Calculus III Part 1

Name: Solutions

1. (a) $\boldsymbol{u} \times \boldsymbol{v} = \left\langle -\sqrt{2}, \sqrt{2}, 0 \right\rangle$

   (b) $\boldsymbol{u} \cdot \boldsymbol{v} = 2$

   (c) Let $\theta \in [0, \pi]$ be the angle between the two vectors

   $$\cos\theta = \frac{\boldsymbol{u} \cdot \boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|} = \frac{2}{2\sqrt{2}} \implies \theta = 45^0$$

   (d)

   $$\frac{\boldsymbol{u} \cdot \boldsymbol{v}}{|\boldsymbol{v}|^2}\boldsymbol{v} = \boldsymbol{v}$$

   (e)

   $$|(\boldsymbol{u} \times \boldsymbol{v}) \cdot \langle 1, 0, 0 \rangle| = \sqrt{2}$$

2. (a) $\boldsymbol{j}$

   (b) $-\boldsymbol{i}$

   (c) 0

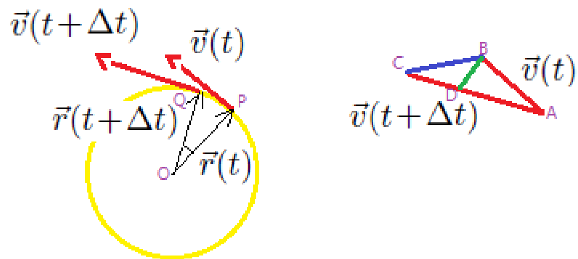
   (d) $\boldsymbol{i}$

   (e) $-\boldsymbol{j} + \boldsymbol{k} = \langle 0, -1, 1 \rangle$

3. Unit tangent vector, $\boldsymbol{T}$, gives the direction of the velocity, and unit normal vector, $\boldsymbol{N}$, gives the direction of the normal acceleration which is responsible for the change of the direction of the velocity.

   Recall there are two parts of acceleration: tangential acceleration $\vec{a}_T$ changes the magnitude of the velocity only, and it is parallel to the velocity; and normal acceleration $\vec{a}_N$ changes the direction of the velocity only (that is the reason why $\boldsymbol{T}$ has to be normalized).

   Here is a picture that illustrates $\vec{a}_T$ and $\vec{a}_N$. Suppose the motion is circular, and we can look at velocities at two instances, $\vec{v}(t)$ and $\vec{v}(t + \Delta t)$, separated by time Δt.



To find the difference of two velocities, we shift $\vec{v}(t)$ and $\vec{v}(t + \Delta t)$ so that the two tails are coincide, so the blue line $\overrightarrow{BC} = \vec{v}(t + \Delta t) - \vec{v}(t) \approx \vec{a}\Delta t$. Now mark a point D on $\overline{AC}$ such that $\overline{AD} = \overline{AB} = |\vec{v}(t)|$. Now we find that $\Delta POQ \sim \Delta BAD$, because both are isosceles and $\angle POQ = \angle BAD$. That is because $\overline{OP} \perp \overline{AB}$ and $\overline{OQ} \perp \overline{AD}$.

Let $\Delta t \to 0$, hence $\angle BAD \to 0$, so $\angle DBA \to \pi/2$, i.e. $\overline{BD} \perp \overline{AB}$, so $\overline{BD} \parallel \overline{OP}$, i.e. $\overline{BD}$ is in the normal direction. Since $\overrightarrow{BC} = \overrightarrow{BD} + \overrightarrow{DC}$ and clearly $|\overrightarrow{DC}| = |\vec{v}(t + \Delta t)| - |\vec{v}(t)|$, it is natural to define $\vec{a}_T$ and $\vec{a}_N$ so that

$$\vec{a} = \vec{a}_T + \vec{a}_N$$

and

$$\overrightarrow{DC} = \vec{a}_T \Delta t \qquad \overrightarrow{BD} = \vec{a}_N \Delta t$$

Since $\vec{a}_N$ is perpendicular to the motion, centripetal forces do no work, i.e. $\vec{a}_N$ doesn't contribute to the change of the speed.

Furthermore if the particle moves in a circular motion with constant speed, i.e. $\overline{CD} = 0$, using $\Delta POQ \sim \Delta BAD$, we get

$$\frac{\overline{OP}}{\overline{PQ}} = \frac{\overline{AB}}{\overline{BD}} \implies \frac{v\Delta t}{r} = \frac{a\Delta t}{v} \implies a = \frac{v^2}{r}$$

For arbitrary "smooth" motion in 3D, we can always approximate the trajectory at every instance by circle with radius, 1/curvature. And $\boldsymbol{T}$ is tangent to the circle, $\boldsymbol{N}$ radially points to the center of the circle, and $\boldsymbol{B}$ gives the normal direction of the plane in which the circle lies. We take $\boldsymbol{B} = \boldsymbol{T} \times \boldsymbol{N}$ but not $\boldsymbol{N} \times \boldsymbol{T}$ because under right hand rule $\boldsymbol{B}$ also gives the direction of the rotation of the particle. Two for the price of one.

One way to memorize the formula for curvature κ is to think the special case above: uniform circular motion.

We learned for uniform circular motion with constant speed v and radius r, the magnitude of the acceleration is given by

$$a = \frac{v^2}{r} = \kappa v^2$$

and

$$\vec{a} = \frac{d\vec{v}}{dt} = v\frac{d\vec{T}}{dt}$$

so it is natural to define κ as

$$\kappa = \frac{|d\vec{T}/dt|}{v} = \left|\frac{d\vec{T}}{ds}\right|$$

The presentation given above is of course not a proof, but a good trick to use on a exam. Special cases help memorizing formulas.

(a)

$$\vec{T}' = \left\langle \frac{4t}{(t^2+2)^2}, -\frac{4t}{(t^2+2)^2}, -\frac{2(t^2-2)}{(t^2+2)^2} \right\rangle$$

$$\vec{N} = \frac{\left\langle 2t, -2t, -(t^2-2) \right\rangle}{\sqrt{8t^2+(t^2-2)^2}} = \frac{\left\langle 2t, -2t, -(t^2-2) \right\rangle}{t^2+2}$$

(b)

$$\kappa = \frac{\left|\left\langle \frac{4t}{(t^2+2)^2}, -\frac{4t}{(t^2+2)^2}, -\frac{2(t^2-2)}{(t^2+2)^2} \right\rangle\right|}{\frac{1}{2}t^2+1} = \frac{\frac{2}{(t^2+2)^2}(t^2+2)}{\frac{1}{2}t^2+1} = \frac{4}{(t^2+2)^2}$$

(c)

$$\vec{B}(t=0) = \langle 0,1,0 \rangle \times \langle 0,0,1 \rangle = \hat{\imath}$$

4. (a) No

   (b) Yes

   (c) Should read $\frac{\partial z}{\partial t}$ not $\frac{dz}{dt}$. ANS yes

   (d) Yes

   (e) Should read $f(x,y)$ is a non-constant function... ANS yes (cf problem 6 below)

5. (a) Clearly

$$f(0,0) = f(0,x) = f(0,y) = 0$$

so all points on the x and y axes give the same value, so (a) goes with (4).

(b) Similarly $f(0,y) = 0$ for all y, and we already used (4), so (b) goes with (8).

(c) For fixed f, if $f > 0$, the level curve is

$$\frac{y^2}{(\sqrt{f})^2} - \frac{x^2}{(\sqrt{f})^2} = 1$$

If $f < 0$

$$\frac{x^2}{(\sqrt{-f})^2} - \frac{y^2}{(\sqrt{-f})^2} = 1$$

So (c) goes to (1)

(e) f is invariant under $x \to x+a$, and $y \to y+a$, for any a, so the contour plot has to have this property, i.e. symmetric under shifting the plot by the vector $\langle 1,1 \rangle$, so (e) goes to (7).

(d) We can do the following transformation

$$\begin{cases} x+y=u \\ x-y=v \end{cases}$$

then

$$f=\frac{x-y}{x^2+y^2+1}=\frac{v}{u^2+v^2+1}$$

which is almost (b).

If you know the transformation

$$\begin{cases} x+y=u \\ x-y=v \end{cases}$$

means to rotate x and y axes by 45^0, then you know the answer. Otherwise use the same trick

$$f(0,0)=f(x,x)$$

so the line $y=x$ must be one of the level curve, and we already used (1), (7), so it has to go to (3).

6. Recall $\vec{u}=\langle \Delta x, \Delta y, \Delta z\rangle$

$$\Delta f=\frac{\partial f}{\partial x}\Delta x+\frac{\partial f}{\partial y}\Delta y+\frac{\partial f}{\partial z}\Delta z=\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right\rangle \cdot \langle \Delta x, \Delta y, \Delta z\rangle$$

So if one chooses $\vec{u}=\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right\rangle$, then Δf is maximum, i.e. $\vec{u}=\nabla f$ gives the direction that maximally increases f. The direction perpendicular to ∇f gives $\Delta f=0$, which makes up the tangent plane.

So the normal direction at point $(1,-1,1)$ is

$$\langle 2x, 4y, 2z\rangle \sim \langle 1,-2,1\rangle$$

So the equation of the plane

$$x-2y+z=d$$

Since it passes $(1,-1,1)$,

$$x-2y+z=4$$

7. (a)

$$\begin{cases} 4x^3-4y=0 \\ 4y^3-4x=0 \end{cases} \implies x=y=\pm 1, 0$$

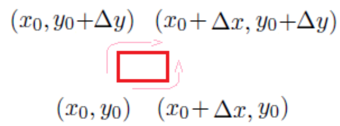
ANS $(0,0)$, $(1,1)$, $(-1,-1)$

(b) We are going to apply second derivative test. Recall second derivative test says suppose f has continuous second derivatives and at the critical points if

$$f_{xx}f_{yy} - f_{xy}^2 > 0 \text{ and } f_{xx} > 0$$

then that critical point is a local minimum. Let's use a crude argument to show why this test makes sense.

Suppose (x_0, y_0) is a critical point. Let us compare $f(x_0, y_0)$ to its neighborhood, say $f(x_0 + \Delta x, y_0 + \Delta y)$



Let us use Taylor. First expand in y then expand in x, and keep up to second order terms (because first order terms are zeros, for (x_0, y_0) is a critical point. Because f has continuous second derivatives, by Clairaut's, expanding in y then expanding in x gives the same answer if we expand in x then expand in y, i.e. following the upper left path is the same as following the lower right path.) We obtain

$$\begin{aligned} f(x_0 + \Delta x, y_0 + \Delta y) &= f(x_0 + \Delta x, y_0) + \left.\frac{\partial f}{\partial y}\right|_{(x_0+\Delta x, y_0)} \Delta y + \frac{1}{2}\left.\frac{\partial^2 f}{\partial y^2}\right|_{(x_0+\Delta x, y_0)} (\Delta y)^2 \\ &= f(x_0, y_0) + \left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)} \Delta x + \frac{1}{2}\left.\frac{\partial^2 f}{\partial x^2}\right|_{(x_0, y_0)} (\Delta x)^2 \\ &\quad + \left[\left.\frac{\partial f}{\partial y}\right|_{(x_0, y_0)} + \frac{\partial}{\partial x}\left.\frac{\partial f}{\partial y}\right|_{(x_0, y_0)} \Delta x\right] \Delta y + \frac{1}{2}\left.\frac{\partial^2 f}{\partial y^2}\right|_{(x_0, y_0)} (\Delta y)^2 \\ &= f(x_0, y_0) + \frac{1}{2}\left.\frac{\partial^2 f}{\partial x^2}\right|_{(x_0, y_0)} (\Delta x)^2 + \left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(x_0, y_0)} \Delta x \Delta y + \frac{1}{2}\left.\frac{\partial^2 f}{\partial y^2}\right|_{(x_0, y_0)} (\Delta y)^2 \end{aligned}$$

We want $f(x_0, y_0)$ to be truly a local minimum, then the sum after $f(x_0, y_0)$ had better to be positive for any direction $\langle \Delta x, \Delta y \rangle$ we pick, i.e.

$$f_{xx}(\Delta x)^2 + 2f_{xy}\Delta x \Delta y + f_{yy}(\Delta y)^2 > 0$$

If we view above as a parabola in variable Δx, then we know the entire parabola lives above the x axis iff the parabola is concave up and no real roots, so the requirements are

$$f_{xx} > 0 \;\&\; 4f_{xy}^2(\Delta y)^2 - 4f_{xx}f_{yy}(\Delta y)^2 < 0$$

That is what we want

$$f_{xx} > 0 \;\&\; f_{xx}f_{yy} - f_{xy}^2 > 0$$

And the requirements for local maximum are that the entire parabola lives below the x axis, i.e. the parabola is concave down and no real roots.

Now we do the test on $(0,0)$, $(1,1)$, and $(-1,-1)$

$$f_{xx} = 12x^2, \quad f_{xx}f_{yy} - f_{xy}^2 = 144x^2y^2 + 4 > 0$$

So $(1,1)$ and $(-1,-1)$ are minimum, and $(0,0)$ is inconclusive by the test, so we will have to use other methods. So we can stop here.

[If you have the luxury of time, you can work out the problem for extra credits:

> Is $(0,0)$ a min, max or saddle point?

Hint: use the 45^0 rotation transformation mentioned in problem 5(d) above with proper normalization (i.e. Jacobian $= 1$), so f is reduced into a equation with 2nd degrees in x and y, then go to polar coordinate to find a level curve passing through the origin, then rotate 45^0 back to the normal xy plane.

ANS: the level curve passing through the origin is given by

$$r^2 = \frac{4\sin 2\theta}{2 - \sin^2 2\theta}$$

Hence $(0,0)$ is not min nor max, is a saddle point.]