SUBADDITIVE ERGODIC THEOREMS

CHEN

1. Introduction

Lemma 1.1. Let (a_n) be a subadditive sequence of non-negative terms a_n. Then $(\frac{a_n}{n})$ is bounded below and converges to $inf[\frac{a_n}{n} : n \in \mathbb{N}]$

Above is the famous Fekete's lemma which demonstrates that the ratio of subadditive sequence (a_n) to n tends to a limit as n approaches infinity. This lemma is quite crucial in the field of subadditive ergodic theorems because it gives mathematicians some general ideas and guidelines in the non-random setting and leads to analogous discovery in the random setting. Kingman's Subadditive Ergodic Theorem, for instance, is a perfect analogy in the random setting. This paper will introduce Kingman's theorem and its variation in details as well as describing the concepts within the subadditive ergodic setting. In the latter section of the paper, the paper attempts to draw a parallel between the shape theorems and Kingman's theorem so that random multidimensional distance functions satisfy similar properties like those of random single-dimensional subadditive sequences.

2. Subadditivity and Ergodicity

Before we attempt to understand the theorems and prove their veracity, we need to clarify the definition of a couple of key concepts:

2.1. Subadditivity.

- A sequence (a_n), $n \geq 1$ is called subadditive if for all m and n, there is an inequality such that $a_{n+m} \leq a_n + a_m$
- A subadditive function is a function $f : A \to B$ having a domain A and an Ordered codomain B that are both closed under addition, and satisfies $\forall x, y \in A, f(x + y) \leq f(x) + f(y)$. This property resembles the triangle inequality.

2.2. Ergodicity.

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- We use the term "ergodic" to refer to a dynamical system which has the same behaviour averaged over the probabilistic averaging and over the transformation in homogeneous space of all states of the system.

3. An in-depth look at Fekete's lemma

Before proving the Fekete's lemma, we can alternate it slightly to get a different version:

Lemma 3.1 (Fekete's lemma)**.** let f be a subadditive function on the set of the positive integers. Then the limit, $\lim_{n\to\infty} \frac{f(n)}{n}$ exists and is equal to inf $\frac{f(n)}{n}$ for $n \geq 1$.

Proof. (Ramdinmawia, 2011)
Let l:= inf $\frac{f(n)}{n}$ for $n \geq 1$;
let $\epsilon > 0$. $\exists K \in \mathbb{N}$ such that
$|\frac{f(K)}{K} - l| < \frac{\epsilon}{2}$
Let L be large enough to guarantee that $\frac{f(r)}{KL} < \frac{\epsilon}{2}$ for $r < K$. Let $n \geq KL$. Then there exist nonnegative integers q, r such that n = Kq + r, where $r < K$. We also know that $q \geq L$. Also
$\frac{f(n)}{n} \leq \frac{f(Kq)}{Kq+r} + \frac{f(r)}{Kq+r} \leq \frac{qf(K)}{Kq} + \frac{f(Kq)}{Kq} = \frac{f(K)}{K} + \frac{f(Kq)}{Kq} < l + \frac{\epsilon}{2} + \frac{\epsilon}{2} = l + \epsilon$,
Above shows that $|\frac{f(n)}{n} - l| < \epsilon$ for $n \geq KL$; this shows $l = \lim_{n\to\infty} \frac{f(n)}{n}$.

□

4. Kingman's Subadditive Ergodic Theorem

4.1. **Historical Perspective**

A significant number of people tend to overlook the history of mathematics when examining fascinating mathematical discoveries. Mathematical history, in reality, is as crucial as other areas of math because for example, by learning the background and personal experiences of certain mathematicians, we can perhaps grasp what really inspired them to come up with such magnificent discoveries. In that case, I would like to provide a little bit of background information about John Kingman, who was responsible for the Subadditive Ergodic theorems:

John Kingman was born on Aug 28, 1939 in Beckenham, Kent, England. He was educated at Christ's College and earned an M.A. at Pembroke College Cambridge in 1956. Even though Kingman has collaborated with Peter Whittle and David Kendall, he never obtained a Ph.D degree(www.history.mcs.st-and.ac.uk).

In 1967, Kingman became a member of the International Statistical Institute and he commenced the study of the Subadditive Ergodic theorem. The idea of such theorem was first brought forth to him by JM Hammersley and DJA Welsh. Kingman eventually proved the theorem in his papers "The Ergodic Theory of Subadditive Stochastic Process" in 1968 and "An Ergodic Theorem" in the "Bulletin of the London Mathematical Society" in 1969. To add upon his contributions, Kingman provided a beautiful description of the development in this subject matter in "Subadditive Ergodic Theory," published in 1973.

4.2. **Building Towards the Theorem**

Kingman made three essential assumptions so that his theorem could stand and they were subadditivity, ergodicity and stationarity. Both subadditivity and ergodicity have been briefly explained earlier in this paper and they are the more apparent conditions because they both appear in the name of the theorem. Stationarity, however, seems more covert and indirect. Nontheless, it is still a powerful underlying concept for the theorem to hold true.

A question then emerges: what is Stationarity? Based on the definition given by `http://iridl.ldeo.columbia.edu/dochelp/StatTutorial/Stationarity/`, "a random variable or random process is stationary if all of its statistical parameters are independent of time." To elaborate, any time series is stationary if the distribution of variables $X_1, X_2..., X_n$ is the same as the distribution of the variables shifted by some time lag k,(resulting in new distribution of $X_{1+k}, X_{2+k}, ..., X_{n+k}$. In that case, the distribution of the variables does not depend on time t.

A major characteristic that differentiates Kingman's theorem from Fekete's lemma on subadditivity is the concept of randomness. It is evident that the word "ergodic" appears in the name of Kingman's theorem. When attempting to define the concept of ergodicity, we normally place ourselves in a probability space $(\Omega, \mathcal{F}, \mathbb{P}, T)$. Ω, $\mathcal{F}$, $\mathbb{P}$ and T represent the sample space, the set of events, the assignment of probability to the events and a measurable map. As we learned previously from statistics and probability classes, randomness is a central concept within the Theory of Probability. In order to better understand the randomness aspect of Kingman's theorem, it is necessary to introduce Kolmogorov's zero-one law concerning random variables.

4.3. **Kolmogorov's Zero-One Law**

This 20th century Soviet/Russian mathematician, Andrey Nikolaevich Kolmogorov made significant contributions to probability theory, topology, intuitionistic logic, turbulence, classical mechanics and computational complexity. Among his many achievements, the zero-one Law is of great importance in the study of randomness.

There are a few versions of the zero-one law. However, the law essentially argues that **there is a certain type of event, called a tail event which will only have the probability of 1 or 0.** In other words, such tail event will either always occur or never occur.

What is a tail event then? Suppose $X_1, X_2, X_3, ...$ is a infinite sequence of random variables. A tail event is an event whose occurrence or failure is determined by the values of these random variables but also probabilistically independent of each finite subset of these random variables.

Now we can finally move on to the main course and examine more thoroughly about the theorem itself.

4.4. **Two Versions of Kingman's Theorem.**

Theorem 4.1 (Version 1)**.** Let$(\Omega, \mathcal{F}, \mathbb{P}, T)$ be a probability space with a measurable map $T : \Omega \to \Omega$ which preserves P. The set Ω is defined such that there is a two parameter family of random variables,

$(X(m,n), 0 \leq m < n < \infty)$. Each $X(m,n)$ is integrable with respect to $\mathbb{P}$ for $0 \leq m < n < \infty$. If $X(m,n)$ satisfies:

- $X(m+1, n+1) = X(m,n) \circ T$ and
- $X(0,n) \leq X(0,m) + X(m,n)$

then there exists an a.s.(almost surely, with probability 1)

$\lim_{n\to\infty} \frac{X(0,n)}{n} = Y$

where $Y \in [-\infty, \infty)$ and Y is T-invariant.

Note: If T is ergodic, then Y is a constant (Pitman, 2003).

Theorem 4.2 (Version 2)**.** If T is a measure-preserving transformation of the probability space $(\Omega, \mathcal{F}, \mu)$ and $\{g_n, 1 \leq n < \infty\}$ is a sequence of integrable functions satisfying $g_{n+m}(x) \leq g_n(x) + g_m(T^n x)$ then, with the probability one, we have

$lim_{n\to\infty} \frac{g_n(x)}{n} = g(x) \geq -\infty$

where g(x) is an invariant function (J.M. Steele, 1989).

Kingman's theorem has inspired a number of elegant proofs such as in Burkholder(1973), Derriennic(1975), Katznelson and Weiss(1982), Neveu(1983), and Smeltzer(1977). In Steele's paper, he summarizes earlier proofs and comes up with arguably the simplest proof in both conception and calculation. The next subsection will describe this proof in details.

4.5. **Proof of Kingman's Theorem.**

Proof. Let's define a new process $g\prime_m$ by $g\prime_m(x) = g_m(x) - \sum_{i=1}^{m-1} g_1(T^i x)$ then $g\prime_m(x) \leq 0$ for all x, and $g\prime_m$ again satisfies $g_{n+m}(x) \leq g_n(x) + g_m(T^n x)$.

At this point, we need to introducew Birkhoff's ergodic theorem so that we can continue the proof.

Theorem 4.3 (Birkhoff's theorem)**.** Let $f_1 : X \to \mathbb{R}$ be an integrable function and let $f_n = \sum_{j=1}^{n} f_1 \circ T^j$ for all $n \geq 1$ Then $\frac{f_n}{n}$ converges a.e.(almost everywhere) to an integrable function f s.t. $\int f = \int f_1$.

If we apply Birkhoff's ergodic theorem to the second term of $g\prime_m$, the a.s. convergence of $g\prime_m/m$ implies the a.s. convergence of g_m/m. Then without loss of generality, we can assume $g_m(x) \leq 0$.

Then it is essential to check check that $g(x) = \lim_{n\to\infty} inf g_n(x)/n$ is an invariant function.

Steele thoroughly outlines the proof in his book "Kingman's Subadditive Ergodic Theorem" and he applies Birkhoff's theorem a couple of times.

□

4.6. Liggett's version.

Theorem 4.4 (Liggett's Theorem)**.** Suppose X(m,n) is a collection of random variables indexed by integers satisfying $0 \leq m < n$.
Assume:

- $X(0,n) \leq X(0,m) + X(m,n) whenever 0 < m < n$.
- The joint distribution of $\{X(m+1, m+k+1) : k \geq 1\}$ are same as those of $\{X(m, m+k) : k \geq 1\}$ for each $m \geq 0$.
- For each $k \geq 1$, $\{X(nk, (n+1)k) : n \geq 1\}$ is a stationary process.
- For each n, $E|X(0,n)| < \infty and E|X(0,n)| \geq (-c * n)$ where c is a constant.

Then $\lim_{n\to\infty} \frac{X(0,n)}{n}$ a.s.

The above theorem is an improved version of Kingman's subadditive ergodic theorem developed by Thomas M. Liggett in 1985. In Liggett's version, the subadditivity and stationarity assumptions are relaxed without weakening the conclusions. This new result applies to a number of situations that were not covered by Kingman's original theorem.

More specifically speaking, in Kingman's subadditive ergodic theorem, there is the following condition: The joint distributions of $\{X_{m+1,n+1} : 0 \leq m < n\}$ are the same as those of $\{X_m, n : 0 \leq m < n\}$. Liggett, however, gives an extension of those conditions in his version of the theorem. In his paper, Liggett explains the importance of this extension in applications (Levental, 1988). As far as proof to Liggett's theorem is concerned, Shlomo Levental gives an elegant proof in his paper "A Proof of Liggett's Version of The Subadditive Ergodic Theorem." To begin with, Levental uses a proved lemma and then attempts to express the theorem in the terminology of ergodic theory. This powerful lemma can lead to the proof of both Birkhoff's ergodic theorem and Liggett's theorem.

The lemma is following:

Lemma 4.5. Let $\{f_n\}$ be a sequence of $L^1(\mu)$ functions and assume that T is $\sigma\{f_k : k \geq 1\}$ m.p. If

- $f_{n+m}(x) \leq f_n(x) + f_m(T^n(x))$, $n, m \geq 1$
- For each $x \in \Omega$ there exists an integer $n(x)$ such that $f_n(x) \leq 0$, then $\lim_{n\to\infty} sup_n E(f_n/n) \leq 0$.

5. Application of Kingman's theorem and Its Relationship to the Shape Theorem

Since Kingman's theorem is essentially used to prove the ergodic theorems for vector valued stationary processes, it is a good time to introduce the Shape theorem and try to find a connection between them.

Before introducing the Shape theorem, I'd like to discuss the concept of distance functions. A distance function, aka a metric on a set X is a function $d : X * X \to \mathbb{R}$ (where R is the set of real numbers). For all x, y, z in X, this function is required to satisfy the following conditions:

- $d(x, y) \geq 0$ (non-negativity)
- $d(x, y) = 0$ if and only if x = y (identity of indiscernibles)
- d(x,y) = d(y,x) (symmetry)
- $d(x, z) \leq d(x, y) + d(y, z)$ (Subadditivity/triangle inequality)

As we can see from above, subadditivity plays an important role in the distance function.

Now, let's define the following function $\mu(x) = \mu_{x/|x|}|x|$, if $x \neq 0$ and $= 0$ if x = 0. We can prove that μ is continuous and for $x \neq 0$, it is strictly positive. By the triangle inequality for the distance function, $\mu(x)$ is a norm on $\mathbb{R}^d$. Consider the unit ball in this norm, $A = \{x : \mu(x) \leq 1\} = \{x : |x| \leq \mu^{-1}_{x/|x|}\}$, as well as the random Riemannian ball of radius t centered at the origin, $B_t = \{x : d(0, x) \leq t\}$.

Theorem 5.1 (Shape Theorem)**.** For all $\epsilon > 0$, with probability of 1, there exists T s.t. if $t \leq T$, then $(1 - \epsilon)A \subseteq 1/tB_t \subseteq (1 + \epsilon)A$.

The set A is called the limiting shape of the random Riemannian metric g. Dr. Thomas LaGatta provided a systematic proof of the shape theorem for Riemannian first-passage percolation in his paper(2010). It is very technically advanced in the setting of this paper and I won't discuss it in great details. The most fascinating aspect of the shape theorem is that it generalizes Kingman's theorem.

To elaborate, Kingman's theorem focuses on a single-dimensional subadditive sequence while the shape theorem deals with such a sequence for every direction. For instance, if we consider the space of $\mathbb{R}^d$, the directions are the $d-1$ dimensional sphere S^{d-1}. For every direction vector $v \in S^{d-1}$, we can apply Kingman's theorem and get a sequence. The shape theorem states that we can get the convergence of sequence indicated by Kingman's theorem from every direction and such convergence is uniform in the direction.

6. Conclusion

So far, this paper explores a series of subadditive ergodic theorems and focuses on Kingman's theorem in particular. Subadditivity, ergodicity and stationarity are three crucial underlying conditions in those theorems. Building upon Kingman's theorem, many mathematicians made practical extensions to accommodate applications in other fields. The shape theorem, for example, extends Kingman's theorem to the vector valued stationary process and expands the horizon to Riemannian first-passage percolation.

7. Works Cited

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