Math for Economics 1
New York University
FINAL EXAM, Fall 2013
VERSION A

Name:________________________________ ID:________________

Circle your instructor and lecture below:

Jankowski-001 Jankowski-006 Ramakrishnan-013

Read all of the following information before starting the exam:

- For multiple choice questions, only the answer is required. No work is required and no partial credit will be awarded. You must clearly circle your answer.
- For free response questions, you must show all work, clearly and in order, if you want to get full credit. We reserve the right to take off points if we cannot see how you arrived at your answer (even if your final answer is correct).
- The exam is closed book. You are not allowed to use a calculator or consult any notes while taking the exam.
- The exam time limit is 1 hour and 50 minutes. **Good luck!**

SCORES

MC (32 points)	
1 (18 pts)	
2 (16 pts)	
3 (16 pts)	
4 (18 pts)	
TOTAL	

(32 points) This parts consists of 16 multiple choice problems. Nothing more than the answer is required; consequently no partial credit will be awarded.

1. Find $\lim_{x \to 1^-} \frac{|x-1|}{x-1}$.

   (a) 1
   (b) the limit does not exist
   (c) $\boxed{-1}$
   (d) $-\infty$
   (e) none of the above

2. Suppose f is differentiable everywhere. Which of the following formulas are equal to $f'(a)$, for every a?

   I. $\lim_{h \to 0} \frac{f(h)-f(a)}{h-a}$ II. $\lim_{x \to a} \frac{f(x)-f(a)}{x-a}$ III. $\lim_{h \to 0} \frac{f(a+h)-f(h)}{a}$ IV. $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$

   (a) I and III
   (b) II only
   (c) II and III
   (d) $\boxed{\text{II and IV}}$
   (e) I and IV
   (f) none of the above

3. Find the value for the constant c which makes the function $f(x)$ continuous on $(-\infty, \infty)$, where f is given by

$$f(x) = \begin{cases} x^2 + cx & \text{if } x \leq 2 \\ c + x & \text{if } x > 2. \end{cases}$$

   (a) any value of c will work
   (b) -1
   (c) 0
   (d) $\boxed{-2}$
   (e) none of the above

4. The slope of the line tangent to the curve $e^{2y} = x^2 + y$ at $(-1, 0)$ is

   (a) $\boxed{-2}$
   (b) 4
   (c) 0
   (d) 3
   (e) none of the above

5. Let $f(x)$ be a smooth function (smooth means derivatives of all orders exist and are continuous) with exactly two inflection points, which occur at $x = -1$ and $x = 2$. If f has a local maximum at $x = 0$, then which of the following must be true?

I. $f''(x) \geq 0$ on $(-1, 2)$ II. $f(x) \geq 0$ on $(-1, 2)$ III. $f''(x) \leq 0$ on $(-1, 2)$

(a) I only
(b) I and II
(c) II only
(d) II and III
(e) $\boxed{\text{III only}}$

6. Find the values of A (if any) so that $f(x, y) = y^3 + Ay - x^2$ has a local maximum at $(0, -1)$.

(a) $A = 0$
(b) $\boxed{A = -3}$
(c) $A = -1$
(d) $A = 1$ and $A = -1$
(e) there is no value of A so that f has a local max at $(0, -1)$

7. In an economics model, the number of families with income less than x is given by

$$f(x) = a + 4(1 - e^{-cx}),$$

where a and c are positive constants. Find the differential df.

(a) $\boxed{4ce^{-cx}dx}$
(b) $a - 4c(1 - e^{-cx})$
(c) $-4(1 - e^{-cx})dx$
(d) $-4e^{-cx}dx$
(e) None of the above

8. Assume f is differentiable, and let $g(x) = \dfrac{f(x)}{1 - f(x)}$. Find $g'(x)$.

(a) $\dfrac{1 - 2f(x)f'(x)}{(1 - f(x))^2}$

(b) $\boxed{\dfrac{f'(x)}{(1 - f(x))^2}}$

(c) $\dfrac{1}{(1 - f(x))^2}$

(d) $\dfrac{f'(x) - 2f(x)f'(x)}{1 - f(x)}$

(e) none of the above

9. The demand function for a product is $q = e^{4-p} + 1$. Find the inverse demand function.

(a) $p = \ln(4 - q) + 1$

(b) $p = \ln(4 - q) - 1$

(c) $p = 1 - \ln(q - 4)$

(d) $p = \ln(q - 1) - 4$

(e) $\boxed{\text{none of the above}}$

10. For which of the following regions does the extreme value theorem guarantee that any continuous function attains an absolute maximum on the region?

I. $D = \{(x, y) : \ 0 \le x \le 3, \ y^2 \le 1\}$

II. $D = \{(x, y) : \ x^2 + y^2 < 3\}$

III. $D = \{(x, y) : \ 0 \le x \le 4, \ y \le 1\}$

IV. $D = \{(x, y) : \ 1 \le x^2 + y^2 \le 10\}$

(a) I and II

(b) I and III

(c) $\boxed{\text{I and IV}}$

(d) I, III, and IV

(e) I, II, and III

11. The critical points of $x^3 - y^2$ are

(a) All points where $x = 0$
(b) All points where $y = 0$
(c) All points where $x = \pm y$
(d) All points where $x = 0$
(e) $\boxed{\text{The point } (0,0)}$
(f) none of the above

12. The cross-section of the function $f(x, y) = xy^2 + 2e^{xy}$ at $y = 3$ is:

(a) The set of all points (x, y) such that $3 = xy^2 + 2e^{xy}$
(b) The set of all x such that $0 = 9x + 2e^{3x}$
(c) The set of all x such that $3 = 9x + 2e^{3x}$
(d) The function $9x + 2e^{x}$
(e) $\boxed{\text{The function } 9x + 2e^{3x}}$

13. Which of the following is / are x-coordinates of inflection points of $f(x) = x^3 - 2x + 2$?

(a) $x = \sqrt{2}/3$
(b) $x = \sqrt{2}/3$ and $x = -\sqrt{2}/3$
(c) $x = 2$
(d) $\boxed{x = 0}$
(e) none of the above

14. If the supply and demand equations satisfy

$$q_S = 10p - 15, \qquad q_D = 13 - 4p,$$

then the equilibrium *quantity* is

(a) 2
(b) 3
(c) 4
(d) $\boxed{5}$
(e) none of the above

15. If (a, b) is a saddle point of the smooth function $f(x, y)$, then which of the following *must* be true? (by smooth, we mean that all partial derivatives of all orders exist and are continuous)

I. At (a, b), $f(x, y)$ is increasing in x and decreasing in y

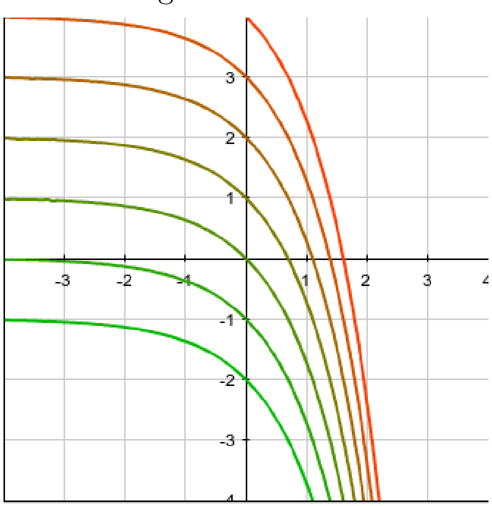
II. At (a, b), $f(x, y)$ is increasing in y and decreasing in x

III. $f_{xx}(a, b)$ and $f_{yy}(a, b)$ have opposite signs

IV. The partial derivatives of f with respect to x and with respect to y are both zero at (a, b).

(a) I and III
(b) II and III
(c) I and IV
(d) III and IV
(e) $\boxed{\text{IV only}}$

16. Which of the following functions could yield the contour diagram below? Each curve is exactly one level higher than the curve on its left.



(a) $f(x, y) = -x^2$.
(b) $f(x, y) = e^x + e^y$.
(c) $f(x, y) = y^2 - e^x$.
(d) $\boxed{f(x, y) = e^x + y}$.
(e) $f(x, y) = e^x - e^y$.

(68 points) Problems 1-4 are free response problems. Put your work/explanations in the space below the problem.

- **Read and follow the instructions of every problem.**
- **Show all your work for purposes of partial credit. Full credit may not be given for an answer alone.**
- **Justify your answers.**

1. (a) (12 pts) Let $f(x,y) = x^2 + 2x + 2y^2$. Find the global maximum and global minimum values that f attains on the region

$$D = \{(x,y) : x^2 + y^2 \leq 4\}.$$

At what point (or points) is the global maximum attained?
At what point (or points) is the global minimum attained?

Solution: Solving $f_x = f_y = 0$ gives us $x = -1$ and $y = 0$, so $(-1,0)$ is the only critical point in the interior. $f(-1,0) = -1$. On the boundary, $y^2 = 4 - x^2$, so

$$f(x,x^2) = x^2 + 2x + 2(4 - x^2) = -x^2 + 2x + 8, \qquad -2 \leq x \leq 2$$

When $x = -2$ we get $y = 0$ and $f(-2,0) = 0$. When $x = 2$ we get $y = 0$ and $f(2,0) = 8$. The critical number for the function of x is $x = 1$, giving is 9. Thus $f(1,\sqrt{3}) = 9$ and $f(1,-\sqrt{3}) = 9$. We summarize the results below.

Global min: $f(x,y) = -1$ at $(-1,0)$.

Global max: $f(x,y) = 9$ at $(1,\sqrt{3})$ and $(1,-\sqrt{3})$.

(b) (6 pts) For each function below, determine whether $(0,0)$ corresponds to a local max, a local min, or a saddle point. You must justify your answer to receive full credit.

I. $g(x,y) = x^4 + y^2$ II. $h(x,y) = x^2 + y^2 - 4$ III. $r(x,y) = x^2 + y^5$

Solution: I. has local min (in fact, global min!) at $(0,0)$ since $g(0,0) = 0$ and $g(x,y) \geq 0$ for all (x,y).
II. Has local min at $(0,0)$ by 2nd der. test: $f_{xx} = 2 > 0$ and $D = f_{xx}f_{yy} - [f_{xy}]^2 = 4 > 0$.
III. saddle point (fix $x = 0$ and change y, get positive and negative values for f near origin)

2. (a) (8 pts) Let

$$f(x, y) = \frac{y}{x} \ .$$

Find $f_x(a, b)$ using the limit definition of the partial derivative with respect to x. *If you do not use the limit definition of the derivative, you will receive zero credit.*

Solution:

$$f_x(a, b) = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h} \quad = \quad \lim_{h \to 0} \frac{\dfrac{b}{a + h} - \dfrac{b}{a}}{h}$$

$$= \lim_{h \to 0} \frac{\dfrac{ba}{a(a + h)} - \dfrac{b(a + h)}{a(a + h)}}{h} = \lim_{h \to 0} \frac{\dfrac{-bh}{a(a + h)}}{h}$$

$$= \lim_{h \to 0} \frac{-bh}{ha(a + h)} = \lim_{h \to 0} -\frac{b}{a(a + h)} = \boxed{-\frac{b}{a^2}}.$$

(b) (8 pts) Use an appropriate linearization to estimate $\sqrt{27}$. Express your answer as a fraction or as a decimal. Is this estimate an overestimate or underestimate? Justify your answer. (hint: consider the second derivative of the function you linearized)

Solution: Use $f(x) = \sqrt{x}$. The formula for linearization is $L(x) = f(a) + f'(a)(x - a)$. Here $a = 25$ (a perfect square close to 27). Note $f'(x) = \frac{1}{2\sqrt{x}}$, so

$$L(x) = f(25) + f'(25)(x - 25) = 5 + \frac{1}{2 \cdot 5}(x - 25), \qquad \sqrt{27} \approx L(27) = 5 + \frac{1}{10}(27 - 25) = \boxed{5.2}.$$

(or alternatively, 26/5)

This is an $\boxed{\text{overestimate}}$, since f is concave downward and concave functions lie below their tangent lines.

3. Given K units of capital and L units of labor, a company produces

$$Q = 2K^{1/2}L^{1/3} \quad (K > 0,\ L > 0)$$

units of its product. Each unit of capital costs \$3 and each unit of labor costs \$1. The selling price for the product is \$3 per unit.

(a) (4 pts) Write the profit function $\pi(K, L)$.

(b) (12 pts) Find the critical points of the profit function. Classify each critical point as a local maximum, local minimum, or neither.

Solution: (a) $\pi(K, L) = \text{price}\cdot\text{quantity} - \text{cost} = 3\cdot 2K^{1/2}L^{1/3} - (3K + L) = \boxed{6K^{1/2}L^{1/3} - 3K - L}$.

(b) The function has partial derivatives and mixed partials of all orders on its domain, so we tet f_K and f_L equal to zero. to find the critical points.

$$f_K = 3K^{-1/2}L^{1/3} - 3 = 0, \qquad K^{-1/2}L^{1/3} = 1, \qquad K^{1/2} = L^{1/3},$$

and below we substitute $K^{1/2} = L^{1/3}$ into the equation for $f_L = 0$.

$$f_L = 2K^{1/2}L^{-2/3} - 1 = 0, \qquad 2L^{1/3}L^{-2/3} - 1 = 0, \qquad 2L^{-1/3} = 1, \qquad L^{1/3} = 2, \qquad \boxed{L = 8}.$$

We recall $K^{1/2} = L^{1/3}$, so $K^{1/2} = 2$, hence $\boxed{K = 4}$.

We apply the 2nd derivative test to see whether our critical point is a local max, min, or neither:

$$f_{KK} = -\frac{3}{2}K^{-3/2}L^{1/3} < 0, \qquad f_{LL} = -\frac{4}{3}K^{1/2}L^{-5/3}, \qquad f_{KL} = K^{-1/2}L^{-2/3},$$

so

$$D = f_{KK}f_{LL} - f_{KL} = -\frac{3}{2}K^{-3/2}L^{1/3} \cdot -\frac{4}{3}K^{1/2}L^{-5/3} - (K^{-1/2}L^{-2/3})^2,$$

$$D = 2K^{-1}L^{-4/3} - K^{-1}L^{-4/3} = K^{-1}L^{-4/3} > 0.$$

We found $f_{KK} < 0$ and $D > 0$, so by the 2nd derivative test, $\boxed{f \text{ has a local maximum at } (4, 8)}$.

4. The amount of money spent on gold jewelry by a consumer is taken to be

$$f(I, p) = \frac{I^2}{p + 2I},$$

where I is the consumer's yearly income and p is the price of an ounce of gold.

(a) (5 pts) What is the partial elasticity of f with respect to income? Fully simplify your answer.
Solution:

$$\text{El}_I f(I,p) = \frac{I}{f(p,I)} \cdot \frac{\partial}{\partial I}[f(I,p)] = \frac{I(p+2I)}{I^2} \cdot \frac{(p+2I)(2I) - 2I^2}{(p+2I)^2}$$

$$= \frac{p+2I}{I} \cdot \frac{(p+2I)(2I) - 2I^2}{(p+2I)^2} = \frac{2I(p+2I)^2 - 2I^2(p+2I)}{I(p+2I)^2}$$

$$= \boxed{2 - \frac{2I}{p+2I}}.$$

Alternate looking answer, depending on simplification: $\boxed{\text{El}_I f(I,p) = 1 + \frac{p}{p+2I}}$.

Another alternate answer: $\boxed{\text{El}_I f(I,p) = \frac{2p+2I}{p+2I}}$

(b) (4 pts) What is the partial elasticity of f with respect to price?
Fully simplify your answer.
Solution:

$$\text{El}_p\ f(I,p) = \frac{p}{f(p,I)} \cdot \frac{\partial}{\partial p}[f(p,I)] = \frac{p(p+2I)}{I^2} \cdot \frac{-I^2}{(p+2I)^2} = \boxed{\frac{-p}{p+2I}}.$$

(c) (4 pts) Explain in words what the partial elasticity of f with respect to price represents.
It represents the approximate percent change in f if income is held fixed and the price is increased by 1%.

(d) (5 pts) Find the partial elasticity of f with respect to p when $I = 2$ and $p = 4$. Use this to estimate the percent change in f when p increases from $p = 4$ to $p = 4.08$ and I is held fixed at 2.

Solution: When $I = 2$ and $p = 4$, we get

$$\text{El}_p\ f(I,p) = \frac{-p}{p+2I} = \frac{-4}{4 + 2 \cdot 2} = -\frac{1}{2}.$$

When p increases from 4 to 4.08, it increases by 2%, so we multiply our elasticity above by 2. The approx. percent change in f is $2 \cdot -\frac{1}{2} = \boxed{-1\%}$.
So f goes down by approx. 1%.

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