Exam 1, Spring 2014
Math for Economics II (Lecture 6)
New York University, Jankowski

Name:______________________________ Recitation Section:____________________

Read all of the following information before starting the exam:

- For multiple choice and true / false questions, you do not need to show work, and no partial credit will be awarded.
- For free response questions, you must show all work, clearly and in order, if you want to get full credit. A correct answer without supporting work will receive little or no credit.
- The exam is 75 minutes. **Good luck!**

SCORES

1-3 (6 pts)	
4-7 (14 pts)	
Free Response 1 (11 pts)	
Free Response 2 (10 pts)	
Free Response 3 (8 pts)	
Free Response 4 (10 pts)	
Free Response 5 (11 pts)	
TOTAL (70 pts)	

1-6 are multiple choice, 2 points each. No work is required, and no partial credit is awarded.

1. If $f(x,y) = Ax^a y^b$ (where A, a, b are positive constants), the differential of f is

   (a) $df = Ax^a\,dx + Ay^b\,dy$

   (b) $df = aAx^a y^b\,dx + bAx^a y^b\,dy$

   (c) $df = aAx^{a-1}y^b + bAx^a y^{b-1}$

   (d) $\boxed{df = aAx^{a-1}y^b\,dx + bAx^a y^{b-1}\,dy}$

   (e) none of the above

2. The equation of the plane tangent to the graph of $f(x,y) = 4 - x^2 + y^2$ at $(2,1,1)$ is

   (a) $z = 7 - 4(x-2) + 2(y-1)$

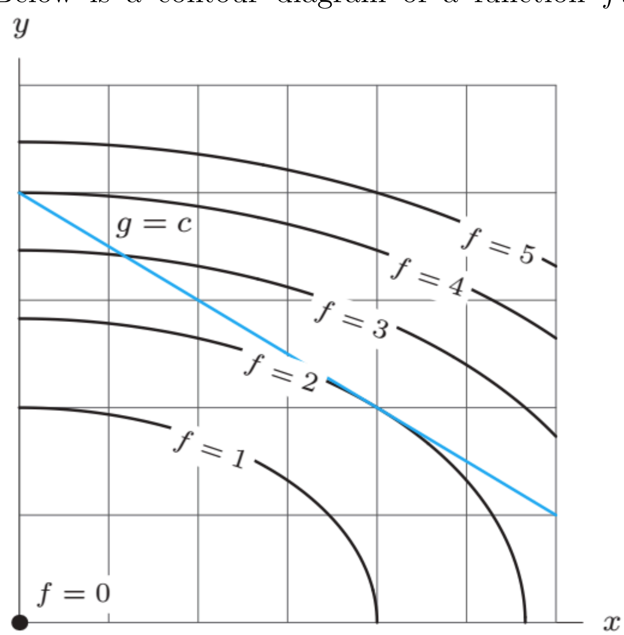
   (b) $\boxed{z = 7 - 4x + 2y}$

   (c) $z = 3 - 2x + 2y$

   (d) $0 = 6 - 4x + 2y$

   (e) none of the above

3. Below is a contour diagram of a function f. The constraint $g = c$ is the straight line.



   Which of the following is true?

   (a) The maximum value of f on $g = c$ is 2 and it occurs at the point $(4,2)$

   (b) $\boxed{\text{The minimum value of } f \text{ on } g = c \text{ is 2 and it occurs at the point } (4,2)}$

   (c) The minimum value of f on the region below the line $g = c$ is 2.

   (d) The maximum value of f on the region below the line $g = c$ is 2.

   (e) Items (b) and (d) are both true.

4. Let $\mathbf{w} = \langle -2, 7 \rangle$. If $\mathbf{u}$ is a unit vector, then the maximum possible value for $\mathbf{w} \cdot \mathbf{u}$ is

   (a) $\boxed{\sqrt{53}}$
   (b) 53
   (c) 9
   (d) there is not enough information
   (e) none of the above

5. The angle θ formed by the vectors $\langle 4, 3 \rangle$ and $\langle -3, -4 \rangle$ is

   (a) $90°$
   (b) $\cos^{-1}(\frac{7}{25})$
   (c) $\cos^{-1}(-24)$
   (d) $\boxed{\cos^{-1}(-\frac{24}{25})}$
   (e) none of the above

6. Let $f(x, y) = x^2 + 2e^y$. Which of the following vectors is perpendicular to the level curve $f = 6$ at the point $(2, 0)$?

   (a) $\langle 4, 0 \rangle$
   (b) $\langle 4, -2 \rangle$
   (c) $\boxed{\langle 4, 2 \rangle}$
   (d) $\langle 4, 1 \rangle$
   (e) none of the above

7. True or false, 2 pts each. If the statement is *ever* false, circle false as your answer.
   No work is required, and no partial credit will be given.

   In each case, assume f is a smooth function (its derivatives of all orders exist and are continuous). If f has a constraint $g = c$, assume that g is smooth and that ∇g is never $\mathbf{0}$.

   (a) If (x_0, y_0) is the point where f attains its minimum subject to the constraint $g(x, y) = c$, then ∇f and ∇g point in opposite directions at (x_0, y_0). TRUE $\boxed{\text{FALSE}}$

   (b) The function $f(x, y) = x - y^2$ has a minimum subject to the constraint $x - y = 1$.
   TRUE $\boxed{\text{FALSE}}$

   (c) If A is a 4×3 matrix and B is a 2×3 matrix, then the product AB is undefined.
   $\boxed{\text{TRUE}}$ FALSE

   (d) If $\mathbf{u}$ is a unit vector which is tangent to the level curve of f at (a, b), then $D_{\mathbf{u}} f(a, b) = 0$.
   $\boxed{\text{TRUE}}$ FALSE

Free response. Show your work and justify your answers!

1. Let $f(x,y) = xe^{y-x}$.

 (a) (5 pts) Find the derivative of f at $(1,1)$ in the direction from $(1,1)$ to $(5,-2)$.

 Solution: $\nabla f = \langle e^{y-x} - xe^{y-x}, xe^{y-x}\rangle = \langle (1-x)e^{x-y}, xe^{y-x}\rangle$, so $\nabla f(1,1) = \langle 0, 1\rangle$

 The vector from $(1,1)$ to $(5,-2)$ is $\langle 4,-3\rangle$, which has associated unit vector $\langle \frac{4}{5}, -\frac{3}{5}\rangle$.

 Therefore,

$$D_{\mathbf{u}}f(1,1) = \nabla f(1,1)\cdot \mathbf{u} = \langle 0,1\rangle \cdot \langle \frac{4}{5}, -\frac{3}{5}\rangle = \boxed{-\frac{3}{5}}.$$

 (b) (2 pts) Find the unit vector pointing in the direction of greatest increase of f at $(2,2)$.

 Solution: $\nabla f(2,2)$ points in the direction of greatest increase, and

$$\nabla f(2,2) = \langle (1-2)e^{2-2}, 2e^{2-2}\rangle = \langle -1, 2\rangle.$$

 The unit vector in this direction is $\boxed{\mathbf{v} = \left\langle -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right\rangle}$.

 (c) (4 pts) Is there some (a,b) such that $\langle 2,-1\rangle$ points in the direction of greatest increase of f at (a,b)? If so, find one such point (a,b). If not, show that no such (a,b) exists.

 Solution: For this to work, we need $\nabla f(a,b) = \lambda\langle 2,-1\rangle$ for some $\lambda > 0$ (same direction!).

 So

$$\langle (1-a)e^{b-a}, ae^{b-a}\rangle = \langle 2\lambda, -\lambda\rangle.$$

 Thus $\lambda = \frac{(1-a)e^{b-a}}{2} = -ae^{b-a}$. Since $e^{b-a} \neq 0$, we cancel it out and get

$$\frac{1-a}{2} = -a, \qquad 1-a = -2a, \qquad \boxed{a=-1}.$$

 Thus, $a=-1$, and in fact b *can be any number we desire*!
 We check that $\lambda > 0$: $\lambda = -ae^{b-a} = -(-1)e^{b+1} = e^{b+1} > 0$ no matter what b is.

2. (10 pts) Find the extreme values of $f(x,y) = (x-1)^2 + (y-3)^2$ in the region $x^2 + y^2 \le 40$, as well as the points where the extreme values occur.
If f does not have an absolute maximum in the region, justify why.
If f does not have an absolute minimum in the region, justify why.

Solution: The region $x^2 + y^2 \le 40$ is closed and bounded and $f$is smooth, so f is guaranteed to have an absolute max and min in the region.

Also, g is smooth and $\nabla g \ne 0$ on the circle $g = 40$, so we apply the 2-step process given by the Extreme Value Theorem.

Step 1: Solve $\nabla f = 0$ within $x^2 + y^2 < 40$. This gives us $\langle 2(x-1), 2(y-3)\rangle = \langle 0, 0\rangle$ and therefore $x = 1$, $y = 3$. We check

$$\boxed{f(1,3) = 0}.$$

Step 2: We solve $\nabla f = \lambda \nabla g$ on $x^2 + y^2 = 40$.

$$\langle 2(x-1), 2(y-3)\rangle = \langle 2\lambda x, 2\lambda y\rangle.$$

If $\lambda = 0$ then $x = 1$ and $y = 3$ (each left-side coordinate is 0), which is not on the constraint. If $x = 0$ then $2(x-1) = 0$ so $x = 1$, which is absurd. If $y = 0$ then immediately we get $y = 3$ which is also absurd.

Thus, we may divide by x and y to solve for λ:

$$\lambda = \frac{x-1}{x} = \frac{y-3}{y}, \qquad \frac{x-1}{x} = \frac{y-3}{y}, \qquad xy - y = xy - 3x, \qquad \boxed{y = 3x}.$$

We plug this into the constraint $x^2 + y^2 = 40$ to get

$$x^2 + (3x)^2 = 40, \qquad 10x^2 = 40, \qquad x = \pm 2.$$

Since $y = 3x$ we get the points $(2,6)$ and $(-2,-6)$. We test

$$\boxed{f(2,6) = 1^2 + 3^2 = 10, \qquad f(-2,-6) = (-3)^2 + (-9)^2 = 90}.$$

Thus, the min in the region is $f(1,3) = 0$, and the max is $f(-2,-6) = 90$.

3. (Each part is worth 4 points) A video game offers participants the chance to play as one of three characters: Archer, Barbarian, or Cleric.

The game has 10 million customers when it is released in 2013, and it maintains a steady customer base.

In 2013:
Archer is played by 5 million customers.
Barbarian is played by 2 million customers.
Cleric is played by 3 million customers.

One year later, in 2014:

- 50% of the people who started with the Archer still play with the Archer, while 30% have switched to Barbarian and 20% have switched to Cleric.
- 60% of the customers who stared with the Barbarian still play with the Barbarian, while 10% have switched to Archer and 30% have switched to Cleric.
- 70% of the customers who stared with the Cleric still play with the Cleric, while 10% have switched to Archer and 20% have switched to Barbarian.

(a) Write down the transition matrix which represents the change in each character's popularity from 2013 to 2014. Make it clear which order you are listing the characters.

Solution: Viewing the characters in alphabetical order A, B, C (A is char 1, B is char 2, C is char 3):

$$P = \begin{bmatrix} 0.5 & 0.1 & 0.1 \\ 0.3 & 0.6 & 0.2 \\ 0.2 & 0.3 & 0.7 \end{bmatrix}.$$

(b) Using your matrix from (a), find the number of people who played with each character in 2014.

$$\begin{bmatrix} 0.5 & 0.1 & 0.1 \\ 0.3 & 0.6 & 0.2 \\ 0.2 & 0.3 & 0.7 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} (0.5)(5) + (0.1)(2) + (0.1)(3) \\ (0.3)(5) + (0.6)(2) + (0.2)(3) \\ (0.2)(5) + (0.3)(2) + (0.7)(3) \end{bmatrix} = \begin{bmatrix} 3 \\ 3.3 \\ 3.7 \end{bmatrix}$$

3 million played with Archer
3.3 million played with Barbarian
3.7 million played with Cleric

4. (10 points) Given x, y, and z units of three different inputs, a company produces

$$P(x,y,z) = 6x^{1/2}y^{1/3}z \quad \text{units of its product.}$$

Each unit of the first input costs \$1, each unit of the second input costs \$2, and each unit of the third input costs \$1. The company's production budget is \$55.

Use Lagrange multipliers to find the values of each input which maximize production. (you may assume the method of Lagrange multipliers gives the maximum)

Solution: If any of the variables is 0 then $P = 0$, so we can assume x, y, z are all nonzero numbers.
The constraint is $x + 2y + z = 55$. Setting $\nabla f = \lambda \nabla g$, we get

$$\langle \frac{3y^{1/3}z}{x^{1/2}}, \frac{2x^{1/2}z}{y^{2/3}}, 6x^{1/2}y^{1/3} \rangle = \langle \lambda, 2\lambda, 1\lambda \rangle.$$

The first and third coordinate equal λ, while the second coordinate equals 2λ. Thus

$$\lambda = \frac{3y^{1/3}z}{x^{1/2}} = \frac{x^{1/2}z}{y^{2/3}} = 6x^{1/2}y^{1/3}.$$

We cross-multiply the equations given by the second equals sign for λ (recalling $z \neq 0$):

$$\frac{3y^{1/3}z}{x^{1/2}} = \frac{x^{1/2}z}{y^{2/3}}, \qquad 3yz = xz, \qquad \boxed{3y = x}.$$

We now cross-multiply the equations for the third equals sign for λ:

$$\frac{x^{1/2}z}{y^{2/3}} = 6x^{1/2}y^{1/3}, \qquad zx = 6xy, \qquad \boxed{z = 6y}.$$

The constraint $x + 2y + z = 55$ becomes

$$(3y) + 2y + (6y) = 55, \qquad 11y = 55, \qquad \boxed{y = 5}.$$

Thus $x = 3y = 15$ and $z = 6y = 30$.

Therefore, when production is maximized:

$$\boxed{x = 15, \quad y = 5, \quad z = 30}.$$

5. Mark does not pay attention to how much fruit costs at the store, but he keeps track of the quantity of fruit he buys and the total amount of money he spends.

He recalls three recent trips to the store and makes the following table for the food he bought:

	Cartons of blueberries	Cartons of strawberries	Cartons of clementines	Total Cost (in $)
Trip 1	16	32	4	208
Trip 2	9	12	6	87
Tip 3	14	35	14	224

(a) (4 pts) Let b, s, and c be the cost of blueberries, strawberries, and clementines, respectively. Write a system of equations that will allow us to solve for b, s, and c. You may write the system in longhand form or in matrix form.

In longhand form, it is

$$16b + 32s + 4c = 208,$$

$$9b + 12s + 6c = 87,$$

$$14b + 35s + 14c = 224.$$

(b) (7 pts; This part is unrelated to part (a)!)
Solve the following system of equations, if it has a solution. Make sure your steps are clear!

$$x + y - 2z = -2$$

$$-4x + 2y + 4z = -12$$

$$3x + y - 3z = 4$$

Solution:

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & -2 \\ -4 & 2 & 4 & -12 \\ 3 & 1 & -3 & 4 \end{array}\right] \to_{\text{row 2+4(row 1)}} \left[\begin{array}{ccc|c} 1 & 1 & -2 & -2 \\ 0 & 6 & -4 & -20 \\ 3 & 1 & -3 & 4 \end{array}\right] \to_{\text{row 3-3(row 1)}} \left[\begin{array}{ccc|c} 1 & 1 & -2 & -2 \\ 0 & 6 & -4 & -20 \\ 0 & -2 & 3 & 10 \end{array}\right]$$

$$\to_{\text{switch rows 2 and 3}} \left[\begin{array}{ccc|c} 1 & 1 & -2 & -2 \\ 0 & -2 & 3 & 10 \\ 0 & 6 & -4 & -20 \end{array}\right] \to_{\text{row 3 +3(row 2)}} \left[\begin{array}{ccc|c} 1 & 1 & -2 & -2 \\ 0 & -2 & 3 & 10 \\ 0 & 0 & 5 & 10 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 1 & -2 & -2 \\ 0 & -2 & 3 & 10 \\ 0 & 0 & 1 & 2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & -2 & -2 \\ 0 & -2 & 0 & 4 \\ 0 & 0 & 1 & 2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & -2 & 0 & 4 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

So $x = 4$, $y = -2$, and $z = 2$.

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