STEADY INTERFACIAL WAVES OVER A NON-FLAT BOTTOM

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ABSTRACT. In this paper, we consider the existence of two-dimensional steady waves on the interface between two immiscible fluids where the lower fluid is taken to lie above an impermeable, non-flat boundary. We construct an asymptotic model to investigate the case where the bottom is not identically flat. Using Implicit Function and Bifurcation theoretic arguments, we prove the existence of a family of stationary solutions to the approximate model that are of small amplitude and small circulation. We also provide some numerical results.

CONTENTS

1. INTRODUCTION

Stokes [4] study of periodic water waves in a region of infinite depth heralded much interest in the field of steady waves. In the early twentieth century, Nekrasov [3] and Levi-Civita [2] first proved, rigorously, the existence of these waves. Struik [5] later extended the results of Nekrasov

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and Levi-Civita to regions with an impermeable, flat bottom. Since then, the majority of research in the field of steady waves has continued to be focused on the case where the bottom is flat.

In this paper, we extend these classical results by studying the case when the lower boundary of the fluid domain need not be flat. Specifically, we investigate the existence of two-dimensional steady waves on the interface between two immiscible incompressible, and irrotational fluids where the lower fluid is taken to lie above an impermeable boundary. This scenario arises naturally in the study of various physical phenomenon such as waves in lee of a mountain and current flow over non-flat ocean bed.

Let us state the problem mathematically. We define the Cartesian coordinates (x, y) such that the x-axis lies in the direction of wave propagation and the y-axis points vertically upward. Additionally, we assume the free surface between the two fluids is given as the graph of a function $\eta(x)$ and lies about $y = 0$ while the impermeable bottom, $y = \zeta(x)$, lies about $y = -h$. Under this coordinate system, the less dense fluid with density ρ_1 occupies the domain Ω_1 where

$$\Omega_1 := \{(x, y) : y > \eta(x)\}$$

and the denser fluid with density ρ_2 occupies the domain Ω_2 where

$$\Omega_2 := \{(x, y) : \zeta(x) < y < \eta(x)\}.$$

Figure 1 provides a useful graphical representation of the above system.

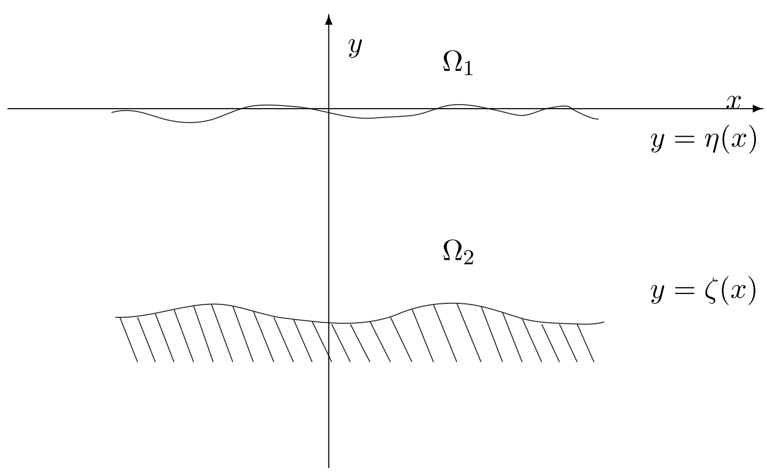

Figure 1. Domain with non-flat bottom and non-flat free surface

The steady Euler's equations for incompressible and irrotational flow govern the motion in our two-fluid system. Within the region Ω_i (for $i = 1, 2$), we have:

$$\partial_x u_i + \partial_y v_i = 0 \tag{1.1a}$$

$$\partial_y u_i - \partial_x v_i = 0 \tag{1.1b}$$

$$u_i \partial_x u_i + v_i \partial_y u_i = -\frac{1}{\rho_i} \partial_x P_i \tag{1.1c}$$

$$v_i \partial_x u_i + v_i \partial_y v_i = -\frac{1}{\rho_i} \partial_y P_i + g \tag{1.1d}$$

where u_i is the i-th fluid velocity in the horizontal x-direction, v_i the velocity in the vertical y-direction, P_i the pressure, and g the gravitational constant of acceleration. In addition, we have Bernoulli's law which states that:

$$u_i^2 + v_i^2 + 2g\rho_i y + 2P_i = Q_i, \tag{1.2}$$

where Q_i is an arbitrary constant, throughout the fluid. On the boundary, we have:

$$P_1 = P_2 \quad \text{on } y = \eta \tag{1.3a}$$

$$v_i = u_i \eta_x \quad \text{on } y = \eta \tag{1.3b}$$

$$v_2 = u_2 \zeta_x \quad \text{on } y = \zeta \tag{1.3c}$$

$$u_1 \to U \quad \text{as } y \to \infty. \tag{1.3d}$$

(1.3a) is the dynamic boundary condition which requires that the pressure, in the absence of surface tension, be continuous across the free-surface. (1.3b) is the kinematic boundary condition that necessitates that a particle which starts on the free-surface stays there. (1.3c) and (1.3d) encapsulate the zero normal-flow conditions at $y - \zeta$ and as $y \to \infty$.

Altogether, Euler's equations in (1.1)–(1.3) are a complicated system that involves multiple unknowns. With some manipulation — which we flesh out in Section 3 — Euler's equations can be rewritten in a more tractable form. First, the incompressibility condition, $u_x + v_y = 0$, permits us to introduce the pseudo relative stream function, ψ, defined, up to a constant, as

$$\psi_x = \sqrt{\rho} v \qquad \text{and} \qquad \psi_y = -\sqrt{\rho} u.$$

In terms of the stream function, our original system in (1.1)–(1.3) is equivalent to:

$$\Delta \psi_1 = 0 \quad \text{in } \Omega_1 \tag{1.4a}$$

$$\Delta \psi_2 = 0 \quad \text{in } \Omega_2 \tag{1.4b}$$

$$\frac{1}{2}(|\nabla \psi_1|^2 - |\nabla \psi_2|^2) + g(\rho_1 - \rho_2)\eta = Q \quad \text{on } y = \eta \tag{1.4c}$$

$$\psi_1 = \psi_2 = C \quad \text{on } y = \eta \tag{1.4d}$$

$$\partial_y \psi_1 = \sqrt{\rho_1} U \quad \text{as } y \to \infty \tag{1.4e}$$

$$\psi_2 = 0 \quad \text{on } y = \zeta \tag{1.4f}$$

where U is the horizontal velocity of the fluid in the upper region as $y \to \infty$ and Q and C are arbitrary constants.

Second, we observe that while the homogeneous Laplace equations in (1.4) is well posed, we do not know, a priori, the domains Ω_1 and Ω_2. To get around this difficulty, we would like to push the problem to the free-surface and hence restate the system in terms of η and C — the trace of ψ on $y = \eta$. Thus, to further simplify (1.4), we can introduce the Dirichlet-Neumann operators, $\mathcal{N}_1 : X \times \mathbb{R} \to \mathscr{L}(\mathbb{R}, Y)$ and $\mathcal{N}_2 : X \times \mathbb{R} \to \mathscr{L}(\mathbb{R}, Y)$, defined as:

$$\mathcal{N}_1(\eta, U)C := \frac{\partial \Psi_1}{\partial n}|_{y=\eta} \quad \text{and} \quad \mathcal{N}_2(\eta, \zeta)C := \frac{\partial \Psi_2}{\partial n}|_{y=\eta}.$$

Here, X and Y denote the Banach spaces:

$$X := \{C^{2+\alpha}(\mathbb{R}), 2\pi\text{-periodic, even functions}\}$$

$$Y := \{C^{1+\alpha}(\mathbb{R}), 2\pi\text{-periodic, even functions}\}, \tag{1.5}$$

Ψ_1 solves the system:

$$\begin{aligned} \Delta\Psi_1 &= 0 \quad \text{in } \Omega_1 \\ \Psi_1 &= C \quad \text{on } y=\eta \\ \partial_y \Psi_1 &= \sqrt{\rho_1} U \quad \text{as } y \to \infty, \end{aligned} \tag{1.6}$$

and Ψ_2 solves the system:

$$\begin{aligned} \Delta\Psi_2 &= 0 \quad \text{in } \Omega_1 \\ \Psi_2 &= C \quad \text{on } y=\eta \\ \partial_y \Psi_2 &= \sqrt{\rho_1} U \quad \text{as } y \to \infty. \end{aligned} \tag{1.7}$$

(1.4) is, in turn, equivalent to:

$$\frac{1}{2}(|\mathcal{N}_1(\eta, U)C|^2 - |\mathcal{N}_2(\eta, \zeta)C|^2) + g(\rho_1 - \rho_2)\eta = Q. \tag{1.8}$$

Therefore, in order to prove the existence of steady waves, we seek solutions:

$$(\eta, \zeta, U, C, Q) \in X \times X \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$$

such that (1.8) holds.

In line with the classical problem, our first theorem proves the existence of steady waves in regions with a non-flat free-surface and flat bottom:

Theorem 1.1 (Existence of small amplitude flows in a region with a non-flat free surface and a flat bottom)**.** *Consider the case where the impermeable bottom is identically flat: $\zeta \equiv -h$. There exists $\delta > 0$ and a C^1 curve of solutions, parametrized by C, given by:*

$$\mathcal{C} = \{(\eta(C), C, Q(C)) : C \in (C_* - \delta, C_* + \delta)\} \subset X \times \mathbb{R} \times \mathbb{R}.$$

bifurcating from the trivial solution at $(0, C_*, Q(C_*))$ such that (1.8) *holds.*

We construct, in Section 6, an approximate model to investigate the case where the bottom is not identically flat. Following the method of Duchêne [1], we let $\zeta = -h + \epsilon\zeta^{(1)}$ and perform an asymptomatic expansion on $\mathcal{N}_2$. We obtain an approximate expression:

$$\frac{1}{2}(|\tilde{\mathcal{N}}_1(\eta, U)C|^2 - |\tilde{\mathcal{N}}_2^{\text{app}}(\eta, \zeta)C|^2) + g(\rho_1 - \rho_2)\eta = Q. \tag{1.9}$$

where $\tilde{\mathcal{N}}_2^{\text{app}}$ is a linear in ζ approximation of $\mathcal{N}_2$ near $\zeta = -h$. See Section 6 for details.

Theorem 1.2 (Non-existence of small-amplitude flows with non-flat bottom and flat free surface)**.** *Consider the approximate problem in a two-fluid region that consists of a flat free interface and a non-flat bottom, and extends infinitely above. There does not exist any steady, incompressible and irrotational flow in such region.*

Theorem 1.3 (Existence of small amplitude flows in a region with a non-flat free surface and a non-flat bottom)**.** *There exists small amplitude steady waves in a region with a non-flat free surface and non-flat bottom. More precisely, there exist a $\delta > 0$ and a curve of solutions $(C, \eta(C))$ such that* (1.9) *holds for all $C \in (-\delta, \delta)$.*

Looking ahead, the structure of this paper is as follows. In Section 2, we first acquaint the reader with the mathematical background necessary to understanding the rest of this paper. Next, in Section 3, we prove the equivalence among (1.1)-(1.3), (1.4), and (1.9) and, in Section 4, flatten the domains Ω_1 and Ω_2 to a strip. At this point, we have all the required tools to furnish the proof of Theorem 1.1; we do so in Section 5. We also presents some numerical results in this section. Moving on, we derive our approximate model in Section 6. Finally, we provide, in Section 7, the proofs of our Theorems 1.2 and 1.3.

2. Mathematical background

This section provides a brief overview of the key concepts covered in this paper.

2.1. Banach spaces.

Definition 2.1. *(Convergent and Cauchy sequences). Let V be a normed vector space, $x \in V$ and a $\{x_n\} \subset V$ be a sequence in V.*

1. *We say that $\{x_n\}$* converges to x in norm *provided that, for all $\epsilon > 0$, there exists $N > 0$ such that $\|x_n - x\| < \epsilon$ for all $n > N$. A sequence $\{x_n\} \subset V$ is said to be* convergent *if there exists $x \in V$ that $\{x_n\}$ approaches in norm.*
2. *A sequence $\{x_n\} \subset V$ is said to be* Cauchy *provided that, for all $\epsilon > 0$, there exists $N > 0$ such that $|x_n - x_m\| < \epsilon$ for all $m, n > N$.*

Definition 2.2. *(Banach Spaces) A normed vector space in which every Cauchy sequence is convergent is called a* Banach space.

Definition 2.3. *(Maps between Banach spaces) If X and Y are Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively, then a function $\mathcal{F} : X \to Y$ is said to be* continuous *if for every $x_1, x_2 \in X$, and every $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\|x_1 - x_2\|_X < \delta \qquad \textit{implies} \qquad \|\mathcal{F}(x_1) - \mathcal{F}(x_2)\|_Y < \epsilon.$$

This is denoted $C^0(X; Y)$ for a set of continuous mappings from X to Y.

2.2. The Fréchet derivative.

Definition 2.4. *(The Fréchet derivative) Let X and Y be Banach spaces and let $\mathcal{F} \in C(X; Y)$ be given. We say that $\mathcal{F}$ is Freéchet differentiable at $x \in X$ provided that there exists a bounded linear operator $\mathcal{L}_x : X \to Y$ such that*

$$\lim_{h \to 0} \frac{\|\mathcal{F}(x + h) - \mathcal{F}(x) - \mathcal{L}_x(h)\|_Y}{\|h\|_X} = 0,$$

where $\|\cdot\|_X$ and $\|\cdot\|_Y$ are the norms on X and Y, respectively, and we denote $(D\mathcal{F})(x) := \mathcal{L}_x$ or sometimes $D_x\mathcal{F}(x)$. If $\mathcal{F}$ is differentiable at every $x \in X$, we say that it is Fréchet differentiable. *If, for each $x \in X$, the derivative $D_x\mathcal{F}(x)$ is continuous, we say that $\mathcal{F} \in C^1(X; Y)$.*

2.3. Implicit Function Theorem.

Theorem 2.1 (Implicit function theorem on Banach spaces)**.** *Let W, X and Y be Banach spaces and $\mathcal{F} \in C^1(W \times X; Y)$. Suppose that for some $(w_0, x_0) \in W \times X$ we have*

$$\mathcal{F}(w_0, x_0) = 0, \qquad \textit{and} \qquad D_xF(w_0, x_0) \textit{ is bijective.}$$

Then there exists an open set $\mathcal{O} \subset W$ with $w_0 \in \mathcal{O}$, and a C^1 function $\varphi : \mathcal{O} \to X$ such that

$$\varphi(w_0) = x_0,$$

and

$$\mathcal{F}(w, \varphi(w)) = 0, \qquad \textit{for all } w \in \mathcal{O}.$$

Moreover, if $\mathcal{F}(w, x) = 0$ and $w \in \mathcal{O}$, then $x = \varphi(w)$.

2.4. **Bifurcation theory.**

Definition 2.5. *Let X and Y be Banach spaces and suppose that $\mathcal{L} : X \to Y$ is a bounded linear operator. Suppose that*

$$\dim \mathscr{N}(\mathcal{L}) < \infty, \qquad \dim \mathscr{R}(\mathcal{L})^c < \infty,$$

where $\mathscr{N}(\mathcal{L})$ is the null space and $\mathscr{R}(\mathcal{L})^c$ is the complement of the range. If we also know that the range is a closed set in Y, then we say that $\mathcal{L}$ is a Fredholm operator*.*

Definition 2.6. *Let $\mathcal{L}$ be a bounded linear operator between Banach spaces. If $\mathcal{L}$ is Fredholm, we define the* index *of $\mathcal{L}$, denoted $\operatorname{ind}\mathcal{L}$, to be the integer*

$$\operatorname{ind}\mathcal{L} := \dim \mathscr{N}(\mathcal{L}) - \operatorname{cod}\mathscr{R}(\mathcal{L}).$$

Theorem 2.2 (Crandall–Rabinowitz)**.** *Let X and Y be Banach spaces and let $\mathcal{F} : \mathbb{R} \times X \to Y$ be a Fredholm operator of index 0. Suppose that for all λ,*

$$\mathcal{F}(\lambda, 0) = 0,$$

and that there exists λ_0 such that

$$\mathscr{N}(D_x\mathcal{F}(\lambda_0, 0)) \text{ is one-dimensional.}$$

If

$$D_\lambda D_x\mathcal{F}(\lambda_0, 0)u \notin \mathscr{R}(D_x\mathcal{F}(\lambda_0, 0)),$$

for any $u \in \mathcal{N}(D_x\mathcal{F}(\lambda_0, 0))$, then, there exists $\epsilon > 0$ and a C^1 curve

$$\mathscr{C} = \{(\lambda, x(\lambda)) : \lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)\} \subset \mathbb{R} \times X,$$

such that $x(0) = 0$ and $\mathcal{F}(\lambda, x) = 0$ for each $(\lambda, x) \in \mathscr{C}$. Moreover, $x(\lambda) \neq 0$ for $\lambda \neq \lambda_0$.

3. Equivalent formulation of the problem

As previously discussed, Euler's equations in (1.1)–(1.3) are an unwieldy system to work with. Instead, it is more convenient to conduct our analysis on (1.8). Thus, in this section, we prove the equivalence among Eulers equations in (1.1)–(1.3), the stream function system in (1.4), and the formulation in terms of the Dirichlet-Neumann operators in (1.8). The blueprint for this section is as follows. We will show, separately, the equivalence between (1.1)–(1.3) and (1.4), and that between (1.4) and (1.8). Combined, these two results prove the equivalence among the three different formulations of the problem.

Proposition 3.1. *Euler's equations in* (1.1)–(1.3) *are equivalent to* (1.4)

Proof. We first show that (1.1)–(1.3) imply (1.4). (1.4a) and (1.4b) arise because of the irrotationality condition which necessitates that

$$\partial_y u_i + \partial_x v_i = \frac{1}{\sqrt{\rho_i}} \Delta\psi_i = 0.$$

Rewriting Bernoulli's law in (1.2) in terms of the pseudo sream function, we have:

$$\frac{1}{2}|\nabla\psi_1|^2 + g\rho_1\eta - Q_1 = -P_1 \quad \text{on } y = \eta$$
$$\frac{1}{2}|\nabla\psi_2|^2 + g\rho_2\eta - Q_2 = -P_2 \quad \text{on } y = \eta.$$

Coupled with the dynamic boudary condition, $P_1 = P_2$, we can readily check that (1.4c) holds, with $Q = Q_1 - Q_2$. Moreover, once we state (1.3b)–(1.3d) in terms of the stream function:

$$\partial_x \psi_i = -\partial_y \psi_i \eta_x \quad \text{on } y = \eta$$
$$\partial_x \psi_2 = -\partial_y \psi_2 \zeta_x \quad \text{on } y = \zeta$$
$$\partial_y \psi_1 \to \sqrt{\rho_1} U \quad \text{as } y \to \infty,$$

it becomes evident that ψ_i must be constant on the free-surface and at bottom; we can pick constants such that (1.4d) – (1.4f) hold.

We next show that (1.4) implies (1.1)–(1.3). Suppose we begin with a solution ψ_i to the system in (1.4) and define the velocity field by $(u, v) = \frac{1}{\sqrt{\rho}} \nabla^\perp \psi$. Moreover, we can define the gradient of its pressure to be:

$$\rho_i (u_i \partial_x u_i + v_i \partial_y u_i) = -\partial_x P_i$$
$$\rho_i (v_i \partial_x u_i + v_i \partial_y v_i - g) = -\partial_y P_i.$$

Then, we can confirm that the irrotationality condition and Euler's momentum equations follow directly from the homogeneous Laplace equation. In addition, we can always pick constants Q in (1.4c) such that the dynamic boundary condition, $P_1 = P_2$, and Bernoulli's law hold. Lastly, to dervive the remaining boundary conditions in (1.3b)–(1.3d), we take tengential derivatives of the traces of ψ_i on the respective boundaroes.

The above results prove the equivalence betwwen Euler's equations and the stream function system. We refer the reader to Yih [6] for a more detailed discussion on the pseudo relative stream function. ■

Proposition 3.2. *The system in* (1.4) *is equivalent to* (1.8).

Proof. We can decompose the trace $\nabla \Psi_i$ on the interface η into its normal and tangential components:

$$\nabla \Psi_i = (\nabla \Psi_i \cdot \mathbf{n})\mathbf{n} + (\nabla \Psi_i \cdot \mathbf{t})\mathbf{t}$$

where $\mathbf{n}$ is the outward unit normal vector and $\mathbf{t}$ is the unit tangent vector. Now consider (1.4c). Because $\Psi_i = C$ on η, the tangential component of its gradient vanishes, and thus

$$|\nabla \Psi_i|^2 = |\nabla \Psi_i \cdot \mathbf{n}|^2 \quad \text{on } y = \eta$$

which in turn allows us to rewrite (1.4c) as:

$$\frac{1}{2}(|\nabla \Psi_1 \cdot \mathbf{n}|^2 - |\nabla \Psi_2 \cdot \mathbf{n}|^2) + g(\rho_1 - \rho_2)\eta = Q.$$

The equivalence of this to (1.8) follows from our definitions of the Dirichlet-Neumann operators. ■

Remark 1. On occasion, it is more convenient to work with the non-normalized Dirichlet-Neumann operator

$$\mathcal{G}(\eta)C = \sqrt{1 + \eta_x^2}\mathcal{N}(\eta)C.$$

4. Flattening the domain

It is difficult to obtain expressions for $\mathcal{N}_1(\eta, U)$ and $\mathcal{N}_2(\eta, \zeta)$ as the systems in (1.6) and (1.7) are unwieldy to solve for general η and ζ. It is therefore convenient to introduce a change in coordinates to flatten the domain to a strip. Before we proceed with this flattening, it is helpful to observe that the systems in (1.6) and (1.7) are independent and we can, thus, flatten the respective domains separately.

4.1. **Fluid in the upper region.** To simplify the problem in (1.6), we let $\xi = \Psi_1 + \sqrt{\rho_1}Uy$ and observe that the original problem is equivalent to

$$
\begin{aligned}
\Delta\xi &= 0 \quad \text{in } \Omega \\
\xi &= C + \sqrt{\rho_1}U\eta \quad \text{on } y = \eta \\
\xi_y &= 0 \quad \text{as } y \to \infty.
\end{aligned} \tag{4.1}
$$

Let $w = x$ and $z = y - \eta$. With this change of variables, we note that the set $\{y = \eta(x)\}$ corresponds to the set $\{z = 0\}$. We aim to rewrite the system in (1.6) in terms of the new variables. Calculating the partial derivatives, we have, by the chain rule:

$$\partial_x = \partial_w - \eta_w \partial_z \quad \text{and} \quad \partial_y = \partial_z.$$

Thus,

$$\Delta_{(x,y)} = \partial_w^2 + \partial_z^2 - 2\eta_w\partial_w\partial_z - \eta_{ww}\partial_z + \eta_w^2\partial_z^2. \tag{4.2}$$

We denote the operator in (4.2) as $\mathcal{L}_1$.

Therefore, (4.1) written in the (w, z)-variables is:

$$
\begin{aligned}
\mathcal{L}_{(w,z)}\tilde{\xi} &= 0 \quad \text{in } \Omega \\
\tilde{\xi} &= C + \sqrt{\rho_1}U\eta \quad \text{on } z = 0 \\
\tilde{\xi}_z &= 0 \quad \text{as } z \to \infty
\end{aligned} \tag{4.3}
$$

where $\tilde{\xi}$ is a function in the new w and z variables.

Also, the Dirichlet-Neumann operator. $\tilde{\mathcal{G}}_1(\eta, U)$, written in our new variables is:

$$
\begin{aligned}
\tilde{\mathcal{G}}_1(\eta, U)C &= [\partial_{n(x,y)}\Psi_1]_{|y=\eta(x)} \\
&= [(-\eta_w(\partial_w + \eta_w\partial_z) + \partial_z)\tilde{\Psi}_1]_{|z=0} \\
&= [(-\eta_w(\partial_w + \eta_w\partial_z) + \partial_z)(\tilde{\xi} - \sqrt{\rho_1}U(z + \eta))]_{|z=0}.
\end{aligned}
$$

4.2. **Fluid in the lower region.** Let $w = x$ and $z = -\frac{h(y-\eta(x))}{\zeta(x)-\eta(x)}$. With this change of variables, we note that the sets $\{y = \eta(x)\}$ and $\{y = \zeta(x)\}$ correspond to the sets $\{z = 0\}$ and $\{z = -h\}$ respectively. We rewrite the system in (1.7) in terms of the new variables. Calculating the partial derivatives, we have, by the chain rule:

$$\partial_x = \partial_w + \frac{h\eta_w - z(\zeta_w - \eta_w)}{\zeta - \eta}\partial_z \quad \text{and} \quad \partial_y = -\frac{h}{\zeta - \eta}\partial_z.$$

Thus,

$$\Delta_{(x,y)} = \left(\partial_w + \frac{h\eta_w - z(\zeta_w - \eta_w)}{\zeta - \eta}\partial_z\right)^2 + \left(-\frac{h}{\zeta - \eta}\partial_z\right)^2. \tag{4.4}$$

We denote the operator in (4.5) as $\mathcal{L}_2$.

Therefore, our system in (1.7) reduces to:

$$
\begin{aligned}
\mathcal{L}_2\tilde{\Psi} &= 0 \quad \text{in } \Omega \\
\tilde{\Psi} &= C \quad \text{on } z = 0 \\
\tilde{\Psi} &= 0 \quad \text{on } z = -h
\end{aligned} \tag{4.5}
$$

where $\tilde{\Psi}$ is Ψ expressed in the new (w, z)-variables.

Also, the Dirichlet-Neumann operator. $\tilde{\mathcal{G}}_2(\eta,\zeta)$, written in our new varibles is:

$$\begin{aligned}\tilde{\mathcal{G}}_2(\eta,\zeta)C &= [\partial_{n(x,y)}\Psi_2]_{|y=\eta(x)} \\ &= [(-\eta_w\partial_w - \frac{h\eta_w^2 - z\eta_w(\zeta_w-\eta_w)-h}{\zeta-\eta}\partial_z)\tilde{\Psi}_2]_{|z=0}.\end{aligned}$$

5. Existence of small amplitude flows in a region with a non-flat free surface and a flat bottom

In this section, we investigate the classical case where the region Ω_2 lies above an impermeable flat bottom, $\zeta \equiv -h$ (See Figure 1). To simplify our analysis, we consider the single fluid case (i.e. $\rho_1 = 0$). Under this scenario, (1.8) reduces to:

$$\frac{1}{2}|\tilde{\mathcal{N}}_2(\eta,-h)C|^2 + g\rho_2\eta = Q. \tag{5.1}$$

With this in mind, it is useful to define a function, $\mathcal{F}$, as

$$\mathcal{F}(\eta,C,Q) := \frac{1}{2}|\tilde{\mathcal{N}}_2(\eta,-h)C|^2 + g\rho_2\eta - Q \tag{5.2}$$

whose zero set corresponds to the solution set, $\{(\eta,C,Q)\in X\times\mathbb{R}\times\mathbb{R}\}$, of (5.1).

When $\eta\equiv 0$, we note that there exist trivial laminar flows. By (4.5), $\tilde{\Psi}_2 = \frac{C(h+z)}{h}$ and, hence, $\tilde{\mathcal{N}}_2(0,-h)C = \frac{C}{h}$. Therefore, the set of trivial solutions is given by:

$$\{(0,C,Q(C))\in X\times\mathbb{R}\times\mathbb{R}\},$$

where $Q(C) = \frac{C^2}{2h^2}$. We would like to show that there exist small amplitude flows such that $\eta\not\equiv 0$.

Proof of Theorem 1.1. We look for solutions to (5.1) near $\eta = 0$, $C = C_0$, and $Q = \frac{C_0^2}{2h^2}$. Taking the Fréchet derivative of $\mathcal{F}$ with respect to η, and evaluating it at $(0, C_0, Q(c_0))$, we have:

$$D_\eta\mathcal{F}(0,C_0,Q(C_0))v = \mathcal{N}(0,-h)C_0\langle D_\eta\mathcal{N}_2(0,-h)C_0,v\rangle + g\rho_2 v.$$

Here:

$$\langle D_\eta\mathcal{N}_2(0,-h)C_0,v\rangle = [\partial_z\dot{\tilde{\Psi}}_2 - \frac{v}{h}\partial_z\tilde{\Psi}_2]_{z=0}$$

where:

$$\tilde{\Psi}_2 = \frac{C_0(h+z)}{h}$$

and, from (4.5), $\dot{\tilde{\Psi}}_2$ solves the system:

$$\begin{aligned}\Delta\dot{\tilde{\Psi}}_2 &= \left[\frac{2}{h}(h+z)v_w\partial_w\partial_z + \frac{1}{h}(h+z)v_{ww}\partial_z - v\partial_z^2\right]\tilde{\Psi}_2 \\ \dot{\tilde{\Psi}}_2 &= 0 \quad \text{on} \quad z=0 \\ \dot{\tilde{\Psi}}_2 &= 0 \quad \text{on} \quad z=-h.\end{aligned}$$

Thus:

$$\dot{\tilde{\Psi}}_2 = -\sum_{k\geq 0}\frac{C_0}{h^2}\hat{v}_k\left[h\cosh(kz) + h\coth(kh)\sinh(kz) - (h+z)\right]\cos(kw).$$

In summary:

$$D_\eta\mathcal{F}(0,C_0,Q(C_0))v = -\frac{C_0^2}{h^2}\sum_{k\geq 0}\left[k\coth(kh) - \frac{gh^2}{C_0^2}\right]\hat{v}_k\cos(kw). \tag{5.3}$$

For C_0 such that $k\coth(kh) - \dfrac{gh^2}{C_0^2} \neq 0$, $D_\eta \mathcal{F}(0, C_0, Q(C_0))v$ is an isomorphism. We can thus conclude by Implcit Function Theorem that there exists $\delta > 0$ and a C^1 curve

$$\mathcal{C} = \{(\eta(C), C, Q(C)) : C \in (C_0 - \delta, C_0 + \delta)\} \subset X \times \mathbb{R} \times \mathbb{R}$$

such that (5.1) holds.

Conversely, for C_0 such that $k\coth(kh) - \dfrac{gh^2}{C_0^2} = 0$, $D_\eta \mathcal{F}(0, C_0, Q(C_0))v$ fails to be an isomorphism and the Implicit Function Theorem, consequently, fails to prove the existence of non-trivial solutions. In this case, we would need to use the Crandall–Rabinowitz theorem to establish the existence of a curve bifurcating from the trivial solutions. Without loss of generality, assume that there exists a

$$C_* = \left(\frac{\rho g h^2}{\coth(h)} \right)^{\frac{1}{2}}. \tag{5.4}$$

such that (5.3) vanishes for $k = 1$. Over the next three lemmas, we will verify the hypothesis of the Crandall–Rabinowitz theorem and thus prove the existence of small amplitude flows over a flat bottom.

Lemma 5.1. *$\mathscr{N}(\mathcal{L}) = \operatorname{span}\{v_*\}$, where $v_*(x) = \cos x$.*

Proof. From our definition of C_* in (5.4), and the computation of $D_\eta \mathcal{F}$ in (5.3), we see that the null space of $\mathcal{L}$ is nontrivial because $\dfrac{\rho g h^2}{\coth(h)}$ vanishes. More specifically, because it vanishes at $k = 1$, $\cos x$ will be the null space. So, $\mathscr{N}(\mathcal{F}_\eta(0, Q_0(C_*))) = \operatorname{Span}\{v_*\}$, where $v_*(x) = \cos x$. Therefore, $\dim \mathscr{N}(\mathcal{L}) = 1$ since v_* vanishes only at $k = 1$. ■

Next, we show that $\mathcal{L}$ is a Fredholm operator of index 0 (cf. Definition 2.6).

Lemma 5.2. *The range of $\mathcal{L}$ is $\operatorname{span}\{v_*\}^\perp$.*

Proof. To identify the range of $\mathcal{L}$, we want to find for which $u \in Y$ one can solve:

$$D_\eta \mathcal{F}(0, C, Q_0(C_*))v = u, \quad \text{for} \quad u \in Y.$$

Taking the Fourier transform yields:

$$-\frac{C_*^2}{h^2} \sum_{k \geq 0} \left(k\coth(kh) - \frac{gh^2}{C_*^2} \right) \hat{v}_k \cos(kw) = \sum_k \hat{u}_k \cos(kw). \tag{5.5}$$

If $k\coth(kh) - \dfrac{gh^2}{C_*^2}$ is non-zero for some $k \geq 0$, then we can solve for $\hat{v}_k$. Since this quantity is 0 for $k = 1$, we have $\hat{u}_1 = 0$. If $\hat{u}_1 \neq 0$, then there is no v satisfying (5.5) can exist. Therefore, we can solve for v if and only if u is orthogonal to $\cos(w)$. Therefore, the range of $\mathcal{L}$ will equal $\operatorname{span}\{v_*\}^\perp$ and we have that $\dim \operatorname{span}\{v_*\} = 1$. ■

Lemma 5.2 implies that the Fredholm index is 0 as $\dim \operatorname{span}\{v_*\} = \operatorname{cod} \mathscr{R} = 1$. Lastly, we need to verify the transcritical condition of the Crandall–Rabinowiz Theorem 2.2, which is done in the following lemma.

Lemma 5.3. *$D_C D_\eta \mathcal{F}(0, Q_0(C_*))v_* \notin \mathscr{R}(D_\eta \mathcal{F}(0, Q_0(C_*)))$; it satisfies the transcritical condition.*

Proof. Taking the derivative of $D_\eta \mathcal{F}$ with respect to C, and then evaluating at $(\eta = 0, Q_0(C_*))$ we find:

$$
\begin{aligned}
D_C D_\eta \mathcal{F}(0, Q_0(C_*))v_* &= D_C \left[-\frac{C_*^2}{h^2} \sum_k \left(k \coth(kh) - \frac{\rho g h^2}{C_*^2} \right) \hat{v}_{*k} \cos(kw) \right] \\
&= -\frac{2C_*}{h^2} \sum_k \left(k \coth(kh) - \frac{\rho g h^2}{C_*^2} \right) \hat{v}_{*k} \cos(kw) + \frac{C_*^2}{h^2} \sum_k \left(\frac{\rho g h^2}{C_*^3} \cdot 2C_* \right) \hat{v}_{*k} \cos(kw) \qquad (5.6) \\
&= -\frac{2C_*}{h^2} \sum_k \left[\left(k \coth(kh) - \frac{\rho g h^2}{C_*^2} \right) + \rho g \right] \hat{v}_{*k} \cos(kw).
\end{aligned}
$$

So by taking inner product with $\cos(w)$,

$$(D_C D_\eta \mathcal{F}(0, Q_0(C_*))v_*, \cos(w))_{L^2} = \coth h\pi \neq 0$$

Hence, $D_C D_\eta \mathcal{F} \notin \mathscr{R}(D_\eta \mathcal{F})$, by Lemma 5.2. ∎

From our first two lemmas, we have shown that the dimension of the null space and the codimension of the range both equal 1. From Definition 2.6, we have

$$\operatorname{ind} \mathcal{L} = 1 - 1 = 0,$$

which proves that $D_\eta \mathcal{F}(0, Q_0(C_*))$ is a Fredholm operator of index 0. Moreover, the transcritical condition of the Crandall–Rabinowitz Theorem is satisfied by Lemma 5.3. Therefore, by Crandall–Rabinowitz, there exists a curve of solutions bifurcating from $\mathcal{T}$ at $(0, Q_{0*}(C_*))$. There exists $\delta > 0$ and a C^1 curve

$$\mathcal{C} = \{(\eta(C), C, Q(C)) : C \in (C_* - \delta, C_* + \delta)\} \subset X \times \mathbb{R} \times \mathbb{R}.$$ ∎

5.1. **Numerical analysis of solutions.** After we have proven existence of solution for the approximate problem presented in Section 5, we show in this section what these solutions look like by using AUTO. AUTO-07p, the newest version of AUTO, is a package that runs on Linux/Unix, Mac OS X and Windows. After proper installation, AUTO-07p can give a bifurcation analysis for algebraic systems. For a detailed overview of AUTO's capabilities, please visit their website at `http://indy.cs.concordia.ca/auto/`.

Before using AUTO to find bifurcation branches, we need to simplify our equations into something that AUTO can understand. Recall that the equation we need to solve is the following

$$\frac{1}{2}|\mathcal{N}(\eta)C|^2 + \rho g \eta + \sigma \kappa - Q = 0,$$

where κ is the mean curvature and $\sigma > 0$ is the coefficient of surface tension. We include the surface tension to simplify the numerics; and take $\sigma = 1$. Multiplying both sides by 2 and writing the Dirichlet-to-Neumann operator as its Fourier cosine series, we have

$$\left[\frac{C}{h} + \frac{C}{h} \sum_{k=0}^{\infty} k \coth(kh) \hat{\eta}_k \cos(kx) \right]^2 + 2\rho g \eta - 2\sigma \eta'' - 2Q = 0,$$

where $-\eta''$ is the approximation of κ It follows that

$$
\begin{aligned}
\frac{C^2}{h^2} \left[1 + 2 \sum_k^{\infty} \coth(kh) \hat{\eta}_k \cos(kh) + \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} k \coth(kh) \hat{\eta}_k j \coth(kh) \hat{\eta} j \cos(kx) \cos(jx) \right] \\
+ 2\rho g \eta - 2\sigma \eta'' - 2Q = 0.
\end{aligned}
$$

Moreover, taking inner product in L^2_x of the left hand side with $\cos(\ell x)$, and using the fact that

$$\cos(kx)\cos(jx) = \frac{1}{2}\left[\cos(k+j)x + \cos(k-j)x\right],$$

we get
(5.7)

$$\frac{C^2}{h^2}\left[\delta_{0,\ell} + 2\ell\coth(\ell h)\hat{\eta}_\ell + \frac{1}{2}\sum_{\substack{\ell=k+j \\ \text{or } \ell=|k-j|}} k\coth(kh) j\coth(jh)\hat{\eta}_k\hat{\eta}_j\right] + 2\rho g\hat{\eta}_\ell + 2\sigma\ell^2\hat{\eta}_l - 2Q\delta_{0,\ell} = 0.$$

For example, for $\ell = 0$, we have $k = 1$ and $j = 1$. Thus, the above expression is the same as

$$\frac{C^2}{h^2}\left[1 + \frac{1}{2}\coth^2(h)\hat{\eta}_1^2\right] - 2Q = 0,$$

which is consistent with the results shown in section 5. That is, for $\hat{\eta}_1 = 0$, the trivial solution satisfies

$$Q = \frac{1}{2}\frac{C^2}{h^2}.$$

Moreover, for $\ell = 1$, $k = 0$ and $j = 1$ (which gives the same outcome as $k = 1$ and $j = 0$). Then 5.7 becomes

$$\left[\frac{C^2}{h^2}\coth(h) + g + \sigma\right]\hat{\eta}_1 = 0.$$

It follows that $\hat{\eta}_1 = 0$ and we get the trivial solution again. If we want to approximate η up to $\hat{\eta}_2$, then we have to solve the following system of equations

$$\frac{C^2}{h^2}\left[4\coth(2h)\hat{\eta}_2 + \coth^2(h)\hat{\eta}_1^2\right] + 2g\hat{\eta}_2 + 8\sigma\hat{\eta}_2 = 0 \quad \text{for } \ell = 1+1 = 2$$

$$\frac{C^2}{h^2}\left\{2\coth(h)\hat{\eta}_1 + \frac{1}{2}\left[2\coth(2h)\coth(h)\hat{\eta}_2\hat{\eta}_1\right]\right\} + 2g\hat{\eta}_1 + 2\sigma\hat{\eta}_1 = 0 \quad \text{for } \ell = 2-1 = 1$$

$$\frac{C^2}{h^2}\left\{1 + \frac{1}{2}\left[\coth^2(h)\hat{\eta}_1^2 + 4\coth^2(2h)\hat{\eta}_2^2\right]\right\} - 2Q = 0 \quad \text{for } \ell = 2-2 = 1-1 = 0.$$

At this point, we can use AUTO-07p to analyse the above algebraic system. First, we need to transform the system into something that AUTO-07p can understand. The input to AUTO-07p is written in Fortran 90 and the corresponding name for the unknowns are given below.

C	$\hat{\eta}_1$	$\hat{\eta}_2$	Q
PAR(1)	U(1)	U(2)	U(3)

TABLE 1.

The source code is shown below as reference.

```
F(1)= (PAR(1)**2)*(1.00 + (U(1)**2)/(DTANH(1D0)**2) + 4.00*(U(2)**2)/(DTANH(2D0)**2))
- 2*U(3)

F(2)= (PAR(1)**2)*(-2.00*U(1)/DTANH(1D0) + U(1)*U(2)/(DTANH(2D0)*DTANH(1D0)))
+ 20.00*U(1) + 2.00*U(1)

F(3)= (PAR(1)**2)*(-4.00*U(2)/DTANH(2D0) + (U(1)**2)/(DTANH(1D0)**2))
+ 20.00*U(2) + 8.00*U(2)
```

AUTO is then told to vary C, solve for $\hat{\eta}_1$, $\hat{\eta}_2$ and Q, and look for bifurcation branches, starting at $C = 2.5$, $\hat{\eta}_1 = 0$, $\hat{\eta}_2 = 0$, and $Q = 3.1$. The results are attached in the appendix.

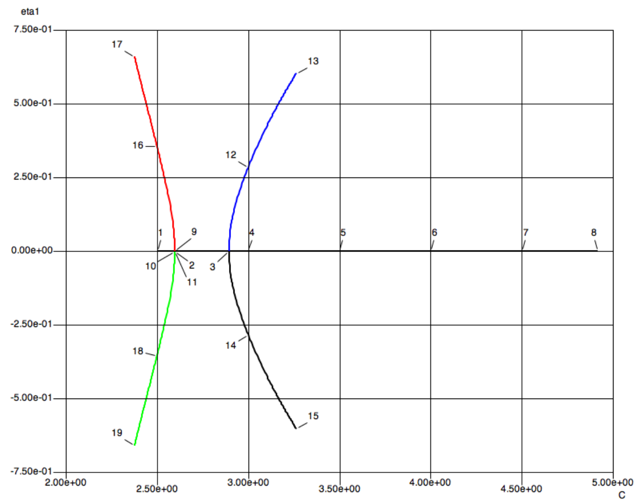

Figure 2. Bifurcation diagram

We can also produce graphs of the corresponding solutions using Matlab. AUTO outputs values of $\hat{\eta}_1$ and $\hat{\eta}_2$ along the bifurcation curve, and Figure 2 is the graph of $\hat{\eta}_1 \cos(x) + \hat{\eta}_2 \cos(2x)$ for these values. Similarly, Figure 3 shows results for the second bifurcation curve. In the appendix, we include the computation of solutions for the three remaining curves.

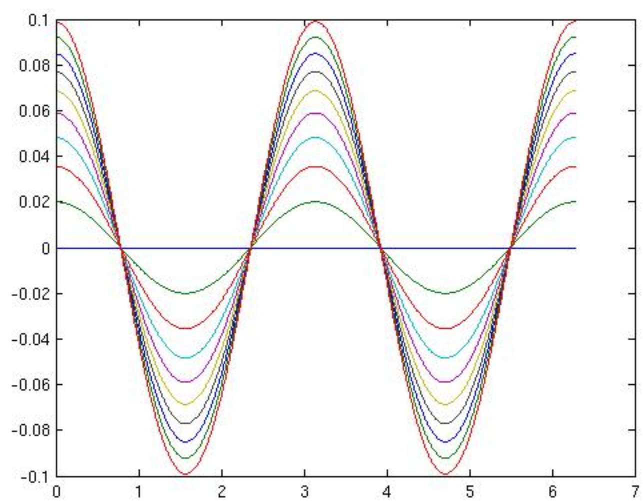

Figure 3. Solution Branch #1

6. Approximate model

In order to investigate the case where the bottom, ζ, is not identically flat, we seek asymptotic expansions, near $\eta = 0$ and $\zeta = -h$, for $\tilde{\mathcal{N}}_2(\eta, \zeta)$.

6.1. **Approximate solutions.** We look for approximate solutions of the form

$$\tilde{\Psi}^{app} = \tilde{\Psi}^{(0)} + \epsilon\tilde{\Psi}^{(1)}.$$

(It is worth noting that $\tilde{\Psi}^{(0)}$ is the solution when $\zeta = -h$.) To this end, we first expand the operators (4.4) in terms of ϵ. Next, we plug in our approximate solution and solve for the respective orders of ϵ to obtain expression for each $\tilde{\Psi}^{(i)}$.

We perform a Taylor series expansion of the operators in (4.4). Let $\zeta = -h + \epsilon\zeta^{(1)}$. Hence, we know that:

$$\begin{aligned}
\partial_x^2 &= \Big(\partial_w + \frac{h\eta_w - z(\zeta_w - \eta_w)}{\zeta - \eta}\partial_z\Big)^2 \\
&= \Big(\partial_w + \frac{(h+z)\eta_w - z\epsilon\zeta_w^{(1)}}{-h + \epsilon\zeta^{(1)} - \eta}\partial_z\Big)^2 \\
&= \Big(\partial_w - \frac{1}{(h+\eta)^2}[(h+z)\eta_w - z\epsilon\zeta_w^{(1)}][h + \eta + \epsilon\zeta^{(1)}]\partial_z\Big)^2 + O(\epsilon^2) \\
&= \Big(\partial_w - \frac{1}{(h+\eta)^2}[(h+z)(h+\eta)\eta_w + \epsilon[(h+z)\eta_w\zeta^{(1)} - (h+\eta)z\zeta_w^{(1)}]]\partial_z\Big)^2 + O(\epsilon^2).
\end{aligned}$$

By the above expression, we mean that ∂_x^2 equals the operator in parenthesis plus ϵ^2 multiplying a bounded operator. So, dropping $O(\epsilon^2)$ terms, we have:

$$\begin{aligned}
&= \partial_w^2 - \frac{2}{(h+\eta)^2}[(h+z)(h+\eta)\eta_w + \epsilon[(h+z)\eta_w\zeta^{(1)} - (h+\eta)z\zeta_w^{(1)}]]\partial_w\partial_z \\
&\quad - \partial_w\Big[\frac{1}{(h+\eta)^2}[(h+z)(h+\eta)\eta_w + \epsilon[(h+z)\eta_w\zeta^{(1)} - (h+\eta)z\zeta_w^{(1)}]]\Big]\partial_z \\
&\quad + \frac{1}{(h+\eta)^4}[(h+z)(h+\eta)\eta_w + \epsilon[(h+z)\eta_w\zeta^{(1)} - (h+\eta)z\zeta_w^{(1)}]]^2\partial_z^2 \\
&\quad + \frac{2}{(h+\eta)^4}[(h+\eta)\eta_w + \epsilon[\eta_w\zeta^{(1)} - (h+\eta)\zeta_w^{(1)}]][(h+z)(h+\eta)\eta_w + \epsilon[(h+z)\eta_w\zeta^{(1)} - (h+\eta)z\zeta_w^{(1)}]]\partial_z.
\end{aligned}$$

Expanding the expression further, we get,

$$\begin{aligned}
&= \partial_w^2 - \frac{2}{(h+\eta)^2}[(h+z)(h+\eta)\eta_w]\partial_w\partial_z - \epsilon\frac{2}{(h+\eta)^2}[(h+z)\eta_w\zeta^{(1)} - (h+\eta)z\zeta_w^{(1)}]\partial_w\partial_z \\
&\quad - \frac{1}{(h+\eta)^2}[(h+z)[(h+\eta)\eta_{ww} + \eta_w^2]]\partial_z + \frac{2}{(h+\eta)^3}(h+z)(h+\eta)\eta_w^2\partial_z \\
&\quad - \epsilon\frac{1}{(h+\eta)^2}[(h+z)(\eta_{ww}\zeta^{(1)} + \eta_w\zeta_w^{(1)}) - \eta_w z\zeta_w^{(1)} - (h+\eta)z\zeta_{ww}^{(1)}]\partial_z \\
&\quad + \epsilon\frac{2}{(h+\eta)^3}[(h+z)\eta_w^2\zeta^{(1)} - (h+\eta)z\eta_w\zeta_w^{(1)}]\partial_z \\
&\quad + \frac{1}{(h+\eta)^4}[(h+z)(h+\eta)\eta_w]^2\partial_z^2 + \epsilon\frac{2}{(h+\eta)^4}[(h+z)(h+\eta)\eta_w][(h+z)\eta_w\zeta^{(1)} - (h+\eta)z\zeta_w^{(1)}]\partial_z^2 \\
&\quad + \frac{2}{(h+\eta)^4}(h+z)(h+\eta)^2\eta_w^2\partial_z + \epsilon\frac{2}{(h+\eta)^4}(h+\eta)\eta_w[(h+z)\eta_w\zeta^{(1)} - (h+\eta)z\zeta_w^{(1)}]\partial_z \\
&\quad + \epsilon\frac{2}{(h+\eta)^4}[\eta_w\zeta^{(1)} - (h+\eta)\zeta_w^{(1)}](h+z)(h+\eta)\eta_w\partial_z
\end{aligned}$$

Furthermore,

$$
\begin{aligned}
\partial_y^2 &= [-\frac{h}{\zeta-\eta}\partial_z]^2 \\
&= [-\frac{h}{-h+\epsilon\zeta^{(1)}-\eta}\partial_z]^2 \\
&= [\frac{h}{(h+\eta)^2}(h+\eta+\epsilon\zeta^{(1)})\partial_z]^2 + O(\epsilon^2) \\
&= \frac{h^2}{(h+\eta)^2}\partial_z^2 + \epsilon\frac{2h^2}{(h+\eta)^3}\zeta^{(1)}\partial_z^2 + O(\epsilon^2).
\end{aligned}
$$

Combining our above results, (4.4) becomes

$$
\mathcal{L}_{(w,z)} = \mathcal{L}^{(0)} + \epsilon\mathcal{L}^{(1)} + O(\epsilon^2)
$$

where

$$
\begin{aligned}
\mathcal{L}^{(0)} = \partial_w^2 &- \frac{2}{(h+\eta)^2}[(h+z)(h+\eta)\eta_w]\partial_w\partial_z - \frac{1}{(h+\eta)^2}[(h+z)[(h+\eta)\eta_{ww}+\eta_w^2]]\partial_z \\
&+ \frac{2}{(h+\eta)^3}(h+z)(h+\eta)\eta_w^2\partial_z + \frac{1}{(h+\eta)^4}[(h+z)(h+\eta)\eta_w]^2\partial_z^2 \\
&+ \frac{2}{(h+\eta)^4}(h+z)(h+\eta)^2\eta_w^2\partial_z + \frac{h^2}{(h+\eta)^2}\partial_z^2,
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{L}^{(1)} = &-\frac{2}{(h+\eta)^2}[(h+z)\eta_w\zeta^{(1)} - (h+\eta)z\zeta_w^{(1)}]\partial_w\partial_z + \frac{2}{(h+\eta)^3}[(h+z)\eta_w^2\zeta^{(1)} - (h+\eta)z\eta_w\zeta_w^{(1)}]\partial_z \\
&- \frac{1}{(h+\eta)^2}[(h+z)(\eta_{ww}\zeta^{(1)}+\eta_w\zeta_w^{(1)}) - \eta_w z\zeta_w^{(1)} - (h+\eta)z\zeta_{ww}^{(1)}]\partial_z \\
&+ \frac{2}{(h+\eta)^4}[(h+z)(h+\eta)\eta_w][(h+z)\eta_w\zeta^{(1)} - (h+\eta)z\zeta_w^{(1)}]\partial_z^2 \\
&+ \frac{2}{(h+\eta)^4}(h+\eta)\eta_w[(h+z)\eta_w\zeta^{(1)} - (h+\eta)z\zeta_w^{(1)}]\partial_z \\
&+ \frac{2}{(h+\eta)^4}[\eta_w\zeta^{(1)} - (h+\eta)\zeta_w^{(1)}](h+z)(h+\eta)\eta_w\partial_z + \frac{2h^2}{(h+\eta)^3}\zeta^{(1)}\partial_z^2.
\end{aligned}
$$

To solve for $\tilde{\Psi}_2^{(0)}$ and $\tilde{\Psi}_2^{(1)}$, we plug our approximate solution into (4.5) and consider the respective $O(\epsilon^i)$ terms. For $O(1)$, we have:

$$
\begin{aligned}
\mathcal{L}^{(0)}\tilde{\Psi}_2^{(0)} &= 0 \\
\tilde{\Psi}_2^{(0)} &= C \qquad \text{on } z=0 \\
\tilde{\Psi}_2^{(0)} &= 0 \qquad \text{on } z=-h.
\end{aligned}
\tag{6.1}
$$

For $O(\epsilon)$, we have:

$$\begin{aligned}\mathcal{L}^{(0)}\tilde{\Psi}_2^{(1)} &= -\mathcal{L}^{(1)}\tilde{\Psi}^{(0)} \\ \tilde{\Psi}_2^{(1)} &= 0 \quad \text{on } z=0 \\ \tilde{\Psi}_2^{(1)} &= 0 \quad \text{on } z=-h.\end{aligned} \tag{6.2}$$

6.2. **Approximate expression for the Dirichlet-Neumann operator.** Using the results from the previous section, we now have an approximation for the Dirichlet-to-Neumann operator, $\tilde{\mathcal{G}}_2$:

$$\begin{aligned}\tilde{\mathcal{G}}_2(\eta, -h+\epsilon\zeta^{(1)})C &= [\partial_{n(x,y)}\Psi_2]_{|y=\eta(x)} \\ &= [(-\eta_x\partial_x + \partial_y)\Psi_2]_{|y=\eta(x)} \\ &= [(-\eta_w[\partial_w - \frac{1}{(h+\eta)^2}[(h+z)(h+\eta)\eta_w + \epsilon[(h+z)\eta_w\zeta^{(1)} - (h+\eta)z\zeta_w^{(1)}]]\partial_z] \\ &\quad + \frac{h}{(h+\eta)^2}[h+\eta+\epsilon\zeta^{(1)}]\partial_z)(\tilde{\Psi}_2^{(0)} + \epsilon\tilde{\Psi}_2^{(1)})]_{|z=0} + O(\epsilon^2) \\ &= [-\eta_w\partial_w\tilde{\Psi}_2^{(0)} + \frac{1}{h+\eta}[(h+z)\eta_w^2 + h]\partial_z\tilde{\Psi}_2^{(0)} - \epsilon(\eta_w\partial_w\tilde{\Psi}_2^{(1)} - \frac{1}{h+\eta}[(h+z)\eta_w^2 + h]\partial_z\tilde{\Psi}_2^{(1)}) \\ &\quad + \epsilon\frac{1}{(h+\eta)^2}[(h+z)\eta_w^2\zeta^{(1)} - (h+\eta)\eta_w z\zeta_w^{(1)} + h\zeta^{(1)}]\partial_z\tilde{\Psi}_2^{(0)}]_{|z=0} + O(\epsilon^2)\end{aligned}$$

Define $\tilde{\mathcal{G}}_2^{\text{app}}$ to be the operator found by taking the expression for $\tilde{\mathcal{G}}_2$ above and dropping the $O(\epsilon^2)$ terms. We define $\tilde{\mathcal{N}}_2^{\text{app}}$ analogously. The approximate model (1.9) is found by replacing $\tilde{\mathcal{N}}_2$ and $\tilde{\mathcal{G}}_2$ by the approximate operators:

7. Small amplitude flows over a non-flat bottom

Having derived our approximate model in Section 6, we are now in a position to investigate small amplitude flows over an impermeable non-flat bottom, $-h+\epsilon\zeta^{(1)}$. In order to ultimately prove the existence of small amplitude, small circulation flows over a region bounded by a non-flat free surface and a non flat-bottom, we first prove an auxiliary, albeit important, result: the non-existence of small amplitude flows over a region bounded by a flat free surface and a non flat-bottom.

7.1. Non-existence of small-amplitude flows with non-flat bottom and flat free surface.

Proposition 7.1. *Consider the approximate problem in a fluid region that is bounded by a flat free surface and a non-flat bottom, there does not exist any steady, incompressible and irrotational flow in such region.*

We will use this proposition to prove Theorem 1.2, which states the non-existence of small-amplitude flows in a two-fluid region with non-flat bottom and flat free surface. The proof for this theorem can be found in Section 7.1.1. The proof of the above proposition is as follows:

Proof. Suppose that free surface is flat, i.e. $\eta \equiv 0$. We need only substitute $\eta = 0$ into (6.1) and (6.2), and we get

$$\begin{aligned}\Delta_{(w,z)}\tilde{\Psi}^{(0)} &= 0 \\ \tilde{\Psi}^{(0)} &= 0 \quad \text{on } \{z=-h\} \\ \tilde{\Psi}^{(0)} &= C \quad \text{on } \{z=0\},\end{aligned} \tag{7.1}$$

which has a unique solution

$$\tilde{\Psi}^{(0)} = \frac{C(z+h)}{h}. \tag{7.2}$$

Also, $\tilde{\Psi}^{(1)}$ solves the system

$$\begin{aligned} -\Delta_{(w,z)}\tilde{\Psi}^{(1)} &= \left[\frac{2z}{h}\zeta^{(1)}_w\partial_z w + \frac{z}{h}\zeta^{(1)}_{ww}\partial_z + \frac{2}{h}\zeta^{(1)}\partial_z^2\right]\tilde{\Psi}^{0)} \\ \tilde{\Psi}^{(1)} &= 0 \quad \text{on } \{z=-h\} \\ \tilde{\Psi}^{(1)} &= 0 \quad \text{on } \{z=0\} \end{aligned} \tag{7.3}$$

Substituting the solution for $\tilde{\Psi}^{(0)}$ into (7.3), we get

$$\Delta_{(w,z)}\tilde{\Psi}^{(1)} = -\frac{Cz}{h^2}\zeta^{(1)}_{ww}.$$

Note that the right hand side can be rewritten

$$\begin{aligned} \Delta_{(w,z)}\tilde{\Psi}^{(1)} &= -\frac{Cz}{h^2}\zeta^{(1)}_{ww} - \frac{Cz}{h^2}\zeta^{(1)}_{zz} \\ &= -\Delta_{(w,z)}\frac{Cz}{h^2}\zeta^{(1)} \end{aligned}$$

Let $u := \tilde{\Psi}^{(1)} + Cz\zeta^{(1)}/h^2$, then we have that

$$\Delta_{(w,z)}u = 0.$$

Moreover, the boundary conditions for $\tilde{\Psi}^{(1)}$ becomes

$$\begin{aligned} u &= -\frac{C}{h}\zeta^{(1)} \quad \text{on } \{z=-h\} \\ u &= 0 \quad \text{on } \{z=0\}. \end{aligned}$$

By assumption, $\zeta^{(1)}$, and hence u, is 2π-periodic and even. Expanding both as cosine series gives

$$u = \sum_{k=0}^{\infty}\widehat{u}(k,z)\cos(kw) \quad \text{and} \quad \zeta^{(1)} = \sum_{k=0}^{\infty}\widehat{\zeta}^{(1)}_k\cos(kw).$$

Since u is harmonic, its Fourier cosine series coefficients satisfy the following ODE

$$-k^2\widehat{u}(k) + \widehat{u}_{zz}(k) = 0.$$

It follows that $\widehat{u}$ is given by

$$\widehat{u} = A_k\cosh(kz) + B_k\sinh(kz),$$

for some constants A_k, $B_k \in \mathbb{R}$. Imposing the boundary conditions, we find that

$$\widehat{u} = \frac{C}{h}\operatorname{csch}(kh)\sinh(kz)\widehat{\zeta}^{(1)}_k.$$

Substituting the above expression into the definition of u, we can solve for $\tilde{\Psi}^{(1)}$

$$\tilde{\Psi}^{(1)} = \sum_{k=0}^{\infty}\frac{C}{h^2}\widehat{\zeta}^{(1)}_k\left[h\operatorname{csch}(kh)\sinh(kz) - z\right]\cos(kw).$$

Next, we evaluate the Dirichlet-to-Neumann operator with $\eta = 0$ and $\zeta = -h + \epsilon\zeta^{(1)}$, and note that the normalized Dirichlet-to-Neumann operator is the same as the non-normalized one, because at $\eta = 0$,

$$\mathcal{N}(0,\zeta)C = \frac{1}{\sqrt{1+0^2}}\mathcal{G}(0,\zeta)C = \mathcal{G}(0,\zeta)C.$$

Thus,

$$
\begin{aligned}
\mathcal{N}(0,-h+\epsilon\zeta^{(1)})C &= \left.\frac{\partial\Psi}{\partial n}\right|_{y=0}\\
&= \left.\frac{\partial\Psi}{\partial y}\right|_{y=0}\\
&= \frac{h}{h-\epsilon\zeta^{(1)}}\left[\tilde{\Psi}_z^{(0)}+\epsilon\tilde{\Psi}_z^{(1)}\right]_{z=0}+O(\epsilon^2)\\
&= (1+\frac{\epsilon}{h}\zeta^{(1)})\left[\tilde{\Psi}_z^{(0)}+\epsilon\tilde{\Psi}_z^{(1)}\right]_{z=0}+O(\epsilon^2)\\
&= (1+\frac{\epsilon}{h}\zeta^{(1)})\left\{\frac{C}{h}+\epsilon\sum_{k=0}^{\infty}\frac{C}{h^2}\widehat{\zeta}_k^{(1)}\left(kh\,\mathrm{csch}(kh)-1\right)\cos(kw)\right\}+O(\epsilon^2).
\end{aligned}
$$

Thus, define the approximate Dirichlet-to-Neumann operator

$$
\tilde{\mathcal{N}}^{\mathrm{app}}(0,-h+\epsilon\zeta^{(1)})C := \frac{C}{h}\left\{1+\epsilon\sum_{k=0}^{\infty}k\,\mathrm{csch}(kh)\widehat{\zeta}_k^{(1)}\cos(kw)\right\}. \tag{7.4}
$$

Substituting (7.4) into the approximate model gives

$$
\frac{C}{h}\left|1+\epsilon\sum_{k=0}^{\infty}k\,\mathrm{csch}(kh)\widehat{\zeta}_k^{(1)}\cos(kw)\right| = \sqrt{Q} \tag{7.5}
$$

for some constants C, h and Q. Since we are only interested in what happens when ϵ is taken sufficiently small, i.e. when

$$
\sup_w\left|\epsilon\sum_{k=0}^{\infty}k\,\mathrm{csch}(kh)\widehat{\zeta}_k^{(1)}\cos(kw)\right|<1.
$$

It follows that

$$
1+\epsilon\sum_{k=0}^{\infty}k\,\mathrm{csch}(kh)\widehat{\zeta}_k^{(1)}\cos(kw)>0.
$$

Therefore, the equality (7.5) is the same the statement as

$$
\frac{C}{h}\left(1+\epsilon\sum_{k=0}^{\infty}k\,\mathrm{csch}(kh)\widehat{\zeta}_k^{(1)}\cos(kw)\right)-\sqrt{Q}=0.
$$

Observe that the inner product in L_w^2 of the left-hand side of the above equality with $\cos(jw)$ for positive integers j also vanishes, i.e.

$$
\left(\frac{C}{h}\left(1+\epsilon\sum_{k=0}^{\infty}k\,\mathrm{csch}(kh)\widehat{\zeta}_k^{(1)}\cos(kw)\right)-\sqrt{Q},\cos(jw)\right)_{L_w^2}=0 \quad \text{for} \quad j\geq 0 \tag{7.6}
$$

First, consider when $j=0$. Then (7.6) becomes

$$
\int_0^{2\pi}\left(\frac{C}{h}\left(1+\epsilon\sum_{k=0}^{\infty}k\,\mathrm{csch}(kh)\widehat{\zeta}_k^{(1)}\cos(kw)\right)-\sqrt{Q}\right)dw=0
$$

Evaluating the integral gives

$$0 = \frac{2\pi C}{h} - 2\pi\sqrt{Q} + \epsilon \sum_{k=0}^{\infty} k \operatorname{csch}(kh)\widehat{\zeta}_k^{(1)} \int_0^{2\pi} \cos(kw)dw$$
$$= \frac{2\pi C}{h} - 2\pi\sqrt{Q} + 2\pi\epsilon\widehat{\zeta}_0^{(1)}.$$

It follows that

$$\widehat{\zeta}_0^{(1)} = \frac{h\sqrt{Q} - C}{\epsilon},$$

which equals to a constant. Next, consider when integer $j > 0$, (7.6) becomes

$$0 = \left(\frac{C}{h} - \sqrt{Q}\right) \int_0^{2\pi} \cos(jw)dw + \epsilon \sum_{k=0}^{\infty} k \operatorname{csch}(kh)\widehat{\zeta}_k^{(1)} \int_0^{2\pi} \cos(kw)\cos(jw)dw$$

The first integral term vanishes for all $j > 0$, and the second integral vanishes for all $k \neq j$, hence,

$$0 = \epsilon j \operatorname{csch}(jh)\widehat{\zeta}_j^{(1)} \int_0^{2\pi} \cos^2(jw)dw.$$

It follows that

$$\widehat{\zeta}_j^{(1)} = 0 \quad \text{for all} \quad j > 0.$$

Hence, only the zeroth order coefficient does not vanish for the Fourier cosine series of $\zeta^{(1)}$, which implies that $\zeta^{(1)}$ must equal some constant. ■

7.1.1. *Non-existence of small amplituder flows over region with flat interface and non-flat bottom.* See Figure 4. for a picture of the domain.

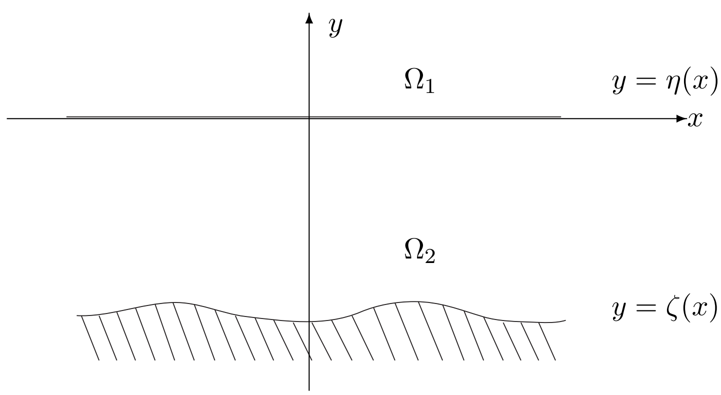

FIGURE 4. Domain with non-flat bottom and free flat surface

Proof of Theorem 1.2. The proof is almost identical to the one-fluid case. The only difference between the two situations is that the Bernoulli condition is now stated as

$$\frac{1}{2}|\nabla\Phi_1|^2 - \frac{1}{2}|\nabla\Phi_2|^2 + g(\rho_1 - \rho_2)\eta = Q.$$

Or equivalently,

$$|\tilde{\mathcal{N}}_1(\eta,U)C|^2 - |\tilde{\mathcal{N}}_2^{\text{app}}(0(\eta,\zeta)C|^2 + 2g(\rho_1-\rho_2)\eta - Q' = 0,$$

for $Q' = 2Q$. Since we are considering the case in which $\eta = 0$, the above simplifies further to

$$|\tilde{\mathcal{N}}_1(\eta,U)C|^2 - |\tilde{\mathcal{N}}_2^{\text{app}}(\eta,\zeta)C|^2 - Q' = 0,$$

where $\tilde{\mathcal{N}}_2^{\text{app}}$ is the same as the approximate form of Dirichlet-to-Neumann operator shown previously:

$$\tilde{\mathcal{N}}_2^{\text{app}}(0,\zeta) = \tilde{\mathcal{N}}(0,-h+\epsilon\zeta^{(1)})C := \frac{C}{h}\left\{1+\epsilon\sum_{k=0}^{\infty} k\,\text{csch}(kh)\widehat{\zeta}_k^{(1)}\cos(kw)\right\}.$$

and $\tilde{\mathcal{N}}_1$ for the system (4.3) can be computed. For $\eta = 0$, (4.3) becomes

$$\begin{aligned}\mathcal{L}_{(w,z)}\tilde{\xi} &= 0\\ \tilde{\xi} &= C \quad \text{on } z=0\\ \tilde{\xi}_z &= 0 \quad \text{as } z\to\infty\end{aligned}$$

Recall that $\tilde{\xi}$ is defined in section 4.1. Solving explicitly, we have

$$\tilde{\xi} = \sum_{k\geq 0}\sqrt{\rho_1}U[\cosh(kz)-\sinh(kz)]\widehat{v}_k\cos(kw).$$

Thus we get

$$\mathcal{N}_1(0,U)C = \frac{1}{\sqrt{1+0^2}}\mathcal{G}_1(0,U)C = \mathcal{G}_1(0,U)C = -\sqrt{\rho_1}U.$$

Plugging the two identities into the approximate model, we get

$$\begin{aligned}0 &= \rho_1U^2 - \frac{C^2}{h^2}\left(1+\epsilon\sum_{k=0}^{\infty}k\,\text{csch}(kh)\widehat{\zeta}_k^{(1)}\cos(kw)\right)^2 - Q'\\ &= \rho_1U^2 - \frac{C^2}{h^2} - Q' - \epsilon\left(\frac{2C^2}{h^2}\sum_{k=0}^{\infty}k\,\text{csch}(kh)\widehat{\zeta}_k^{(1)}\cos(kw)\right).\end{aligned}$$

Intuitively, all the Fourier modes vanish except for $k = 0$ for the above equality to hold. The proof is similar to the proof given for 7.1 and is as follows. Take the inner product of both sides with $\cos(lw)$ for $l \in \mathbb{N}$, and we get

$$0 = \int_0^{2\pi}\left(\rho_1U^2 - \frac{C^2}{h^2} - Q'\right)\cos(lw)dw - \frac{2C^2\epsilon}{h^2}\int_0^{2\pi}\sum_{k=0}^{\infty}k\,\text{csch}(kh)\widehat{\zeta}_k^{(1)}\cos(kw)\cos(lw)dw.$$

For $l = 0$, we have that $\cos(lw) = \cos(0) = 1$. Then, the above becomes

$$\begin{aligned}0 &= 2\pi\left(\rho_1U^2 - \frac{C^2}{h^2} - Q'\right) - \frac{2C^2\epsilon}{h^2}\sum_{k=0}^{\infty}k\,\text{csch}(kh)\widehat{\zeta}_k^{(1)}\int_0^{2\pi}\cos(kw)dw\\ &= 2\pi\left(\rho_1U^2 - \frac{C^2}{h^2} - Q'\right) - 2\pi\frac{2C^2\epsilon\widehat{\zeta}_0^{(1)}}{h^3},\end{aligned}$$

which implies that

$$\widehat{\zeta}_0^{(1)} = \frac{\rho_1U^2h^3 - C^2h - Q'h^3}{2C^2\epsilon}$$

is a constant. Next, for $l > 0$, we have that

$$0 = \left(\rho_1 U^2 - \frac{C^2}{h^2} - Q'\right) \int_0^{2\pi} \cos(lw)dw - \frac{2C^2\epsilon}{h^2} \sum_{k=0}^{\infty} k \operatorname{csch}(kh) \widehat{\zeta}_k^{(1)} \int_0^{2\pi} \cos(kw)\cos(lw)dw$$
$$= -\frac{2C^2\epsilon}{h^2} l \operatorname{csch}(lh) \widehat{\zeta}_l^{(1)} \int_0^{2\pi} \cos^2(lw)dw,$$

which implies that

$$\widehat{\zeta}_l^{(1)} = 0 \quad \text{for all} \quad l > 0.$$

Hence, it is proven $\zeta^{(1)}$ must equal some constant. Moreover, small amplitude flows do not exist for non-flat bottom. ■

7.2. **Existence of small amplitude flows in a region with a non-flat free surface and a non-flat bottom.** We fix $\epsilon = \epsilon_0$, $u_1 = U$, and $\zeta^{(1)} = \zeta_0^{(1)}$ — where $\zeta_0^{(1)}$ is a non-constant, 2π-periodic, even function. It is appropriate that we define a function, $\mathcal{F}(\eta, C, Q)$, as

$$\mathcal{F}(\eta, C, Q) = \frac{1}{2(1+\eta_w^2)}[|\tilde{\mathcal{G}}_1(\eta)C|^2 - |\tilde{\mathcal{G}}_2^{\text{app}}(\eta)C|^2] + g(\rho_1 - \rho_2)\eta - Q$$

whose zero set is exactly the solution to (1.9). At $\eta = 0$, $C = 0$ and $Q = 0$, we observe that

$$\mathcal{F}(0,0,0) = 0 \tag{7.7}$$

and hence $(0,0,0)$ is a solution to (1.9).

Proof of Theorem 1.3. At $C = 0$, we have that, from (6.1) and (6.2) respectively, $\tilde{\Psi}_2^{(0)} = 0$ and $\tilde{\Psi}_2^{(1)} = 0$. Hence, we know that $\dot{\tilde{\Psi}}_2^{(0)} = 0$ and $\dot{\tilde{\Psi}}_2^{(1)} = 0$. Therefore,

$$D_\eta \mathcal{F}(0,0,0)v = \sum_{k\geq 0} [\rho_1 U^2 k + g(\rho_1 - \rho_2)] \widehat{v}_k \cos(kw). \tag{7.8}$$

Therefore, for

$$k_0 \neq \frac{g(\rho_2 - \rho_2)}{\rho_1 U^2}$$

where k_0 is some arbitrary integer, $D_\eta \mathcal{F}(0,0,0)v$ is a bijective mapping from $X \times \mathbb{R} \times \mathbb{R}$ to $Y \times \mathbb{R} \times \mathbb{R}$. Therefore, applying the Implicit Function Theorem to (7.7) and (7.8), we can conclude that there is a curve of solutions $(C, \eta(C), Q(C))$ such that (1.9) holds for all $C \in (-\delta, \delta)$

Furthermore, from Theorem 1.2, we know that, for $C \neq 0$ and $\zeta_0^{(1)} \neq a$ where a is some arbitrary constant, it is impossible to have a flat free surface, η. Thus, $\eta(C)$ is not identically flat. ■

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Appendix A. Numerical Results

```
AUTO> run(dd2)
gfortran -fopenmp -O -c dd2.f90 -o dd2.o
gfortran -fopenmp -O dd2.o -o dd2.exe /Users/AllieWang/Desktop/Research12/auto/07p/lib/*.o
Starting dd2 ...

  BR    PT  TY  LAB       C          L2-NORM        eta1          eta2           Q
   1     1  EP    1   2.50000E+00   3.10000E+00   0.00000E+00   0.00000E+00   3.10000E+00
   1     7  BP    2   2.59773E+00   3.37410E+00   0.00000E+00   0.00000E+00   3.37410E+00
   1    16  BP    3   2.89440E+00   4.18877E+00   0.00000E+00   0.00000E+00   4.18877E+00
   1    20  UZ    4   3.00003E+00   4.50009E+00   0.00000E+00   0.00000E+00   4.50009E+00
   1    38  UZ    5   3.50000E+00   6.12500E+00   0.00000E+00   0.00000E+00   6.12500E+00
   1    58  UZ    6   4.00002E+00   8.00007E+00   0.00000E+00   0.00000E+00   8.00007E+00
   1    80  UZ    7   4.50001E+00   1.01250E+01   0.00000E+00   0.00000E+00   1.01250E+01
   1   100  EP    8   4.91556E+00   1.20814E+01   0.00000E+00   0.00000E+00   1.20814E+01

  BR    PT  TY  LAB       C          L2-NORM        eta1          eta2           Q
   2   100  EP    9   2.59773E+00   1.19535E+01   0.00000E+00   7.67508E-01   1.19288E+01

  BR    PT  TY  LAB       C          L2-NORM        eta1          eta2           Q
   2    46  BP   10   2.59773E+00   6.53791E+00   0.00000E+00  -4.65526E-01   6.52132E+00
   2   100  EP   11   2.59773E+00   1.19374E+01   0.00000E+00  -7.66788E-01   1.19127E+01

  BR    PT  TY  LAB       C          L2-NORM        eta1          eta2           Q
   3    29  UZ   12   3.00000E+00   5.47635E+00   2.83368E-01   1.33351E-01   5.46739E+00
   3   100  EP   13   3.26382E+00   1.25694E+01   6.02664E-01   4.11757E-01   1.25482E+01

  BR    PT  TY  LAB       C          L2-NORM        eta1          eta2           Q
   3    29  UZ   14   3.00000E+00   5.47635E+00  -2.83368E-01   1.33351E-01   5.46739E+00
   3   100  EP   15   3.26382E+00   1.25694E+01  -6.02664E-01   4.11757E-01   1.25482E+01

  BR    PT  TY  LAB       C          L2-NORM        eta1          eta2           Q
   4    41  UZ   16   2.50000E+00   9.62611E+00   3.54835E-01  -6.56321E-01   9.59715E+00
   4   100  EP   17   2.37694E+00   1.55234E+01   6.59941E-01  -9.30855E-01   1.54814E+01

  BR    PT  TY  LAB       C          L2-NORM        eta1          eta2           Q
   4    41  UZ   18   2.50000E+00   9.62611E+00  -3.54835E-01  -6.56321E-01   9.59715E+00
   4   100  EP   19   2.37694E+00   1.55234E+01  -6.59941E-01  -9.30855E-01   1.54814E+01
```

Figure 5. output from AUTO

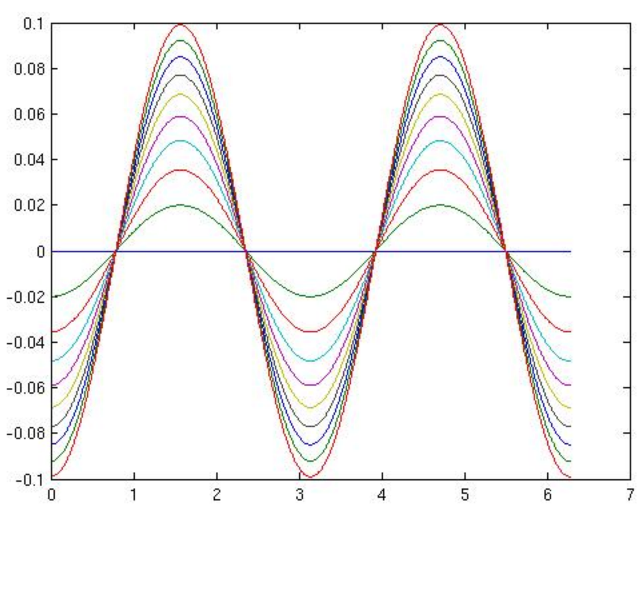

Figure 6. Solution Branch #2

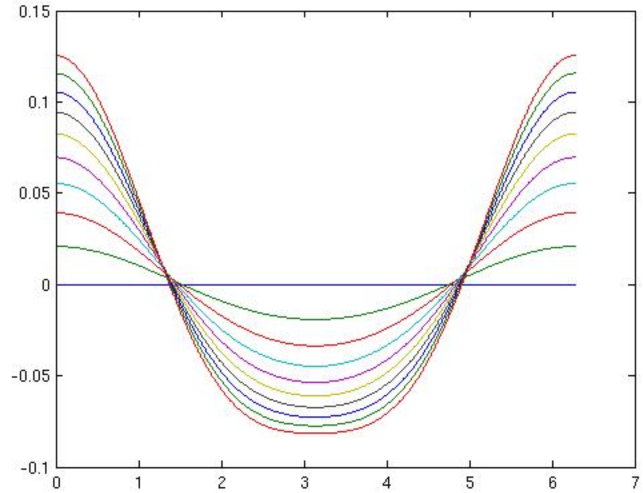

FIGURE 7. Solution Branch #3

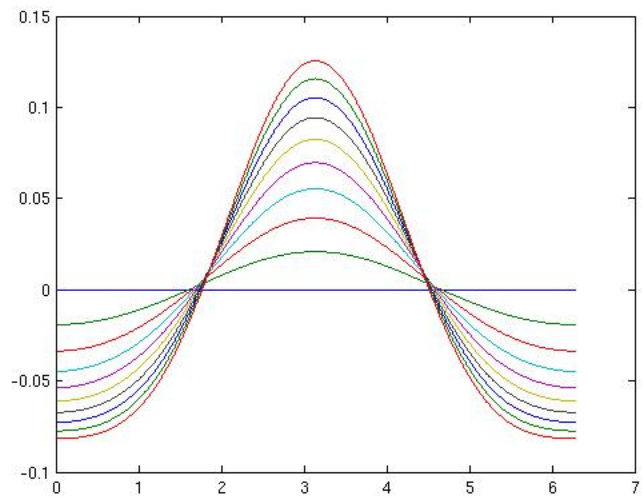

FIGURE 8. Solution Branch #4