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3. (10 Points) *Evaluate the following limits.* Put your answers in the boxes. Show your work.

(i) *(5 points)* $\lim_{x\to 1} \frac{x^4-1}{x^2-1}$.

$\boxed{2}$

Solution. We can factor the numerator:

$$\lim_{x\to 1} \frac{x^4-1}{x^2-1} = \lim_{x\to 1} \frac{(x^2-1)(x^2+1)}{x^2-1} = \lim_{x\to 1}(x^2+1) = 1^2+1 = 2$$

Alternatively, we could use L'Hôpital's Rule:

$$\lim_{x\to 1} \frac{x^4-1}{x^2-1} \overset{\text{H}}{=} \lim_{x\to 1} \frac{4x^3}{2x} = \lim_{x\to 1} 2x^2 = 2$$

▲

(ii) *(5 points)* $\lim_{x\to 0} (\cos x)^{1/x^2}$

$\boxed{e^{-1/2}}$

Solution. Let the limit be L, if it exists. Then

$$\ln L = \lim_{x\to 0} \ln(\cos x)^{1/x^2} = \lim_{x\to 0} \frac{\ln(\cos x)}{x^2}$$

This limit is of the form $\frac{0}{0}$ and so we can try L'Hôpital's Rule:

$$\lim_{x\to 0} \frac{\ln(\cos x)}{x^2} \overset{\text{H}}{=} \lim_{x\to 0} \frac{\frac{1}{\cos x}\cdot(-\sin x)}{2x} = \lim_{x\to 0} \frac{-\tan x}{2x} \overset{\text{H}}{=} \lim_{x\to 0} \frac{-\sec^2 x}{2} = -\frac{1}{2}.$$

So $L = e^{-1/2}$.

▲

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4. (15 Points) *Let* $f(x) = x^4 + 4x^3 - 2$. Explain your answers on each of these parts.

(i) *(4 points) The derivative of* f *is* $f'(x) = 4x^3 + 12x^2$. *On which intervals is* f *increasing? decreasing?*

Solution. We have $f'(x) = 4x^3 + 12x^2 = 4x^2(x+3)$. This is zero when $x = 0$ or $x = -3$. On the interval $(-\infty, -3)$, $f'(x) < 0$, so f is decreasing. On the intervals $(-3, 0)$ and $0, \infty$, $f'(x) > 0$. So f is deccreasing on $(-\infty, -3]$ and increasing on $[-3, \infty)$. ▲

(ii) *(4 points) The second derivative of* f *is* $f''(x) = 12x^2 + 24x$. *On which intervals is* f *concave up? concave down?*

Solution. We have $f''(x) = 12x^2 + 24x = 12x(x+2)$. This is zero when $x = 0$ or $x = -2$. On the interval $(-\infty, -2)$, $f''(x) > 0$, so f is concave up. On the interval $(-2, 0)$, $f''(x) < 0$, so f is concave down. On the interval $(0, \infty)$, $f''(x) > 0$, so f is concave up. Thus f is concave up on $(-\infty, -2]$ and $[0, \infty)$, and concave down on $[-2, 0]$. ▲

(iii) (4 points) Find the absolute maximum and minimum values of f on $[-4, 1]$.

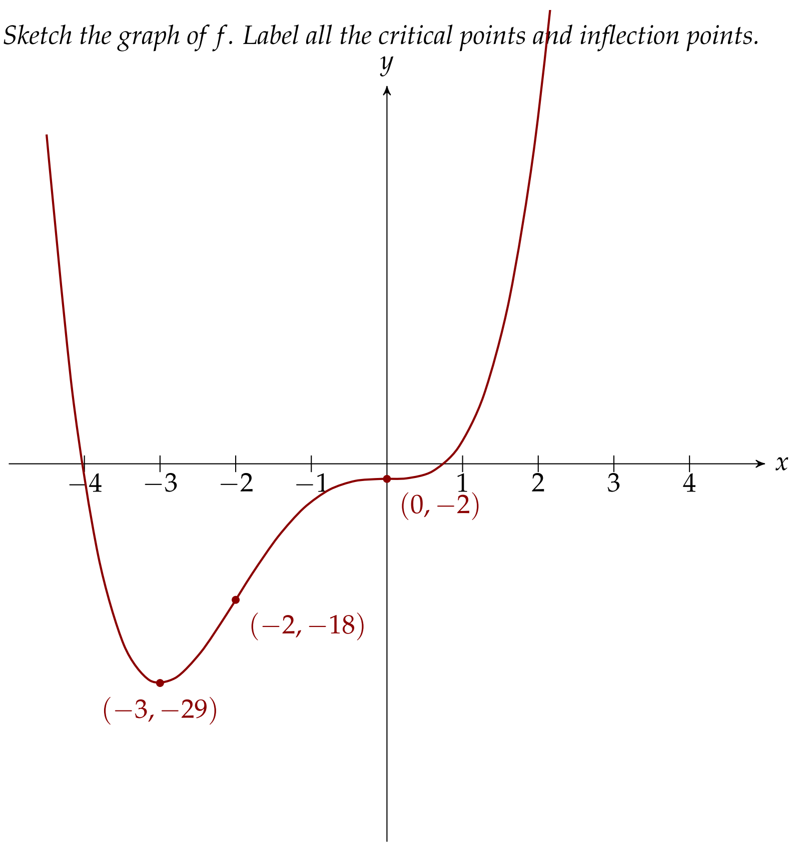
Solution. By the closed interval method, we need only check the values of f at the endpoints of $[-4, 1]$ (that is, at -4 and 1), and the critical points within $(-4, 1)$ (that is, at -3 and 0).

We have

$$
\begin{aligned}
f(-4) &= (-4)^4 + 4(-4)^3 - 2 = -2 & \qquad f(0) &= -2 \\
f(-3) &= (-3)^4 + 4(-3)^3 - 2 = -29 & \qquad f(1) &= 3
\end{aligned}
$$

So the maximum value is $f(1) = 3$, while the minimum value is $f(-3) = -29$. ▲

(iv) (3 points) Sketch the graph of f. Label all the critical points and inflection points.



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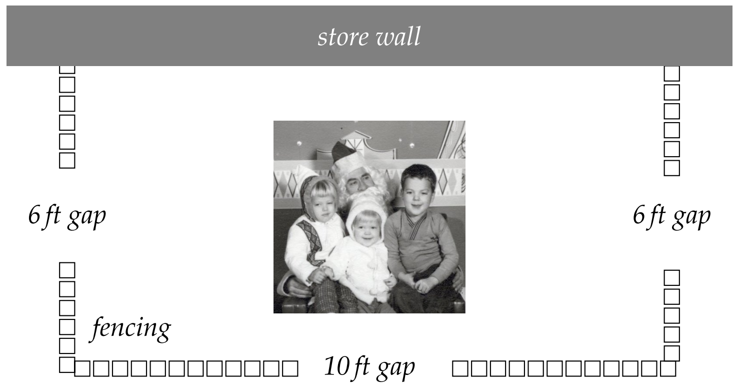
5. (12 Points) *A department store is fencing off part of the store for children to meet and be photographed with Santa Claus. They have decided to fence off a rectangular region of fixed area* $800\,\text{ft}^2$*. Fire regulations require that there be three gaps in the fencing: 6 ft openings on the two facing sides and a 10 ft opening on the remaining wall (the fourth side of the rectangle will be against the building wall). Find the dimensions that will minimize the length of fencing used.*



Dimensions: $\boxed{40}$ ft × $\boxed{20}$ ft

Solution. Let x be the side with the 10 ft gap and y the side with the 6 ft gap. We want to minimize

$$f = (x - 10) + 2(y - 6) = x + 2y - 2$$

subject to the constraint that $xy = 800$. Isolating $y = \dfrac{800}{x}$ gives us a function

$$f(x) = x + 2 \cdot \frac{800}{x} - 2 = x + \frac{1600}{x} - 2$$

The domain of this function is $[10, \infty)$, because we need at least 10 ft to have a gap. To find the critical points, we have

$$f'(x) = 1 - \frac{1600}{x^2}$$

So $f'(x) = 0$ when $x = 40$ (we discard the negative root since it's not in our domain). Now $f''(x) = \dfrac{3200}{x^3}$, which is always positive on our domain. So the unique critical point is the global minimum of f. Thus the dimensions of the rectangle that minimize fence are 40 ft × 20 ft. ▲

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6. (12 Points) *A cannonball is shot into the air. Its velocity is given as a function $f(t)$ m/s, where t measured in seconds since 1:00PM. We know that $f(t)$ takes the following values:*

t	0	7.5	15	22.5	30	37.5	45	52.5	60
$f(t)$	10.0	6.46	5.00	3.88	2.93	2.09	1.34	0.646	0

(i) *(2 points) What does the integral $I = \int_0^{60} f(t)\,dt$ represent?*

Solution. The integral represents the distance traveled between 1:00PM and 1:01PM. ▲

*For the two parts below, let L_n be the Riemann sum for I using n subintervals and **left** endpoints, R_n be the Riemann sum for I using n subintervals and **right** endpoints, and M_n be the Riemann sum for I using n subintervals and **midpoints**.*

(ii) *(2 points) Write out the terms in M_4.*You may leave your answer unsimplified.

Solution. We have

$$M_4 = 6.46 \cdot 15 + 3.88 \cdot 15 + 2.09 \cdot 15 + 0.646 \cdot 15$$

▲

(iii) *(2 points) Assume that $f(t)$ is* decreasing *for all $t \geq 0$. Without computing, put these in order from least to greatest: L_2, R_4, R_2, I, L_4.* Put your answers in the boxes. No justification is necessary.

Hint. A picture might help your thinking here.

$$\boxed{R_2} < \boxed{R_4} < \boxed{I} < \boxed{L_4} < \boxed{L_2}$$

(iv) *(6 points). It turns out $f(t) = 10 - \frac{10}{\sqrt{60}}\sqrt{t}$. Compute I exactly.* $\boxed{200}$

Solution. We have

$$I = \left[10t - \frac{10}{\sqrt{60}} \cdot \frac{2}{3} t^{3/2}\right]_0^{60} = 10 \cdot 60 - \frac{20}{3\sqrt{60}} \cdot 60\sqrt{60} = 600 - 400 = 200$$

▲

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7. (10 Points) *Find the following indefinite integrals.* Your answer should be the most general antiderivative.

(i) *(5 points)* $\displaystyle\int \frac{12x}{\sqrt{2x^2+5}}\,dx$

Solution. Let $u = 2x^2 + 5$, so $du = 4x\,dx$. Then

$$\int \frac{12x}{\sqrt{2x^2+5}}\,dx = \int \frac{3\,du}{\sqrt{u}} = 6\sqrt{u} + C = 6\sqrt{2x^2+5} + C$$

▲

(ii) *(5 points)* $\displaystyle\int 2x\sin(3x)\,dx$

Solution. Let $u = 2x$, so $dv = \sin(3x)\,dx$. Then $du = 2\,dx$ and $v = -\frac{1}{3}\cos(3x)$. So

$$\int 2x\sin(3x)\,dx = -\tfrac{2}{3}x\cos(3x) + \tfrac{2}{3}\int \cos(3x)\,dx = -\tfrac{2}{3}x\cos(3x) + \tfrac{2}{9}\sin(3x) + C$$

▲

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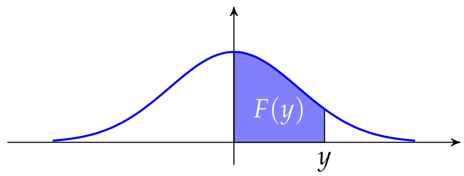
8. (10 Points) *Let*

$$F(y) = \int_0^y e^{-s^2}\,ds$$



(i) *(5 points) In each of these, select "$>$" if the quantity in column A is greater, "$<$" if the quantity in column B is greater, "$=$" if the two quantities are the same, and "?" if it is impossible to determine which is greater.* No justification is necessary. No partial credit will be given. Please fill in the circle completely.

Hint. It is **mathematically impossible** to compute $F(y)$ exactly by antidifferentiation, so please do not try. That is not the point of this problem.

	A	B	Your answer
1)	$F(0)$	0	○> ○< ●= ○?
2)	$F(1)$	$F(2)$	○> ●< ○= ○?
3)	$F(-1)$	$F(-2)$	●> ○< ○= ○?
4)	$F'(0)$	0	●> ○< ○= ○?
5)	$F'(1)$	$F'(-1)$	○> ○< ●= ○?

(ii) *(5 points) Suppose $y(t) = 9\sin(\pi t)$ and let $g(t) = F(y(t))$. In other words,*

$$g(t) = \int_0^{9\sin\pi t} e^{-s^2}\,ds$$

Find $g'\left(\frac{1}{2}\right)$. Put your answer in the box. $\boxed{0}$

Solution. We have $g(t) = F(9\sin\pi t)$, so

$$g'(t) = F'(9\sin\pi t)\cdot 9\pi\cos(\pi t) = 9\pi e^{-81\sin^2\pi t}\cos(\pi t).$$

Therefore

$$g'\left(\tfrac{1}{2}\right) = 9\pi e^{-81\sin^2(\pi/2)}\cos\left(\frac{\pi}{2}\right) = 0$$

▲

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9. (5 Points) *Evaluate the following.* No justification is necessary for this problem. In the first three, express your answer as an integer or a fraction.

(i) $\log_3(27)$

$\boxed{3}$

(ii) $\log_4\left(\frac{1}{2}\right)$

$\boxed{-\frac{1}{2}}$

(iii) $\ln(1)$

$\boxed{0}$

In the next two, express your answer as an angle in radians.

Note. Remember that arcsin is the inverse of sin, sometimes also written as $\sin^{-1}$. But this is **not** the same as $\dfrac{1}{\sin}$.

(iv) $\arcsin\left(\frac{1}{2}\right)$

$\boxed{\pi/6}$

(v) $\arctan(1)$

$\boxed{\pi/4}$

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10. (6 Points) *Determine whether the following statements are **true** (i.e. true in general) or **false** (i.e. not true in all cases). As long as there is one example where the statement does not hold, it is considered false.* Please fill in the circle completely. No justification is necessary. No partial credit will be given.

(i) *If f and g are continuous on $[a,b]$, then* $\int_a^b f(x)g(x)\,dx = \left(\int_a^b f(x)\,dx\right)\left(\int_a^b g(x)\,dx\right)$

T F

Solution. We use integration by parts to integrate a product. ▲

(ii) *If f is differentiable at a, then f is continuous at a.*

T F

Solution. Differentiability is a kind of super-continuity. ▲

(iii) *If $-1 < x < 1$, then* $\arctan(x) = \dfrac{\arcsin(x)}{\arccos(x)}$

T F

Solution. Inverse functions do not work this way. ▲

(iv) *If* $\lim_{x\to 5} f(x) = 0$ *and* $\lim_{x\to 5} g(x) = 0$, *then* $\lim_{x\to 5} [f(x)/g(x)]$ *does not exist.*

T F

Solution. It is possible for the limit to still exist. For instance $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$. ▲

(v) $$\lim_{x\to 1} \frac{x-3}{x^2+2x-4} = \frac{\lim_{x\to 1}(x-3)}{\lim_{x\to 1}(x^2+2x-4)}$$

T F

Solution. Since the denominator approaches something other than 0, the limit of the quotient is the quotient of the limits. ▲

(vi) *If f is continuous on $[a,b]$ and differentiable on (a,b), then there is a point c in (a,b) with* $f'(c) = \dfrac{f(b)-f(a)}{b-a}$.

T F

Solution. This is the statement of the Mean Value Theorem. ▲

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