Index option volatility, Returns Dispersion and Implied Correlations

Speaker: Marco Avellaneda

Dedicated to Nicole El Karoui on her 60th birthday

Avellaneda, Boyer-Olson, Busca and Friz: ‘Reconstructing Volatility’, *RISK* Oct 2002; ‘Large Deviations Methods and the Pricing of Index Options in Finance’, *CRAS Paris 2003*

Outline

• Stylized facts about index options and volatility

• Steepest descent approximation: matching index skew with single-stock option skews

• Implied correlation: skew and term structure

• Modelling correlation skew

• Statistics of implied correlation for different markets/sectors
U.S. Equities: Main Sectors & Their Indices

- **Major Indices:** SPX, DJX, NDX
  SPY, DIA, QQQ  (Exchange-Traded Funds)

- **Sector Indices & Index Trackers:**
  Semiconductors: SMH, SOX
  Biotech: BBH, BTK
  Pharmaceuticals: PPH, DRG
  Financials: BKX, XBD, XLF, RKH
  Oil & Gas: XNG, XOI, OSX
  High Tech, WWW, Boxes: MSH, HHH, XBD, XCI
  Retail: RTH

All these indices have options
Components of NASDAQ 100 Trust (AMEX:QQQ)

- Capitalization-weighted average of 100 largest stocks in NASDAQ

- QQQ trades as a stock

- QQQ options are the most heavily traded contracts in the world
SOX, XNG, XOI

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... & many others
# BBH: Basket of 20 Biotechnology Stocks

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BBH March 2003 Implied Vols
Pricing Date: Jan 22 03 10:42 AM

![Graph showing implied volatility for BBH March 2003 options with strike prices ranging from 75 to 105 and implied volatility values ranging from 55 to 30. The graph compares bid and ask volatility.](image)
Implied Volatility Curve for Options on Dow Jones Average

DJX Mar 03    Pricing Date: 10/25/02
Stylized facts about equity volatility curves

- Implied volatility curves are typically downward sloping.

- Counterexamples: precious metal stocks are upward sloping.

- There is little curvature (or smile). Skew is important.
AOL Jan 2001 Options:
Implied volatility curve on Dec 20, 2000
Market close

Vol.

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Stock probability is not lognormal
The AOL "volatility skew" for several expiration dates
What is the relation between index options and options on the components?

Standard (log-normal) Volatility Formula for Index Options

\[
\sigma_I^2 = \sum_{j=1}^{N} p_j^2 \sigma_j^2 + \sum_{i \neq j} p_i p_j \sigma_i \sigma_j \rho_{ij} \quad (*)
\]

Does not apply when volatilities are strike-dependent

How can we incorporate volatility skew information into (*)?
Volatility Modeling

1. Joint stock-volatility ‘dynamics’

\[ \frac{dS}{S} = \sigma_t dW \]

A. \( \sigma_t = \sigma(S,t) \)  
   Dupire's Local Volatility  
   \( \sigma(s,t) = \sigma(t) \left( \frac{S}{S_0} \right)^\gamma \)

B. \( \frac{d\sigma_t}{\sigma_t} = \kappa dZ_t \)  
   Stochastic Volatility

2. Implied vol. curve

\[ \sigma_{\text{implied}}(K,T) = \sigma_{\text{implied}}(S,T) \cdot \left( 1 + a \ln \left( \frac{K}{S} \right) \right) \]

Joint stock-volatility dynamics gives rise to an implied volatility curve
Relation between Stochastic Volatility and Local Volatility

\[
\frac{dS_t}{S_t} = \sigma_t dZ_t
\]

\[
\sigma_{loc}^2(S,t) = E\left\{ \sigma_t^2 \mid S_t = S \right\}
\]

Application to Index Options

\[ I = \sum_{i=1}^{n} w_i S_i \]

Index = weighted sum of stock prices (constant weights)

Diffusion eq.
for each stock reflects vol skew (local vol)

\[
\begin{aligned}
\frac{dS_i}{S_i} &= \sigma_i(S_i,t)dW_i + \mu_i dt, \\
E(dW_i dW_j) &= \rho_{ij} dt
\end{aligned}
\]

\[
\frac{dI}{I} = \sigma_1(S,t)dZ + \mu_1(S,t)dt
\]

\[
\sigma_1^2(S,t) = \frac{\sum_{ij} \sigma_i(S_i,t)\sigma_j(S_j,t)w_i S_i w_j S_j \rho_{ij}}{I^2}
\]

\[
\mu_1(S,t) = \frac{\sum_i \mu_i w_i S_i}{I}
\]
Characterization of the equivalent local volatility for the index

$$\sigma^2_{I,loc}(I,t) = E \left\{ \frac{\sum_{ij} \sigma_i(S_i(t),t) \sigma_j(S_j(t),t) w_i w_j \rho_{ij} S_i(t) S_j(t)}{I^2} \ \bigg| \sum_i w_i S_i(t) = I \right\}$$

$$= E \left\{ \sum_{ij} p_i(S(t)) p_j(S(t)) \sigma_i(S_i(t),t) \sigma_j(S_j(t),t) \rho_{ij} \ \bigg| \sum_i w_i S_i(t) = I \right\}$$

$$p_i(S) = \frac{w_i S_i}{\sum_j w_j S_j}, \quad i = 1, \ldots, n.$$  

- $\sigma_I$ can be seen as a "stochastic vol" driving the index
- $\sigma_{I,loc}$ is then the "equivalent local vol"
Varadhan’s Formula and Large Deviations

\[
\begin{cases}
  dX_i = \sum_{j=1}^{n} \sigma_i^j(X,t) dW_j & E\{dW_j dW_k\} = \rho_{jk} dt \\
  X_i(0) = x_i
\end{cases}
\]

\[
\log \text{Prob.}\{X(t) = y | X(0) = x\} \approx -\frac{d^2(x, y)}{2t}, \quad (\overline{\sigma})^2 t << 1
\]

\[
d^2(x, y) = \inf_{\gamma(0)=x, \gamma(1)=y} \int_0^1 \sum_{ij=1}^{n} g_{ij}(\gamma(s)) \gamma^i(s) \gamma^j(s) ds
\]

\[
g(x) = a^{-1}(x) \quad a_{ij}(x) = \sigma_i(x,0) \sigma_j(x,0) \rho_{ij}
\]

Dupire local volatility model for each stock

Riemannian metric

In practice: dimensionless time \(\sim 0.02\)
Steepest-descent approximation

Change to log-scale: \[ x_i \equiv \log \left( \frac{S_i}{S_i(0)e^{\mu_i t}} \right) = \log \left( \frac{S_i}{F_i(t)} \right) \quad i = 1, 2, \ldots, n. \]

Formally,

\[ \sigma^2_{I, \text{loc}}(t, I) = \frac{E\left\{ \sigma^2_I \delta(I(t) - I) \right\}}{E\{\delta(I(t) - I)\}} \]

Applying Varadhan’s Formula,

\[ \sigma^2_{I, \text{loc}}(I, t) \approx \sigma^2_I \left( S^*, t \right) \quad S^*_i = S_i(0)e^{\mu_i t} e^{x^*_i} \]

where

\[ x^* = \arg \min \left\{ d^2(0, x) \left| \sum_i w_i S_i(0)e^{\mu_i t} e^{x_i} = I \right. \right\} \]
Steepest Descent = Most Likely Stock Price Configuration

Replace conditional distribution by “Dirac function” at most likely configuration

Level sets multivariate PDF for n stocks

Most likely path for stock price returns

\[ \sum_{i} F_{i}e^{x_{i}} = I \]
Characterization of MLC

Euler-Lagrange equations: find \((x^*, \lambda)\) such that

\[
\begin{aligned}
\int_0^{x_i^*} \frac{du}{\sigma_i(u)} &= \lambda \sum_{j=1}^{n} p_j(x^*_i) \sigma_j(x^*_j) \rho_{ij} \quad i = 1, \ldots, n \\
\sum_{i=1}^{n} w_i S_i(0) e^{x_i^* + \mu_i t} &= I
\end{aligned}
\]

\[
\sigma_{I, \text{loc}}^2(I, t) = \sum_{ij=1}^{n} p_i(x^*_i) p_j(x^*_j) \sigma_i(x^*_i) \sigma_j(x^*_j) \rho_{ij}
\]
Linearization gives CAPM-like characterization

\[ \sigma_i^2(0) \equiv \sum_{ij=1}^{n} p_i(0)p_j(0)\sigma_i(0)\sigma_j(0)\rho_{ij} \]

\[ \bar{x} \equiv \ln \left( \frac{I}{I(0)e^{\mu t}} \right) \]

\[ x_i^* \approx \frac{-x}{\sigma_i^2(0)} \sum_{j=1}^{n} \rho_{ij}p_j(0)\sigma_i(0)\sigma_j(0) = \frac{-x}{\sigma_i^2(0)} \text{Cov}(x_i, \bar{x}) \]

\[ x_i^* = \hat{\beta}_i \bar{x} \]

\[ \hat{\beta} = \text{Cov} \left( \frac{\Delta S}{S} , \frac{\Delta I}{I} \right) / \left[ \text{Var} \left( \frac{\Delta I}{I} \right) \right] \]

Most likely config. : described by the risk-neutral regression coefficients of stock returns with the index return (``micro” CAPM)
From local volatilities back to Black-Scholes implied volatilities

- Seek direct relation between implied volatilities of single-stock options and implied volatility of index options

- Tool: Berestycki-Busca-Florent large-deviations result for single-stock ("1/2 slope rule")

\[
\sigma^{\text{impl.}}(x) \approx \left( \frac{1}{x} \int_0^x \frac{du}{\sigma(u)} \right)^{-1}
\]

\[
\sigma^{\text{impl.}}(x) \approx \frac{1}{2} \left( \sigma^{\text{impl.}}(0) + \sigma(x) \right)
\]
Alternatively: integrate LV along most likely path

\[ (\sigma^{\text{impl}}(x,T))^2 \approx \frac{1}{T} \int_0^T \sigma_{\text{loc}}^2(x^*(s),s) ds \]

\[ \approx \frac{1}{T} \int_0^T \sigma_{\text{loc}}^2(xs, s) ds \]

-- For small dimensionless time, the price diffusion is localized in a neighborhood of the most likely path.

-- this implies the \( 1/2 \) slope rule as trapezoidal approximation to the integral.
Computing The Index Volatility

\[
\left( \sigma_{i}^{\text{impl}}(x, T) \right)^2 \approx \frac{1}{T} \int_{0}^{T} \sum_{ij=1}^{N} \sigma_{i} \left( x^{*}(s), s \right) \sigma_{j} \left( x^{*}(s), s \right) p_{i} p_{j} \rho_{ij} ds
\]

\[
\approx \frac{1}{T} \int_{0}^{T} \sum_{ij=1}^{N} \sigma_{i} \left( \beta_{i} x, s \right) \sigma_{j} \left( \beta_{j} x, s \right) p_{i} p_{j} \rho_{ij} ds
\]

\[
= \sum_{ij=1}^{N} \left[ \frac{1}{T} \int_{0}^{T} \sigma_{i} \left( \beta_{i} x, s \right) \sigma_{j} \left( \beta_{j} x, s \right) ds \right] p_{i} p_{j} \rho_{ij}
\]

\[
\approx \sum_{ij=1}^{N} \sigma_{i}^{\text{impl}}(\beta_{i} x, T) \sigma_{j}^{\text{impl}}(\beta_{j} x, T) p_{i} p_{j} \rho_{ij} Q_{ij}(x, T)
\]

\[
\therefore \quad Q_{ij}(x, T) \equiv \left[ \frac{1}{T} \int_{0}^{T} \sigma_{i} \left( \beta_{i} x, s \right) \sigma_{j} \left( \beta_{j} x, s \right) ds \right]^{1/2} \left[ \frac{1}{T} \int_{0}^{T} \sigma_{i}^{2} (\beta_{i} x, s) ds \right]^{1/2} \left[ \frac{1}{T} \int_{0}^{T} \sigma_{j}^{2} (\beta_{j} x, s) ds \right]^{1/2}
\]
Reconstruction Rule for Index Volatility

-- SD approximation is consistent with \( Q_{ij}(x,T) \approx 1 \)

\[
\left( \sigma_I^{\text{impl}}(x,T) \right)^2 \approx \sum_{ij=1}^{N} \sigma_i^{\text{impl}}(\beta_i x, T) \sigma_j^{\text{impl}}(\beta_j x, T) p_i p_j \rho_{ij}
\]

An \( x \) percent OTM strike for index corresponds to a \( \beta_1 x \) percent OTM strike for stock 1, etc.
T=1 month

DJX Nov 02  Pricing Date: 10/25/02
T = 2 months

Pricing Date: 10/25/02

DJX Dec 02

Vol

BidVol

AskVol

SDA

Delta

Vol

Delta

Expiration

T = 2 months
T=3 months

DJX Jan 03  Pricing Date: 10/25/02
T = 5 months

DJX Mar 03    Pricing Date: 10/25/02
T = 7 months
S&P 100 Index Options
(Quote date: Aug 20, 2002)

Expiration: Sep 02

Market is not pricing with historical correlations
S&P 100 Index Options
(Quote date: Aug 20, 2002)

Expiration: Oct 02

Market is not pricing with historical correlations
Market is not pricing with historical correlations
S&P 100 Index Options
(Quote date: Aug 20, 2002)

Expiration: Dec 02

`Slope` is different
`Level` is different

Market is not pricing with historical correlations
Implied Correlation: a single correlation coefficient consistent with index vol

\[ \left( \sigma_{I}^{\text{impl}} \right)^2 = \sum_{i=1}^{N} p_i \sigma_{i}^{\text{impl}} \sigma_{i}^{\text{impl}} + \rho \sum_{i \neq j} p_i p_j \sigma_{i}^{\text{impl}} \sigma_{j}^{\text{impl}} \]

\[ \bar{\rho} = \frac{\left( \sigma_{I}^{\text{impl}} \right)^2 - \sum_{i=1}^{N} p_i \sigma_{i}^{\text{impl}} \sigma_{i}^{\text{impl}}}{\sum_{i \neq j} p_i p_j \sigma_{i}^{\text{impl}} \sigma_{j}^{\text{impl}}} = \frac{\left( \sigma_{I}^{\text{impl}} \right)^2 - \sum_{i=1}^{N} p_i \sigma_{i}^{\text{impl}} \sigma_{i}^{\text{impl}}}{\left( \sum_{i=1}^{N} p_i \sigma_{i}^{\text{impl}} \right)^2 - \sum_{i=1}^{N} p_i^2 \left( \sigma_{i}^{\text{impl}} \right)^2} \]

Approximate formula: \[ \bar{\rho} \approx \frac{\sigma_{I}^{\text{impl}}}{\sum_{i=1}^{N} p_i \sigma_{i}^{\text{impl}}} \]

Implied correlation can be defined for different strikes, using SDA
Dow Jones Index: Correlation

Skew

Quote Date 9/1/1998 Spot price=78.26

Graph showing implied correlation over different strike prices for various dates.
Quote Date 12/10/2001 Spot=99.21

[Graph showing implied correlation over time with various dates marked on the graph.

- 12/22/2001
- 1/19/2002
- 2/16/2002
- 3/16/2002
- 6/22/2002
- 9/21/2002]

The graph illustrates the implied correlation over different strike values for various dates, showing a downward trend as the strike value increases.
Quote Date 7/25/2002  Spot=81.86

strike

implied correlation

- 8/17/2002
- 9/21/2002
- 10/19/2002
- 12/21/2002
- 3/22/2003
- 6/21/2003
A model for "`Correlation skew": Stochastic Volatility Systems

\[
\frac{dS_i}{S_i} = \sigma_i dW_i \\
E(dW_i dW_j) = \rho_{ij} dt \\
\frac{d\sigma_i}{\sigma_i} = \kappa_i dZ_i \\
E(dW_i dZ_j) = r_{ij} dt
\]

\[-x = \frac{dI}{I}, \quad x_i = \frac{dS_i}{S_i}, \quad y_i = \frac{d\sigma_i}{\sigma_i}\]

Look for most likely configuration of **stocks and vols** \((x_1, \ldots, x_n, y_1, \ldots, y_n)\) corresponding to a given index displacement \(x\).
Most likely configuration for Stochastic Volatility Systems

\[ x_i^* = \beta_i \bar{x} \]
\[ \beta_i = \frac{\sigma_i \rho_{il}}{\sigma_I} \]
\[ y_i^* = \gamma_i \bar{x} \]
\[ \gamma_i = \frac{\kappa_i r_{il}}{\sigma_I} \]

\[ \sigma_{I,loc}^2(\bar{x},t) \cong \sum_{ij=1}^{n} p_i p_j \sigma_i(0,t) \sigma_j(0,t) e^{\gamma_i \bar{x}} e^{\gamma_j \bar{x}} \rho_{ij} \]
Method I: Dupire & Most Likely Configuration for Stock Moves

N-dimensional Equity market

\[
\sigma_1(x_1, t)
\]

\[
\sigma_i(x_i, t)
\]

\[
\sigma_n(x_n, t)
\]

\[
\sigma_{I, loc}(\bar{x}, t)
\]

- **Step 1**: Local volatility for each stock consistent with options market
- **Step 2**: Find most likely configuration for stocks
Method II: Stochastic Volatility System and joint MLC for Stocks and Volatilities

N-dimensional Equity market

\[ \sigma_{I,\text{loc}}(\overline{x}, t) \]

Only one step: compute the most likely configuration of stocks and volatilities at the same time
Methods I and II are not `equivalent'

\[ \sigma_{i,\text{loc}}(x_i, t) \approx \sigma_i(0, t)e^{\omega_i x_i} \]
\[ \omega_i = \frac{K_{r_{ii}}}{\sigma_i} \]

\[ \sigma_{I,\text{loc}}^2(x, t) = \sum_{ij} p_i p_j \sigma_i(0, t) \sigma_j(0, t) \rho_{ij} e^{\omega_i \beta_i x} e^{\omega_j \beta_j x} \]

\[ \sigma_{I,\text{loc}}^2(x, t) = \sum_{ij} p_i p_j \sigma_i(0, t) \sigma_j(0, t) \rho_{ij} e^{\gamma_i x} e^{\gamma_j x} \]
Stochastic Volatility Systems give rise to Index-dependent correlations

\[ \sigma^2_{I, \text{loc}}(\bar{x}, t) \approx \sum_{ij} p_i p_j \sigma_i(0, t) \sigma_j(0, t) \rho_{ij} e^{\gamma_i \bar{x}} e^{\gamma_j \bar{x}} \]

Method II

\[ \approx \sum_{ij} p_i p_j \sigma_i(0, t) e^{\beta_i \bar{\omega}_i \bar{x}} \sigma_j(0, t) e^{\beta_j \bar{\omega}_j \bar{x}} \rho_{ij} e^{\gamma_i \bar{x}} e^{\gamma_j \bar{x}} e^{-\beta_i \bar{\omega}_i \bar{x}} e^{-\beta_j \bar{\omega}_j \bar{x}} \]

\[ \approx \sum_{ij} p_i p_j \sigma_{i, \text{loc}}(\beta_i \bar{x}, t) \sigma_{i, \text{loc}}(\beta_i \bar{x}, t) \rho_{ij}(\bar{x}) \]

\[ \rho_{ij}(\bar{x}) \equiv \rho_{ij} e^{(\gamma_i + \gamma_j - \beta_i \bar{\omega}_i - \beta_j \bar{\omega}_j) \bar{x}} \]
Equivalence holds only under additional assumptions on stock-volatility correlations:

\[ \omega_i \beta_i = \frac{\kappa_i r_{ii}}{\sigma_i} \frac{\sigma_i \rho_{il}}{\sigma_I} = \frac{\kappa_i r_{ii} \rho_{il}}{\sigma_I} \]  

Method I

\[ \gamma_i = \frac{\kappa_i r_{il}}{\sigma_I} \]  

Method II

\[ r_{il} = r_{ii} \rho_{il} \]  

Conditions under which both methods give equivalent valuations:

\[ r_{ij} = r_{ii} \rho_{ij} \]
Numerical Example

\[ \sigma_1 = 20\%, \sigma_2 = 30\%, \rho = 40\% \]

\[ r = \begin{bmatrix} -0.7 & -0.5 \\ -0.6 & -0.7 \end{bmatrix}, \quad \kappa_1 = \kappa_2 = 50\% \]
• Propose a stochastic average correlation

• Linear econometric fit:

\[ \bar{\rho} = \alpha + \beta \ln I + \varepsilon \]

Rho_bar is the `average' correlation

\[ OEX : \quad \beta = -0.66 \]
\[ BKX : \quad \beta = -0.34 \]

This model gives rise to an index-dependent implied correlation via SDA
OEX 60 days correlations

- 9-Feb-1999
- 28-Aug-1999
- 1-Oct-2000
- 19-Apr-2001
- 5-Nov-2001
- 24-May-2002
- 10-Dec-2002
- 28-Jun-2003
- 14-Jan-2004
- 1-Aug-2004

Legend:
- implied
- realized shifted
J. Lim: Statistical distribution of implied correlations

\[ f(\bar{\rho}) = \text{p.d.f. for implied correlation} \]

Parametric model:

\[ \bar{\rho} \sim \frac{2}{\pi} \arctan(X) \]

\[ X \sim N(\mu_0, \sigma^2_0) \]

1 month

2 months

-- Heavy right-tails, low mean
-- characteristic of major indices

3 months
BBH Biotech
Index

Mean=0.5-0.55
Heavy tails
PPH Index correlation
SOX Semiconductor Index
The shapes of implied correlation distribution for different sectors
DJX 60 days correlation

28-Aug-1999
15-Mar-2000
1-Oct-2000
19-Apr-2001
5-Nov-2001
24-May-2002
10-Dec-2002
28-Jun-2003
14-Jan-2004
1-Aug-2004

implied
realized
QQQ 60 days correlation

implied
realized
Conclusions

• Steepest-descent approximation: a simple tool for analyzing the volatility skew of index options

• Implied correlation (for an index): the constant correlation number that makes the index option correctly valued

• In general there is a correlation skew and term-structure

• Statistics of implied correlations: evidence of heavy tails for broad market indices; ‘stability’ for narrow sectors

• Study of market correlations presents several open problems that are interesting in theory and practice