Lecture 12: Asymptotics

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G63.2936.001

Spring Semester 2009

From SABR to Geodesics

A systematic approach for modeling volatility curves with applications to option market-making and pricing multi-asset equity derivatives

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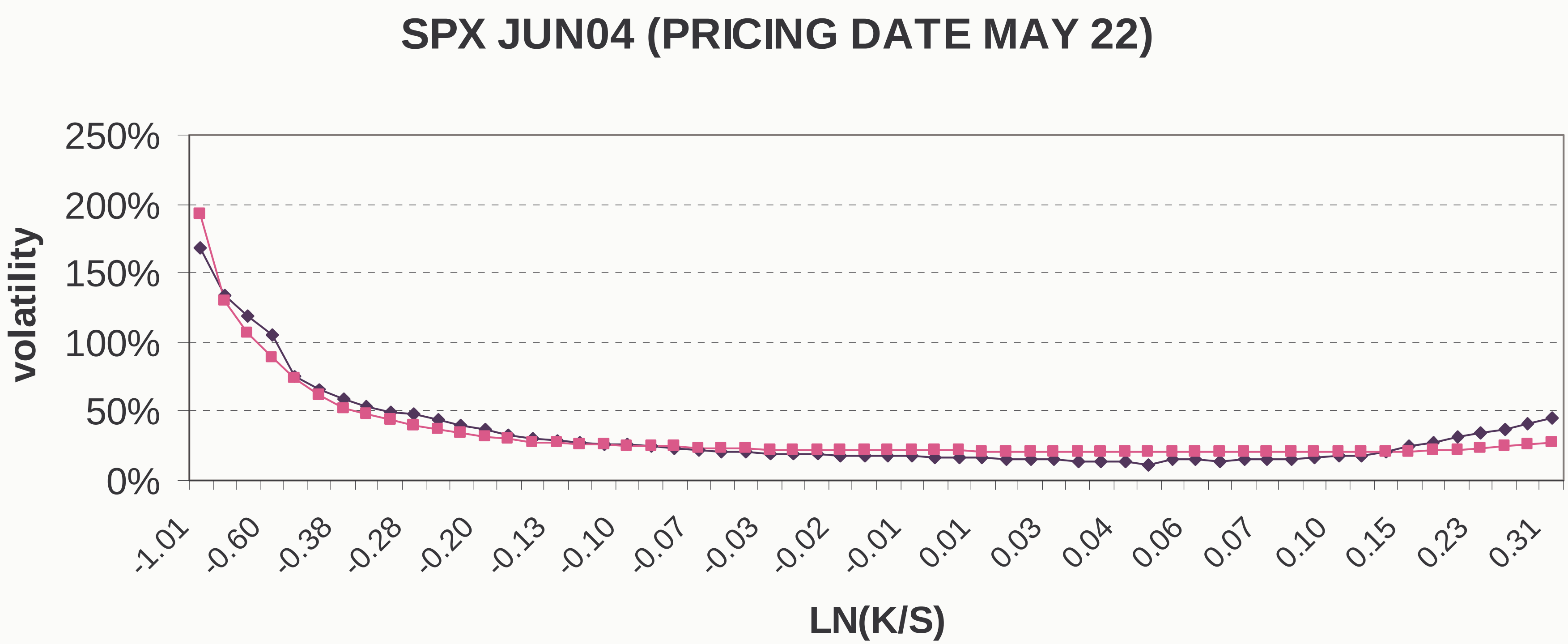
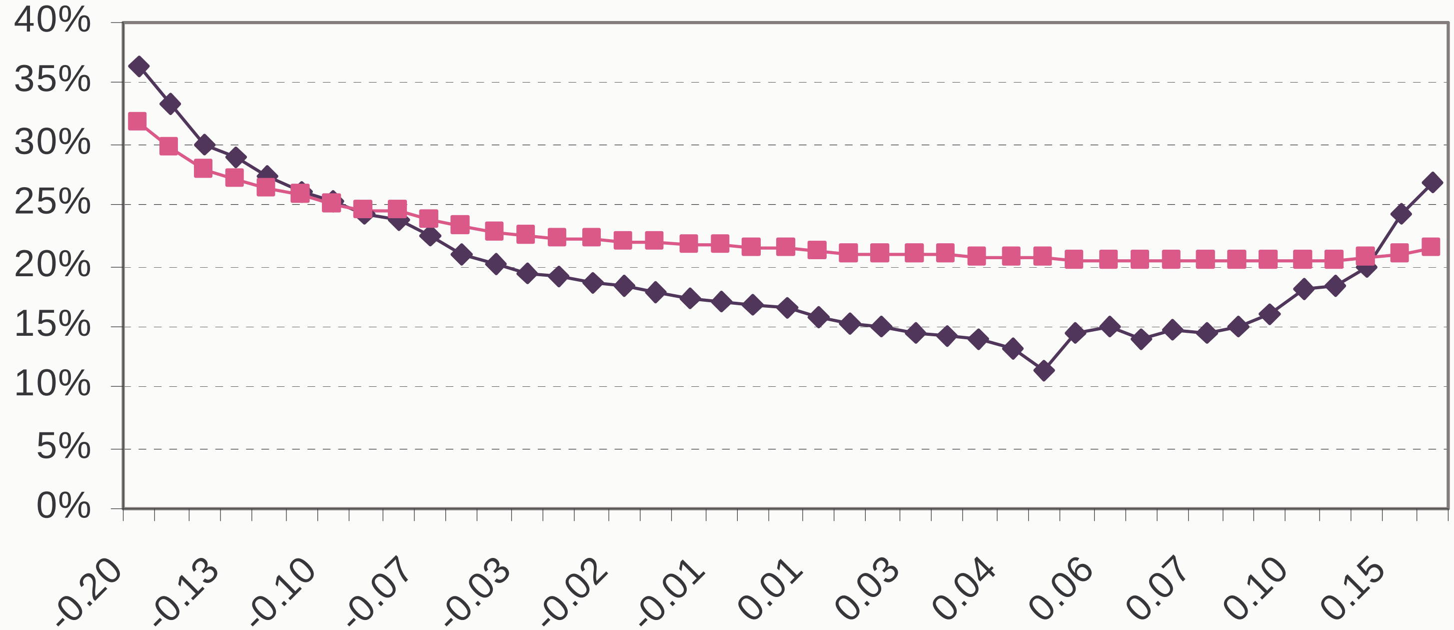
Fitting Volatility Skews



Blue line= average implied vol (puts/calls)

Pink line= fitted parabola

Zoom into the region -0.2<ln(K/F)<0.2



If you zoom into the region of interest, the parabolic fit is seen as clearly inadequate

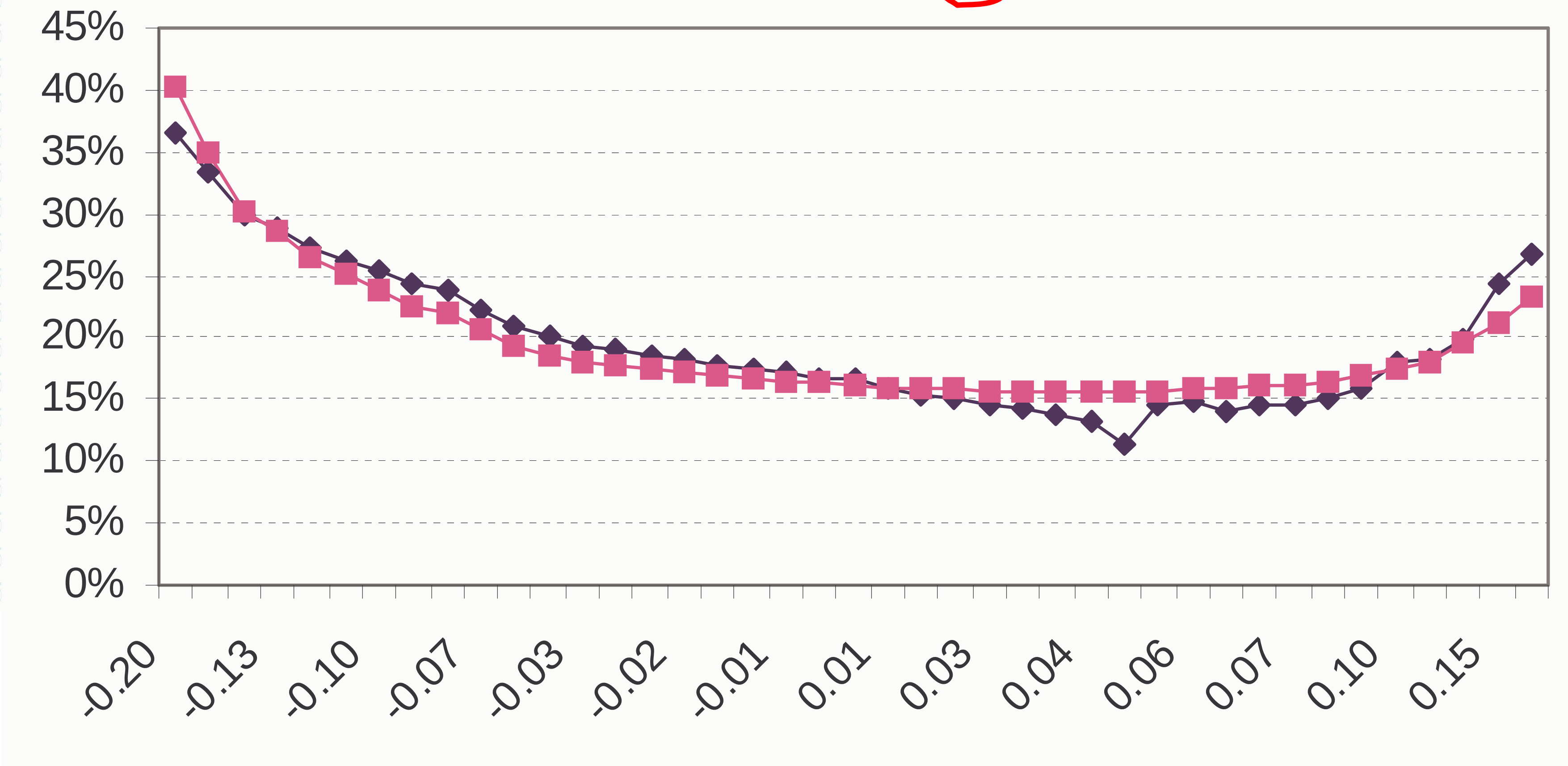
Reason: the out-of-the money options ``lift'' the curve

Parabolic fits are not consistent with arbitrage-free pricing

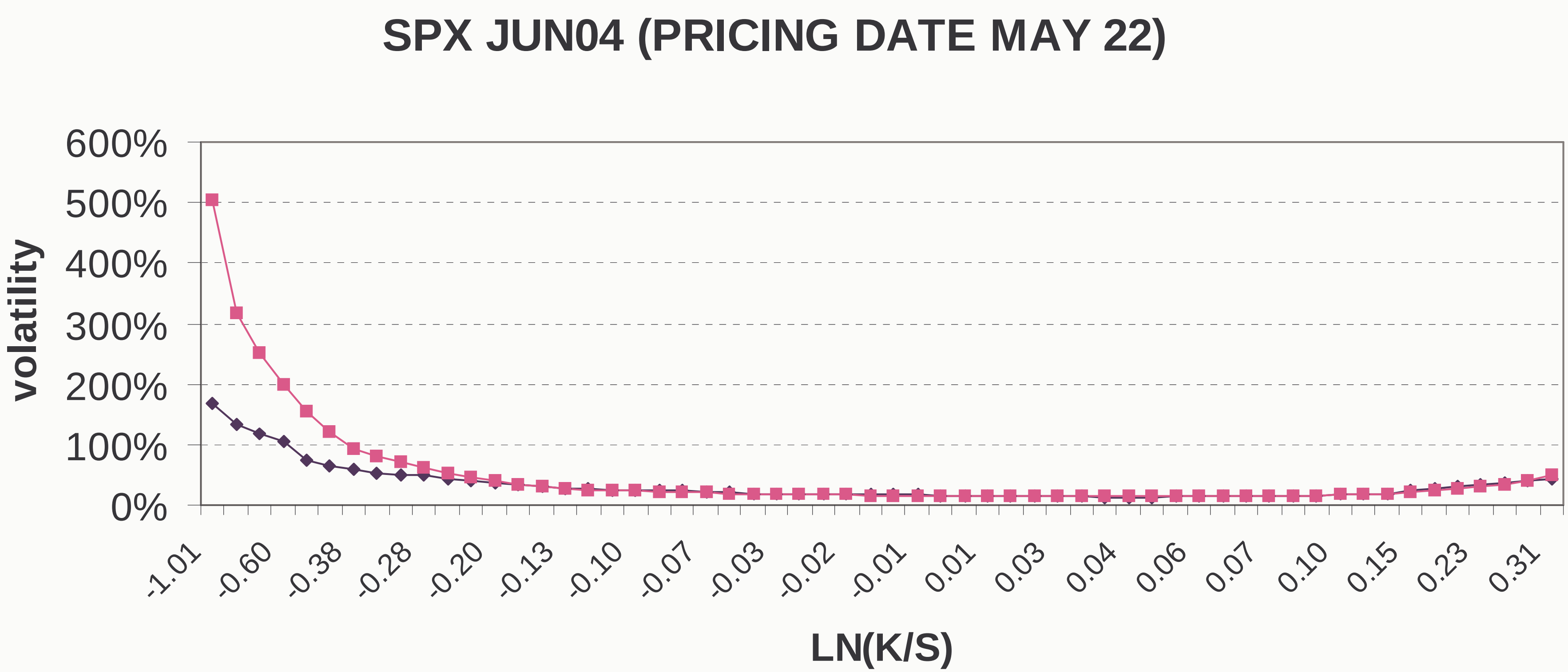
Parabolic fitting requires Delta Truncation!

Fit only volatilities such that -0.2<x<+0.2

$$\sigma_{\text{parabolic}}(x) = 0.16 - 0.34x + 4.45x^2$$
$$= 0.16 \times \left(1 - 2.1\, x + 27.33\, x^2\right)$$



Truncated Parabolic Fit: a look at the full curve



Out of the money options are not guaranteed to be well-fitted

Using a better spline to fit the data (from SABR)

$$\sigma_{\text{imp}}(x) \approx \frac{\kappa|x|}{\ln\left(\kappa\left|\frac{1-e^{-\beta x}}{\sigma_0\beta}\right| + \sqrt{1+\kappa^2\left(\frac{1-e^{-\beta x}}{\sigma_0\beta}\right)^2}\right)}$$

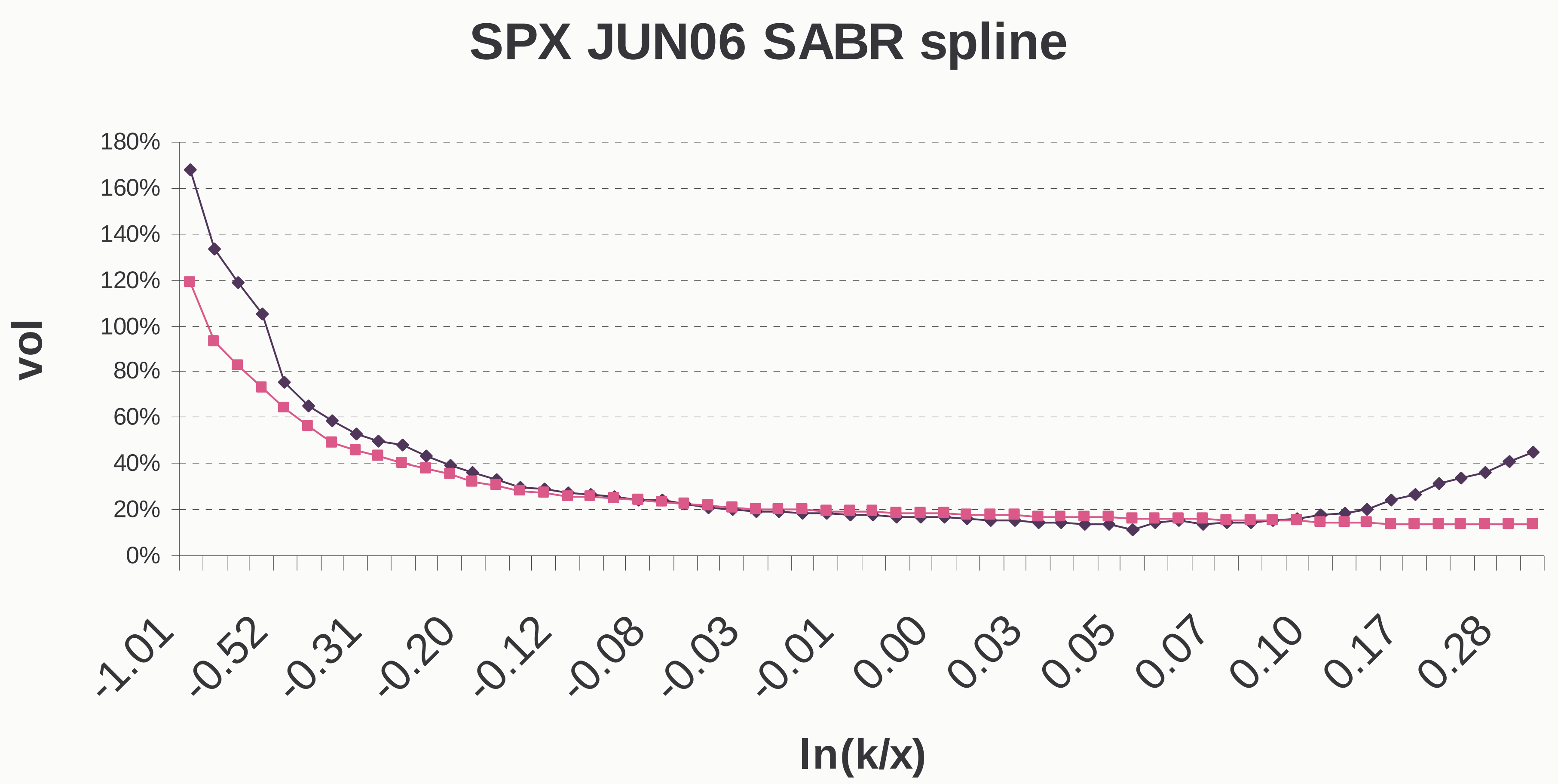
$$x = \ln\left(\frac{K}{F_0}\right)$$

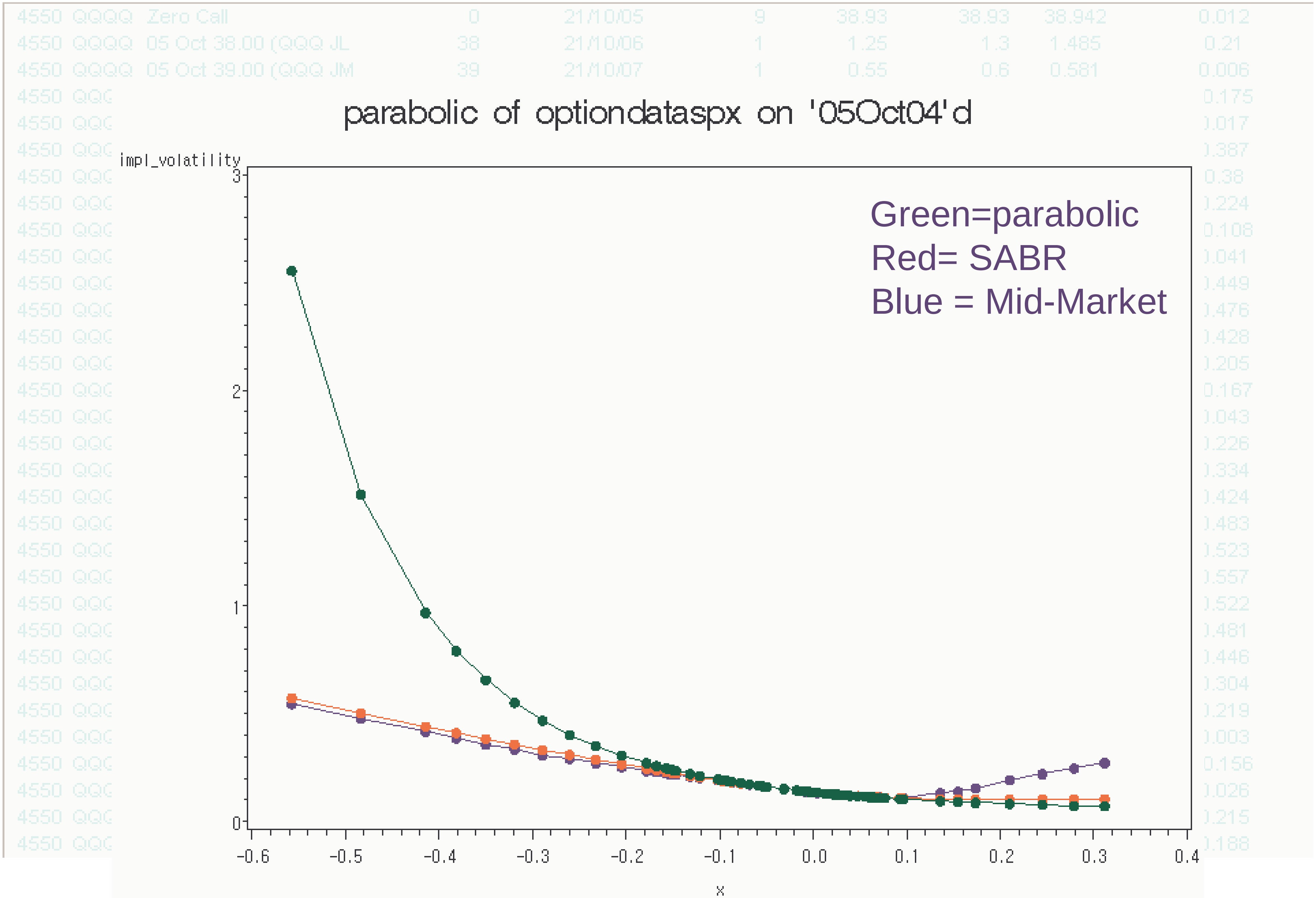
$$\gamma \equiv \text{slope}\left(x=0\right) = \frac{\beta}{2}$$

Sigma, beta and kappa are adjustable parameters

Formula is derived from a **stochastic volatility model** so it does not violate arbitrage conditions

Fitting a SABR-like spline to the SPX front-month curve





parabolic of optiondataspx on '05Oct04'd
impl_volatility
3
2
1
0
Green=parabolic
Red= SABR
Blue = Mid-Market
-0.6
-0.5
-0.4
-0.3
-0.2
-0.1
0.0
0.1
0.2
0.3
0.4
x

Differential Geometry and Implied Volatility Modeling

Factor Models and Diffusion Kernels

CIR-type setting,
X= state variables
W= m-dim Brownian motion

$$\mathrm{x}(t) = (X_1(t),\ldots,X_n(t))$$

$$\mathrm{w}(t) = (W_1(t),\ldots,W_m(t))$$

$$dX_i = \sum_{k=1}^{m} \sigma_j^k dW_k + b_i dt, \qquad i = 1,2,3\ldots,n$$

$$\pi(\mathrm{x},t;\mathrm{y},T) = \mathrm{Prob.}\left\{\mathrm{x}(T) = \mathrm{y} \middle| \mathrm{x}(t) = \mathrm{x}\right\}$$

Diffusion kernel

$$E\left\{F(\mathrm{x}(T)) \middle| \mathrm{x}(t) = \mathrm{x}\right\} = \int_{\mathrm{y}\in R^n} F(\mathrm{y})\pi(\mathrm{x},t;\mathrm{y},T)\,d^n\mathrm{y}$$

Fokker-Planck Equation and Dimensionless Time

$$\frac{\partial \pi}{\partial t} + \frac{1}{2}\sum_{ij=1}^{n} a_{ij}\,\frac{\partial^2 \pi}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i\,\frac{\partial \pi}{\partial x_i} = 0$$

$$\pi(\mathrm{x}, T; \mathrm{y}, T) = \delta(\mathrm{x} - \mathrm{y})$$

$$a_{ij} = \sum_{k=1}^{m} \sigma_i^k \sigma_j^k$$

Covariance matrix of state variables

$$\left(\overline{\sigma}\right)^2 \equiv E\left\{\frac{1}{n}\sum_{i=1}^{n} a_{ii}\right\}$$

$$\tau \equiv \left(\overline{\sigma}\right)^2 t$$

volatility of S&P=0.15
t=1 yr. corresponds to
tau=0.0225<<1

``typical variance'' of x

Dimensionless time

Varadhan Asymptotics for the Diffusion Kernel

$$\lim_{\tau\to 0} \tau \ln \pi\left(\mathrm{x},0;\mathrm{y},T\right) = -\frac{L^2(\mathrm{x},\mathrm{y})}{2}; \qquad \tau = \left(\bar{\sigma}\right)^2 T,$$

$$L(\mathrm{x},\mathrm{y}) = \text{geodesic distance between x and y}$$

$$L(\mathrm{x},\mathrm{y}) = \inf_{\substack{\gamma(0)=x\\ \gamma(1)=y}} \int_0^1 \left\| \frac{\mathrm{d}\gamma}{\mathrm{d}t} \right\|_\gamma dt,$$

$$\left\|v\right\|_{\mathrm{x}}^{2} = \sum_{ij=1}^{n} g_{ij}\left(\mathrm{x}\right) v_i v_j$$

$$g_{ij} = \left(\bar{\sigma}\right)^2 \left(\mathrm{a}^{-1}\right)_{ij}$$

Dimensionless Riemann tensor

Heuristically: Diffusion Kernels ``resemble'' Gaussian Kernels with |x-y| replaced by L(x,y)

$$\pi(x,0;y,T) \approx c(\tau)\cdot e^{-\frac{(L(x,y))^2}{2\tau}} \qquad \tau << 1$$

$$(dL)^2 = \sum_{ij=1}^{n} g_{ij}(\mathrm{x})\,dx_i\,dx_j$$

We shall use this approximation to compute option prices and implied volatilities assuming tau is small

Example 1: Local volatility model

$$\frac{dF_t}{F_t} = \sigma(F_t, t)\,dW_t \qquad x = \ln\left(\frac{F}{F_0}\right)$$

$$dx_t = \sigma(x_t, t)\,dW_t + (\ldots)\,dt$$

$$(dL)^2 = \frac{(\overline{\sigma})^2 dx^2}{(\sigma(x,0))^2} = \frac{dx^2}{\left(\frac{\sigma(x,0)}{\overline{\sigma}}\right)^2} = \left(\frac{dx}{\tilde{\sigma}(x)}\right)^2$$

$$L(x, y) = \left|\int_x^y \frac{du}{\tilde{\sigma}(u)}\right| = \left|G(y) - G(x)\right|$$

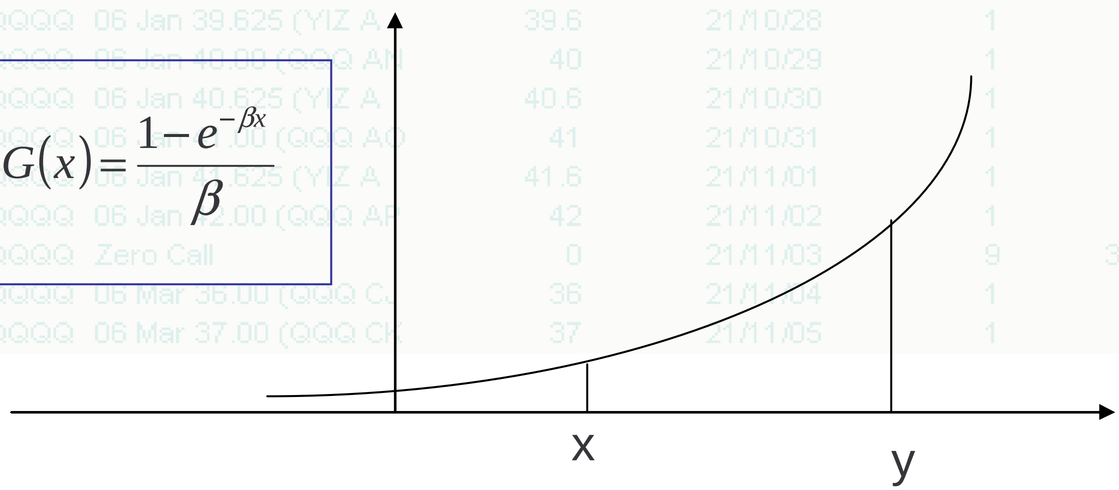
1-dimensional distances are always `trivial'

Special solvable 2-D case: the CEV Model

$$\sigma(F,t) = \bar{\sigma}\left(\frac{F}{F_0}\right)^{\beta} \quad \therefore \quad \sigma(x,t) = \bar{\sigma} e^{\beta x}$$

$$\tilde{\sigma}(x) = e^{\beta x} \qquad L(x,y) = \left|\frac{e^{-\beta x} - e^{-\beta y}}{\beta}\right|$$

$$G(x) = \frac{1 - e^{-\beta x}}{\beta}$$



Negative beta for Equities (leverage)

Distance= area under the curve

Stochastic Volatility Models

$$\frac{dF_t}{F_t} = \sigma_t dW_t$$

Forward price

$$\frac{d\sigma_t}{\sigma_t} = \kappa dZ_t$$

Stochastic vol.

$$E\{dW_t dZ_t\} = \rho dt \qquad \beta \equiv \frac{\kappa\rho}{\sigma}$$

Leverage

$$\frac{d\sigma_t}{\sigma_t} = \beta \frac{dF_t}{F_t} + \varepsilon$$

Beta= regression coefficient of vol on stock returns

Equivalent Model with Independent Brownian Motions (SABR)

$$\sigma_t = \sigma_t^{(0)} \exp(\beta x_t) \qquad x_t = \ln\left(\frac{F_t}{F_0}\right)$$

``Parametric leverage''
SV for tails

$$\frac{d\sigma_t}{\sigma_t} = \frac{d\sigma_t^{(0)}}{\sigma_t^{(0)}} + \beta dx_t$$

``CEV'' with stochastic
<u>independent</u> volatility
is equivalent to SV model
with correlated volatility,
from the Riemann viewpoint

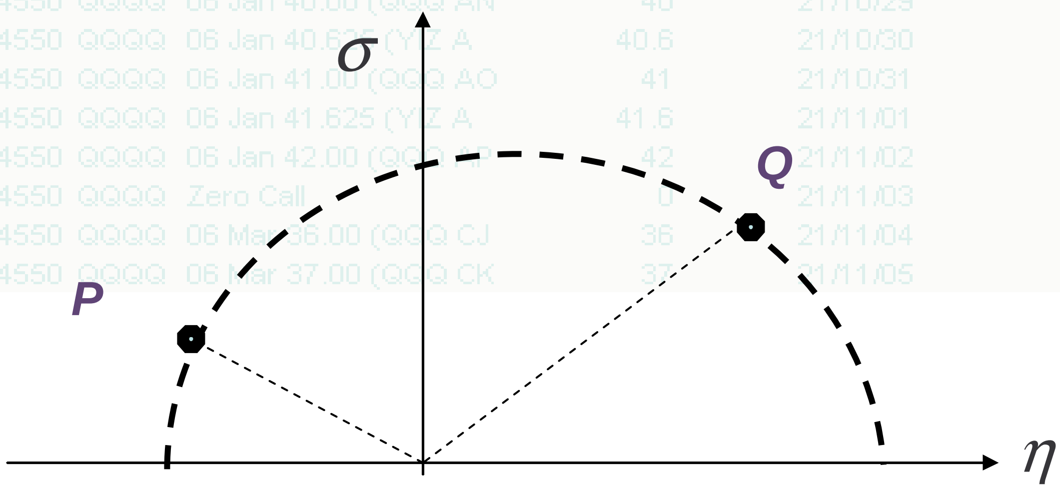
$$\begin{cases} dx_t = e^{\beta x_t} \sigma_t^{(0)} dW_t \\ \dfrac{d\sigma_t^{(0)}}{\sigma_t^{(0)}} = \kappa dZ_t \\ E\left(dW_t dZ_t\right) = 0 \end{cases}$$

Riemann Metric for SV / SABR:
The Poincare Upper Half-Space Model

$$\eta \equiv \kappa \left(\frac{1 - e^{-\beta x}}{\beta} \right), \qquad \sigma \equiv \sigma^{(0)}$$

$$dL^2 = \frac{\overline{\sigma}^2}{\kappa^2} \cdot \frac{d\eta^2 + d\sigma^2}{\sigma^2}$$

Geodesics are half-circles with center on the horizontal axis



$$L(P,Q) = \frac{\overline{\sigma}}{\kappa} \left| \int_{\theta_P}^{\theta_Q} \frac{d\theta}{\sin\theta} \right|$$

Using the asymptotics to compute option prices

$$F_T = F(x_T) \qquad\qquad P_0 = F(0) < K$$

$$CALL = \int_{R^n} \max(F(y) - K, 0)\, \pi(0,0; y, T)\, d^n y$$

$$\approx c \int_{R^n} \max(F(y) - K, 0)\, e^{-\frac{L^2(0,y)}{2\tau}}\, d^n y$$

$$\approx c \int_{\{y:F(y)>K\}} (F(y) - K)\, e^{-\frac{L^2(0,y)}{2\tau}}\, d^n y$$

$$\approx c \int_{\{y:F(y)>K\}} e^{-\left[\ln\left(\frac{1}{F(y)-K}\right) + \frac{L^2(0,y)}{2\tau}\right]}\, d^n y$$

Steepest-descent approximation for computing implied volatilities

$$\int_{\{y:F(y)>K\}} e^{-\left[\ln\left(\frac{1}{F(y)-K}\right)+\frac{L^2(0,y)}{2\tau}\right]} d^n y \approx e^{-\min\limits_{y:F(y)>K}\left[\ln\left(\frac{1}{F(y)-K}\right)+\frac{L^2(0,y)}{2\tau}\right]}$$

$$\min_{y:F(y)>K}\left[\ln\left(\frac{1}{F(y)-K}\right)+\frac{L^2(0,y)}{2\tau}\right] = \frac{1}{\tau}\min_{y:F(y)>K}\left[\tau\ln\left(\frac{1}{F(y)-K}\right)+\frac{L^2(0,y)}{2}\right]$$

$$\approx \frac{1}{2\tau}\min\left\{L^2(0,y) \,\middle|\, y:F(y)>K\right\}, \quad \tau << 1$$

Equate formulas for OTM calls with Black-Scholes …

$$L^*(K) = \min\left\{ L(0,y) \,\middle|\, y : F(y) > K \right\}$$

Minimum distance from 0 to the region {F(y)>0}

$$\ln CALL \approx -\frac{\left(L^*(K)\right)^2}{2\tau}$$

Small-tau asymptotics **(model)**

$$\ln CALL \approx -\frac{\left(\ln\left(K/F_0\right)\right)^2}{2\sigma_{\text{imp}}^2(K)T} = -\frac{\left(\ln\left(K/F_0\right)\right)^2}{2\left(\dfrac{\sigma_{\text{imp}}^2(K)}{\left(\bar{\sigma}\right)^2}\right)\tau}$$

Small-tau asymptotics **(Black-Scholes)**

Approximation for Implied Volatility for general diffusion model

$$\sigma_{\text{imp}}(K) = \bar{\sigma}\,\frac{\left|\ln\left(K / F_0\right)\right|}{\min\left\{L(0,\mathrm{y}) \,\middle|\, \mathrm{y} : F(\mathrm{y}) > K\right\}}$$

$$= \frac{\left|\ln\left(K / F_0\right)\right|}{\min\left\{L_1(0,\mathrm{y}) \,\middle|\, \mathrm{y} : F(\mathrm{y}) > K\right\}}$$

$$L_1(x,y) \equiv \min_{\substack{\gamma(0)=\mathrm{x} \\ \gamma(1)=\mathrm{y}}} \int_0^1 \sqrt{\sum_{ij=1}^{n} \left(\mathrm{a}^{-1}\right)_{ij} \gamma_i(t)\gamma_j(t)}\,dt$$

Example 1: Local Volatility Model

$$\sigma_{\text{imp}}(K) \approx \frac{\ln(K/F_0)}{\displaystyle\int_0^{\ln(K/F_0)} \frac{du}{\sigma(u,0)}} = \left(\frac{1}{x} \int_0^x \frac{du}{\sigma(u,0)} \right)^{-1}$$

$$x = \ln\left(\frac{K}{F_0}\right)$$

Implied Volatility= Harmonic Mean of Local Volatility

Berestycki, Busca and Florent, 2001

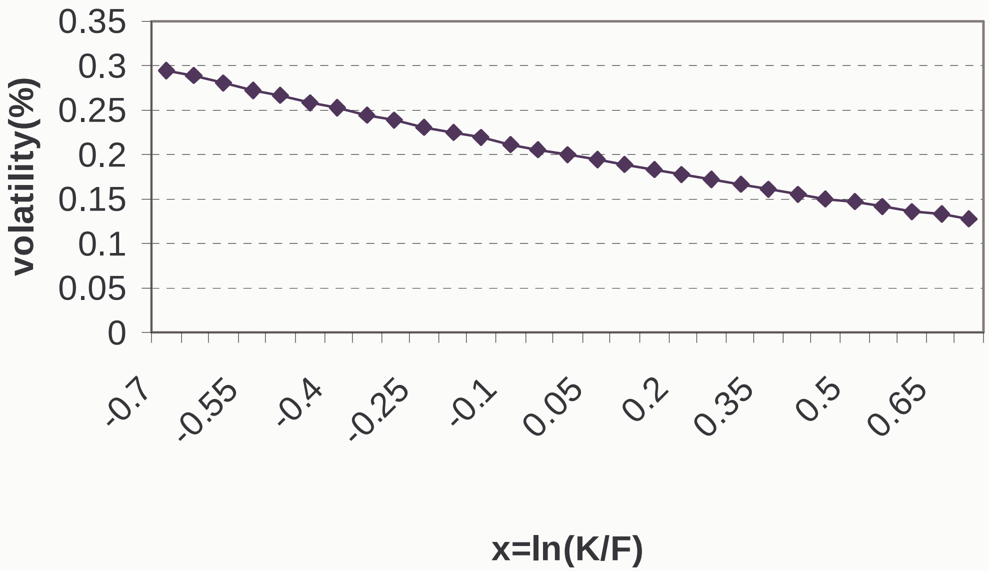
Example 2: Constant Elasticity of Variance

$$\sigma(x,t) = \sigma_0 e^{\beta x}$$

CEV

$$\sigma_{\text{imp}}(x) \approx \sigma_0 \left| \frac{\beta x}{1 - e^{-\beta x}} \right|$$

Implied volatility

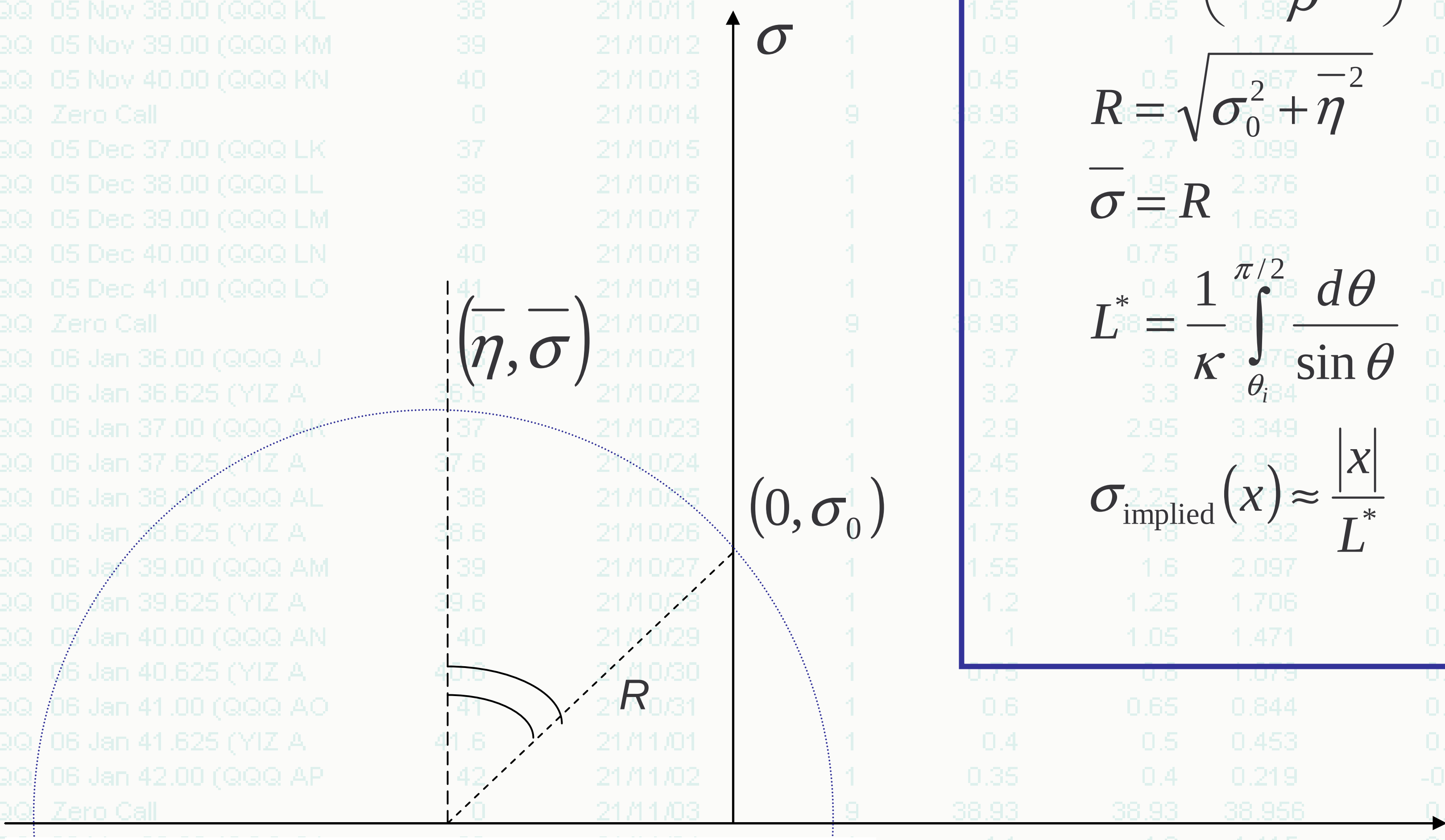


Example 3: Stochastic Volatility / SABR

$$\sigma_{\mathrm{imp}}(x) \approx \frac{\kappa |x|}{\ln\left(\kappa \left| \frac{1-e^{-\beta x}}{\sigma_0 \beta} \right| + \sqrt{1+\kappa^2 \left(\frac{1-e^{-\beta x}}{\sigma_0 \beta} \right)^2} \right)}$$

$$x = \ln\left(\frac{K}{F_0} \right)$$

Minimizing the distance to the line eta =const. in the Poincare plane



$$\bar{\eta} = \kappa\left(\frac{1-e^{-\beta x}}{\beta}\right)$$

$$R = \sqrt{\sigma_0^2 + \bar{\eta}^2}$$

$$\bar{\sigma} = R$$

$$L^* = \frac{1}{\kappa}\int_{\theta_i}^{\pi/2} \frac{d\theta}{\sin\theta}$$

$$\sigma_{\text{implied}}(x) \approx \frac{|x|}{L^*}$$

Example 3bis: Stochastic Volatility / Hull-White

$$\frac{dS}{S} = \sigma dW, \qquad x = \ln(S/S_0)$$

$$\frac{d\sigma}{\sigma} = \kappa dZ, \quad E\left(dWdZ\right) = \rho dt$$

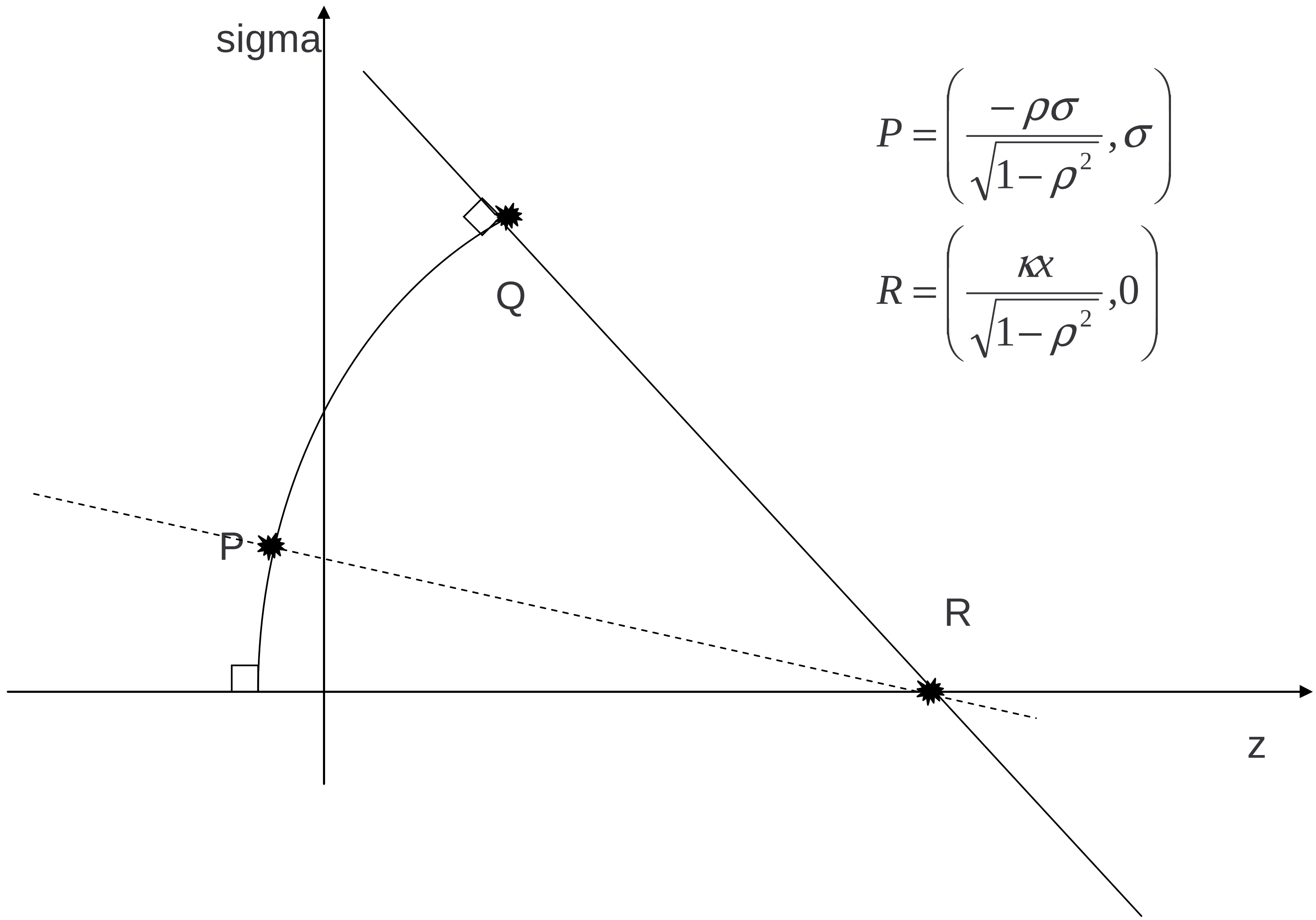
$$dL^2 = \frac{1}{\kappa^2\left(1-\rho^2\right)} \cdot \frac{\kappa^2 dx^2 - 2\rho\kappa d\sigma dx + d\sigma^2}{\sigma^2}$$

$$z = \frac{\kappa x - \rho\sigma}{\sqrt{1-\rho^2}}$$

$$dL^2 = \frac{1}{\kappa^2} \cdot \frac{dz^2 + d\sigma^2}{\sigma^2}$$

Poincare plane after change of variables

Geodesic Distance



$$P = \left(\frac{-\rho\sigma}{\sqrt{1-\rho^2}}, \sigma \right)$$

$$R = \left(\frac{\kappa x}{\sqrt{1-\rho^2}}, 0 \right)$$

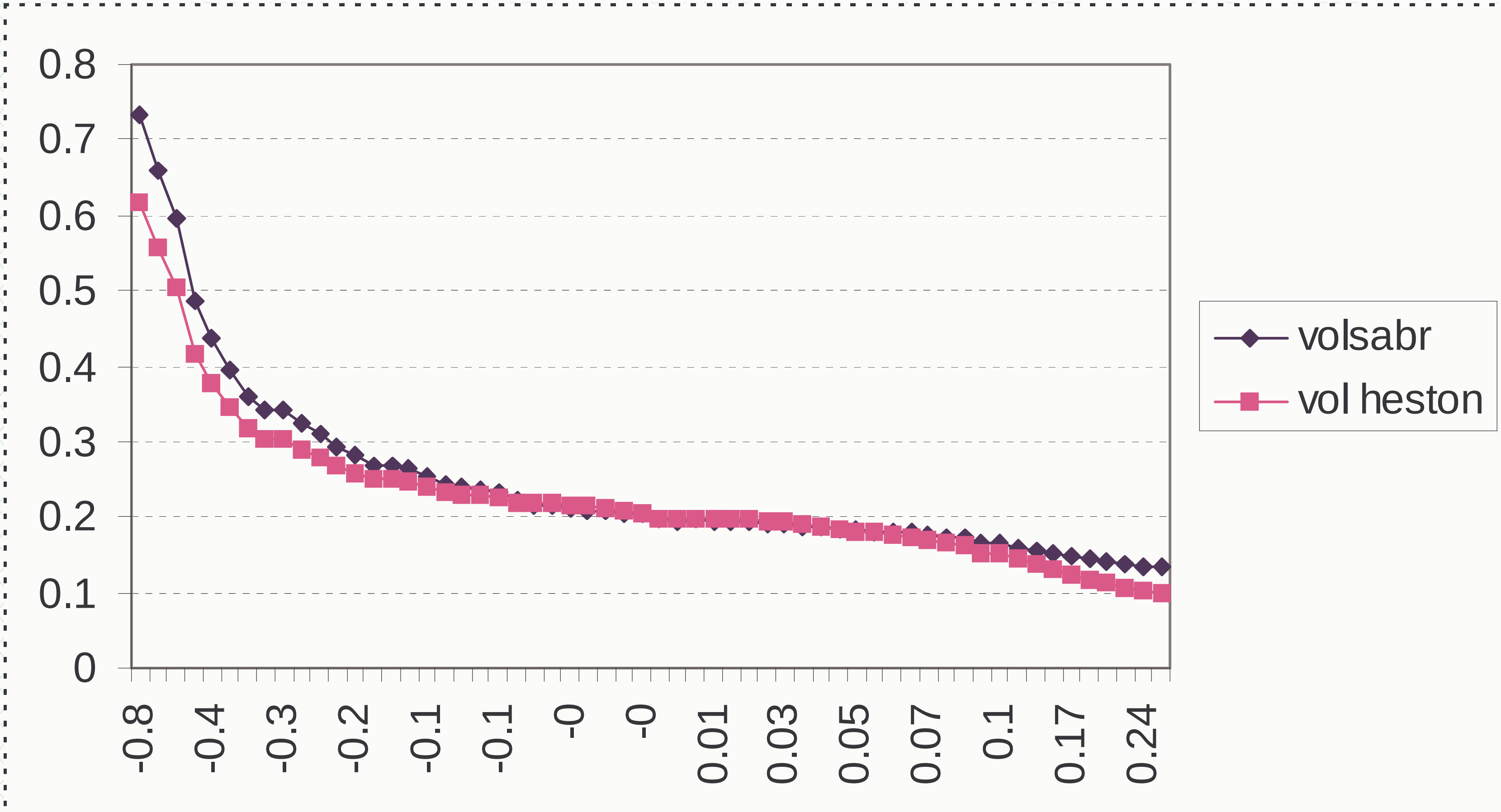
Approximation for Implied Volatility Curves

$$L^* = d(P,Q) = \frac{1}{\kappa}\int_{\vartheta_P}^{\vartheta_Q} \frac{du}{\sin u}$$

$$L^* = \frac{1}{\kappa}\left|\ln\left(\frac{\kappa x + \rho\sigma + \sqrt{(\kappa x + \rho\sigma)^2 + \sigma^2(1-\rho^2)}}{\sigma(1+\rho)}\right)\right|$$

$$\sigma_{imp}(x) \approx \frac{\kappa|x|}{\left|\ln\left(\frac{\kappa x + \rho\sigma + \sqrt{(\kappa x + \rho\sigma)^2 + \sigma^2(1-\rho^2)}}{\sigma(1+\rho)}\right)\right|}$$

Auto-calibration of SABR and Heston



$$\sigma_0 = 20\% \qquad \kappa_{\text{sabr}} = 0.5$$

$$\beta = -4 \qquad \kappa_{\text{Heston}} = 2\sigma_0 \kappa_{\text{sabr}} = 0.2$$

Example 4: the Heston Model
A variant of the Poincare Half-Space

$$\frac{dS_t}{S_t} = \sqrt{V_t}\, dW_t$$

$$dV_t = \kappa \sqrt{V_t}\, dZ_t \qquad E\left(dW_t dZ_t\right) = \rho\, dt$$

$$dL_1^2 = \frac{1}{\kappa^2} \frac{d\xi^2 + dV^2}{V} \qquad \xi = \frac{\kappa\left(1 - e^{-\beta x}\right)}{\beta}$$

Note: V, <u>not V squared</u>

Closed-form solution for geodesics

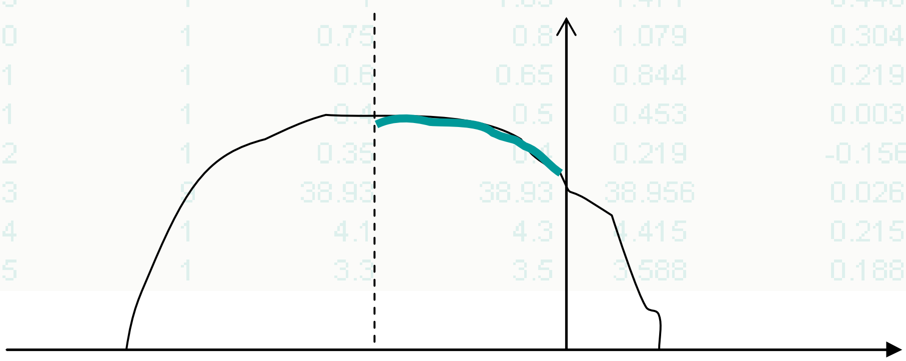
$$\xi = \kappa\left(\frac{1-e^{-\beta x}}{\beta}\right) \qquad dL^2 = \frac{d\xi^2 + dV^2}{\kappa^2 V}$$

$$\xi(\theta) = \frac{R^2}{2}(\theta - \sin\theta\cos\theta) + \xi(0)$$

$$V(\theta) = R^2\sin^2\theta \qquad 0 \le \theta \le \pi$$

$$dL = \frac{2R^2}{\kappa}\sin\theta d\theta$$

Geodesics are cycloids



Implied volatility curve for Heston model is obtained as an algebraic system

$$\xi = \frac{\sigma_0^2}{\sin^2\theta_{\text{init}}}\left(\frac{\pi}{2} - \theta_{\text{init}} + \sin\theta_{\text{init}}\cos\theta_{\text{init}}\right)$$

$$\sigma(\xi) = \frac{\kappa|\xi|\sin^2\theta_{\text{init}}}{2\sigma_0^2|\cos\theta_{\text{init}}|}$$

Given xi, solve for theta_init, and substitute in the second equation

Multi-Asset Derivatives

4550	QQQQ	Zero Call	0	21/10/05	9	38.93	38.93	38.942	0.012
4550	QQQQ	05 Oct 38.00 (QQQ JL	38	21/10/06	1	1.25	1.3	1.485	0.21
4550	QQQQ	05 Oct 39.00 (QQQ JM	39	21/10/07	1	0.55	0.6	0.581	0.006
4550	QQQQ	05 Oct 40.00 (QQQ JN	40	21/10/08	1	0.15	0.2	0	-0.175
4550	QQQQ	Zero Call	0	21/10/09	9	38.93	38.93	38.947	0.017
4550	QQQQ	05 Nov 37.00 (QQQ KK	37	21/10/10	1	2.35	2.45	2.787	0.387
4550	QQQQ	05 Nov 38.00 (QQQ KL	38	21/10/11	1	1.55	1.65	1.98	0.38
4550	QQQQ	05 Nov 39.00 (QQQ KM	39	21/10/12	1	0.9	1	1.174	0.224
4550	QQQQ	05 Nov 40.00 (QQQ KN	40	21/10/13	1	0.45	0.5	0.367	-0.108
4550	QQQQ	Zero Call	0	21/10/14	9	38.93	38.93	38.971	0.041
4550	QQQQ	05 Dec 37.00 (QQQ LK	37	21/10/15	1	2.6	2.7	3.099	0.449
4550	QQQQ	05 Dec 38.00 (QQQ LL	38	21/10/16	1	1.85	1.95	2.376	0.476
4550	QQQQ	05 Dec 39.00 (QQQ LM	39	21/10/17	1	1.2	1.25	1.653	0.428
4550	QQQQ	05 Dec 40.00 (QQQ LN	40	21/10/18	1	0.7	0.75	0.93	0.205
4550	QQQQ	05 Dec 41.00 (QQQ LO	41	21/10/19	1	0.35	0.4	0.208	-0.167
4550	QQQQ	Zero Call	0	21/10/20	9	38.93	38.93	38.973	0.043
4550	QQQQ	06 Jan 36.00 (QQQ AJ	36	21/10/21	1	3.7	3.8	3.976	0.226
4550	QQQQ	06 Jan 36.625 (Y[illegible]	36.6	[illegible]	[illegible]	[illegible]	[illegible]	3.584	0.334
4550	QQQQ	06 Jan 37.00 (QQQ AK	37	21/10/23	1	2.9	2.95	3.349	0.424
4550	QQQQ	06 Jan 37.625 (YIZ A	37.6	21/10/24	1	2.45	2.5	2.958	0.483
4550	QQQQ	06 Jan 38.00 (QQQ AL	38	21/10/25	1	2.15	2.25	2.723	0.523
4550	QQQQ	06 Jan 38.625 (YIZ A	38.6	21/10/26	1	1.75	1.8	2.332	0.557
4550	QQQQ	06 Jan 39.00 (QQQ AM	39	21/10/27	1	1.55	1.6	2.097	0.522
4550	QQQQ	06 Jan 39.625 (YIZ A	39.6	21/10/28	1	1.2	1.25	1.706	0.481
4550	QQQQ	06 Jan 40.00 (QQQ AN	40	21/10/29	1	1	1.05	1.471	0.446
4550	QQQQ	06 Jan 40.625 (YIZ A	40.6	21/10/30	1	0.75	0.8	1.079	0.304
4550	QQQQ	06 Jan 41.00 (QQQ AO	41	21/10/31	1	0.6	0.65	0.844	0.219
4550	QQQQ	06 Jan 41.625 (YIZ A	41.6	21/11/01	1	0.4	0.5	0.453	0.003
4550	QQQQ	06 Jan 42.00 (QQQ AP	42	21/11/02	1	0.35	0.4	0.219	-0.156
4550	QQQQ	Zero Call	0	21/11/03	9	38.93	38.93	38.956	0.026
4550	QQQQ	06 Mar 36.00 (QQQ CJ	36	21/11/04	1	4.1	4.3	4.415	0.215
4550	QQQQ	06 Mar 37.00 (QQQ CK	37	21/11/05	1	3.3	3.5	3.588	0.188

Multi-Asset Derivatives: Index Options, Rainbows

Derive index volatility skew from **single-stock skews** and **correlation matrix**

$$dx_i = \sigma(x_i, t)\, dW_i, \qquad i = 1, 2, \ldots, n$$

$$E(dW_i dW_j) = \rho_{ij}\, dt$$

N equations for the index components

$$I = \sum_{i=1}^{n} w_i S_i = \sum_{i=1}^{n} w_i S_i(0) e^{x_i} \qquad \bar{x} = \ln\left(\frac{F_I}{I(0)}\right)$$

BBH: ETF of 20 Biotechnology Stocks

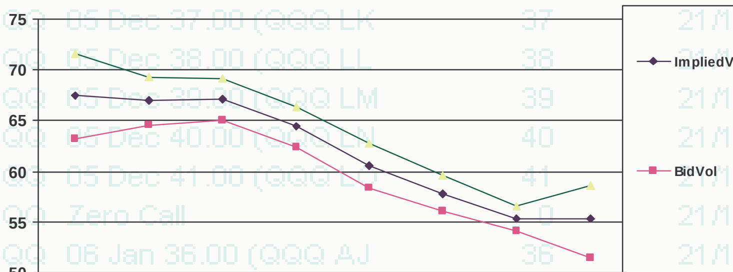
(Components of IBH)

Ticker	Shares	ATM ImVol	Ticker	Shares	ATM ImVol
ABI	18	55	GILD	8	46
AFFX	4	64	HGSI	8	84
ALKS	4	106	ICOS	4	64
AMGN	**46**	40	IDPH	12	72
BGEN	13	41	MEDI	15	82
CHIR	16	37	MLNM	12	92
CRA	4	55	QLTI	5	64
DNA	44	53.5	SEPR	6	84
ENZN	3	81	SHPGY	6.8271	47
GENZ	14	56	BBH	-	32

Implied Volatility Skews

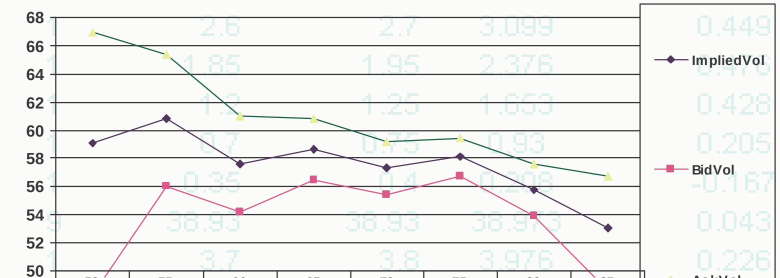
Multiple Names, Multiple Expirations

AMGN



	60	65	70	75	80	85	90	95
ImpliedVol	67.5042	66.93523	67.08418	64.42438	60.53124	57.80586	55.33041	55.29034
BidVol	63.23	64.55163	65.02824	62.43146	58.36119	56.02341	54.08097	51.39689
AskVol	71.59664	69.29996	69.1395	66.41618	62.68222	59.54988	56.54053	58.64633
VarSwap								

BGEN



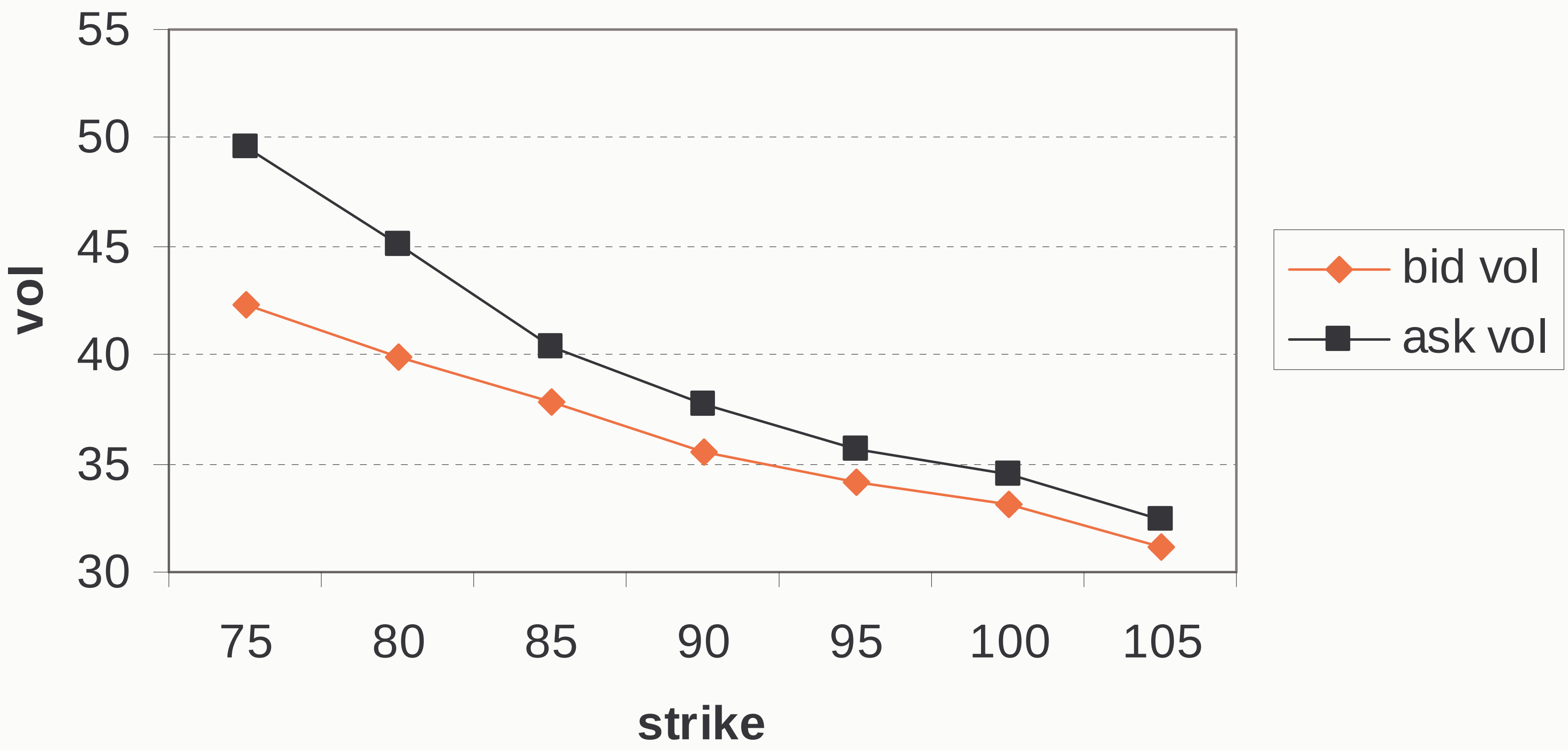
	50	55	60	65	70	75	80	85
ImpliedVol	59.10441	60.82449	57.59728	58.64378	57.3007	58.09035	55.77914	53.02048
BidVol	48.36397	55.99293	54.16418	56.42753	55.39614	56.75332	53.91233	48.60503
AskVol	66.93634	65.38443	60.98989	60.86071	59.19764	59.41203	57.57386	56.72775
VarSwap								

MEDI

	53.4	56.6	58.4	60	65	70	75	80	85	90
ImpliedVol	74.2145	73.3906	71.4854	68.4688	68.7068	64.2811	65.1807	64.3257	62.4619	63.1047
BidVol	57.9276	60.658	59.7874	57.4033	62.3039	58.7537	62.3079	59.6994	59.4478	60.8186
AskVol	0	0	81.818	78.353	74.8939	69.7241	68.0496	68.9602	65.4771	65.3854
VarSwap										

BBH March 2003 Implied Vols

Pricing Date: Jan 22 03 10:42 AM



What is the `fair value' of the index volatility reconstructed from the components?

Riemannian metric
for the multi-D local vol model

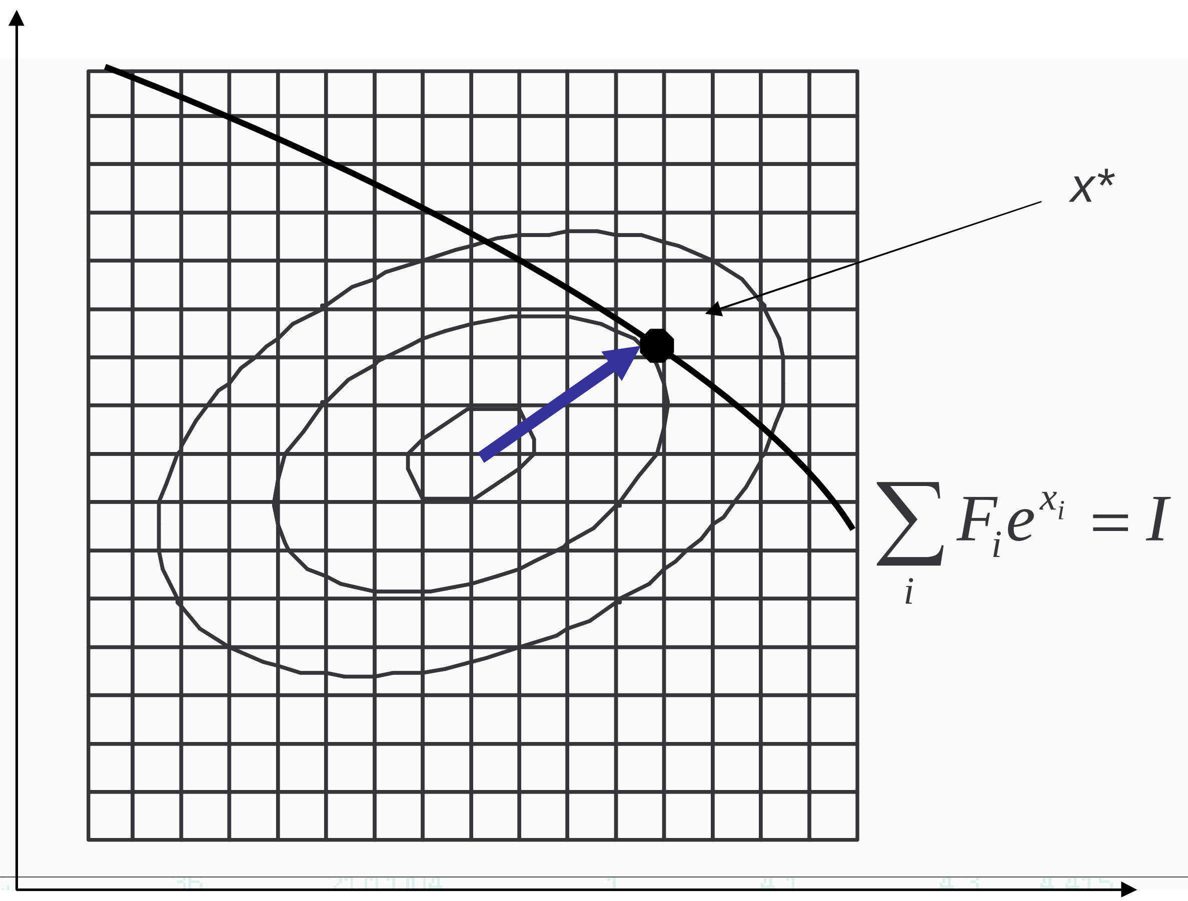
$$dL^2 = \sum_{ij=1}^{n} \left(\rho^{-1}\right)_{ij} \frac{dx_i}{\sigma(x_i,0)} \frac{dx_j}{\sigma(x_j,0)}$$

$$= \sum_{ij=1}^{n} \left(\rho^{-1}\right)_{ij} dy_i dy_j , \qquad dy_i \equiv \frac{dx_i}{\sigma(x_i,0)}$$

If correlations are constant, the metric is ``flat'': it is Euclidean metric after making the change of variables x->y.

Geodesics are straight lines in the y-coordinates

Steepest Descent=Most Likely Stock Price Configuration



Replace conditional distribution by “Dirac function” at most likely configuration

Exact solution: Euler-Lagrange Equations

$$\sigma_{\text{impl.}\,I}\left(\bar{x}\right)=\frac{\left|\bar{x}\right|}{\sqrt{\sum_{ij=1}^{n}\left(\rho^{-1}\right)_{ij}\int_{0}^{x_i^*}\frac{du}{\sigma_i\left(u,0\right)}\int_{0}^{x_j^*}\frac{du}{\sigma_j\left(u,0\right)}}}$$

$$=\frac{\left|\bar{x}\right|}{\sqrt{\sum_{ij=1}^{n}\left(\rho^{-1}\right)_{ij}\frac{x_i^*}{\sigma_{\text{impl.},i}\left(x_i^*\right)}\frac{x_j^*}{\sigma_{\text{impl.},j}\left(x_j^*\right)}}}$$

Euler - Lagrange equations

$$\int_{0}^{x_i^*}\frac{du}{\sigma_i\left(u,0\right)}=\Lambda\sum_{j=1}^{n}\rho_{ij}\,p_j\left(x_j^*\right)\sigma_j\left(x_j^*,0\right)\qquad i=1,2,\ldots n$$

Approximate solution: introduce the stock betas

$$x_i = \beta_i \bar{x} + \varepsilon_i$$

Regression relation between stock and index returns

$$x_i^* = \beta_i \bar{x}$$

Approximate formula for the optimal stock configuration

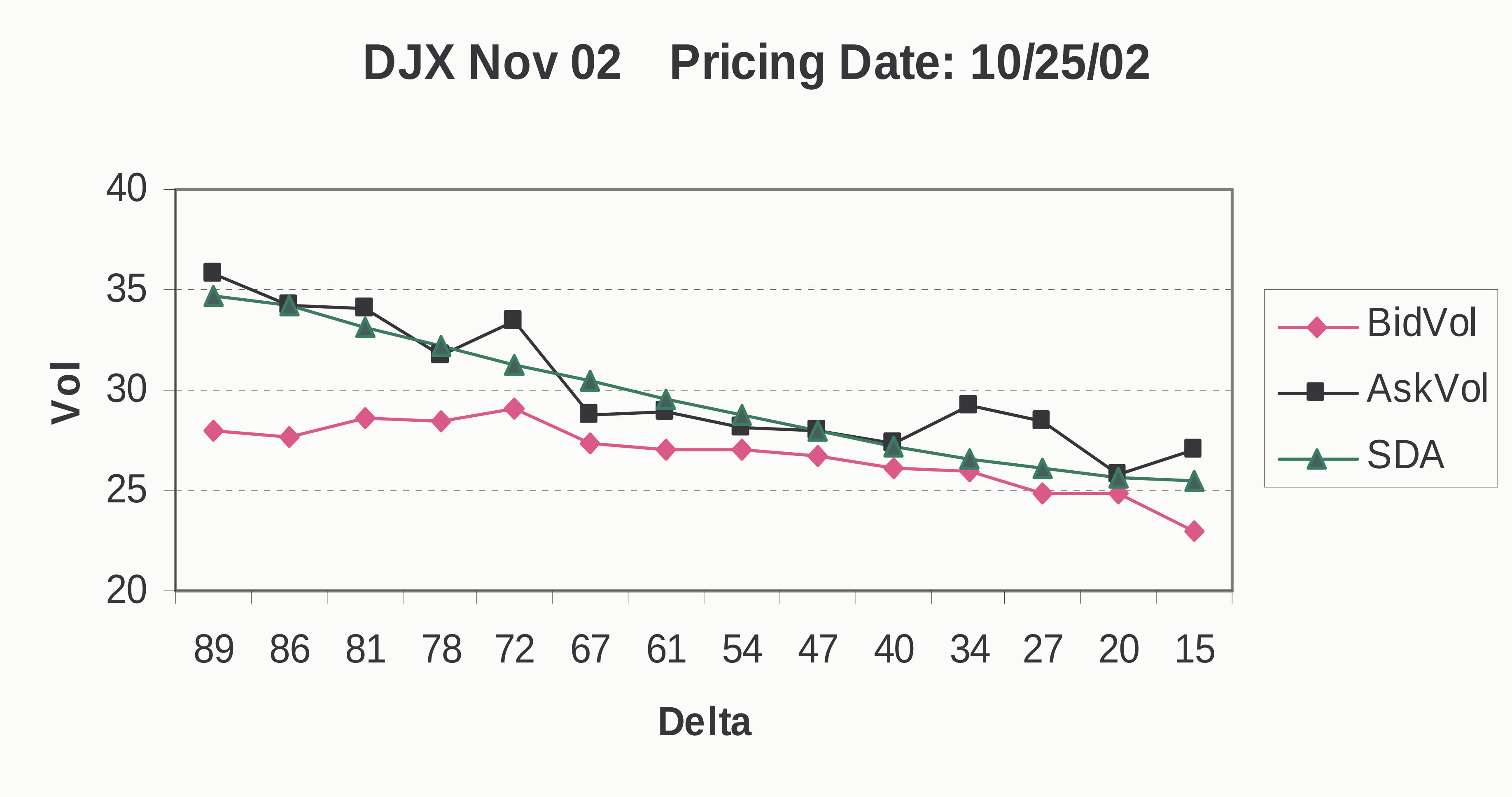
$$\frac{1}{\sigma_{\mathrm{imp},I}\left(\bar{x}\right)} \approx \sqrt{\sum_{ij=1}^{n} \frac{\left(\rho^{-1}\right)_{ij} \beta_i \beta_j}{\sigma_{\mathrm{imp},i}\left(\beta_i \bar{x}\right) \sigma_{\mathrm{imp},i}\left(\beta_i \bar{x}\right)}}$$

$$\sigma_{\mathrm{imp},I}\left(\bar{x}\right) \approx \sqrt{\sum_{ij=1}^{n} \rho_{ij}\, p_i\, p_j\, \sigma_{\mathrm{imp},i}\left(\beta_i \bar{x}\right) \sigma_{\mathrm{imp},i}\left(\beta_i \bar{x}\right)}$$

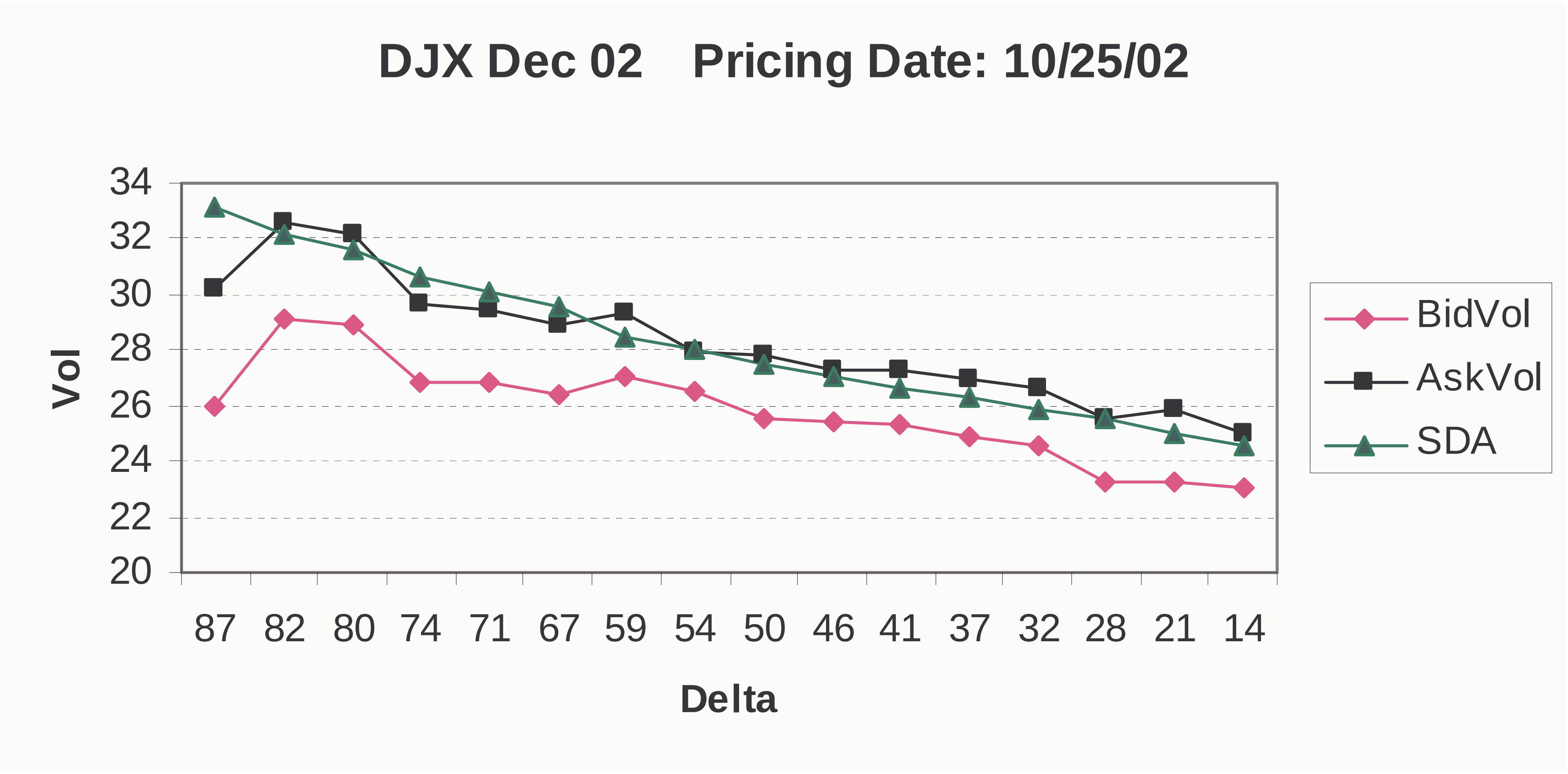
Performs well in the range -0.2<x<+0.2

DJX: Dow Jones Industrial Average

T=1 month

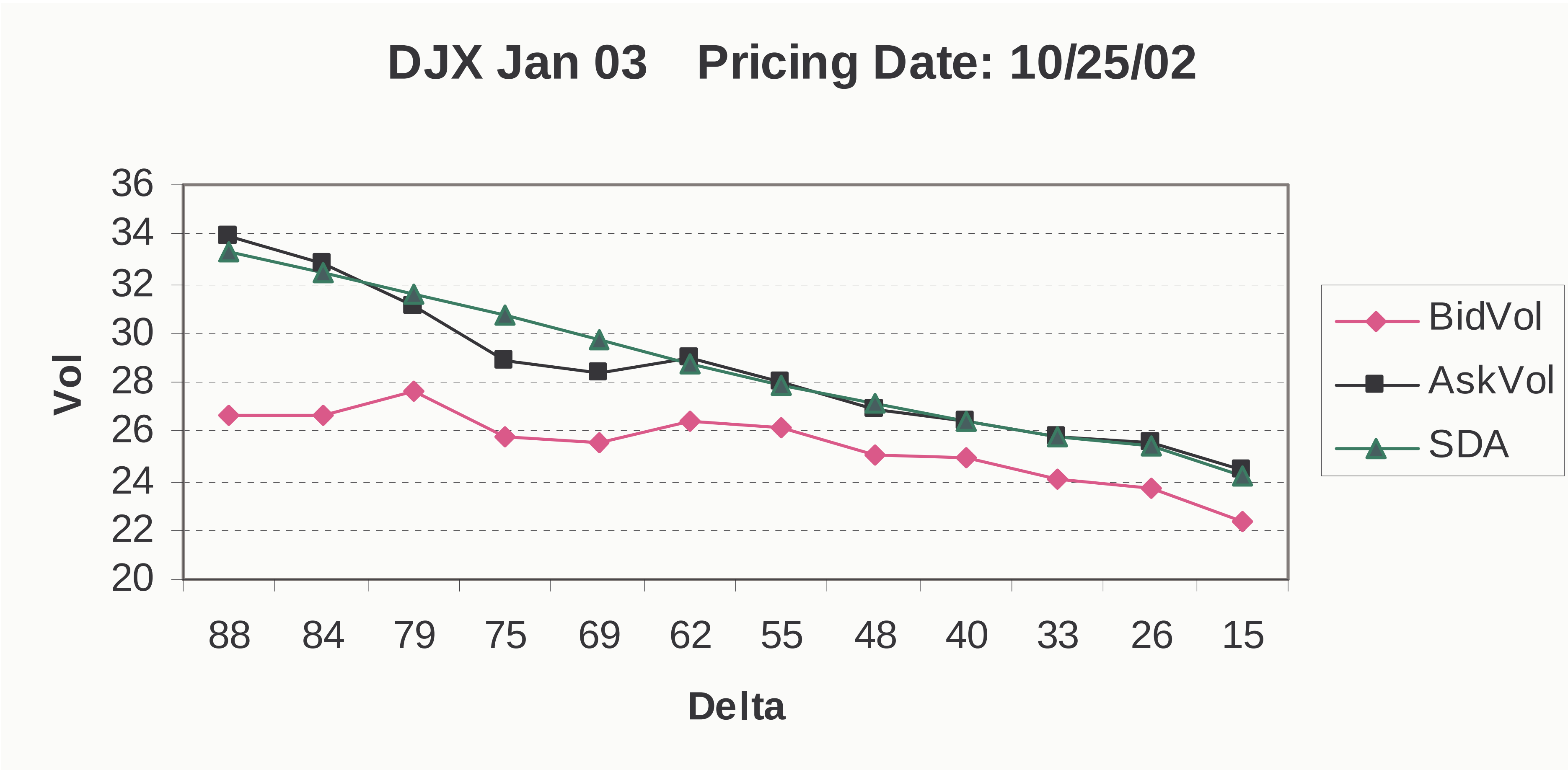


T= 2 months



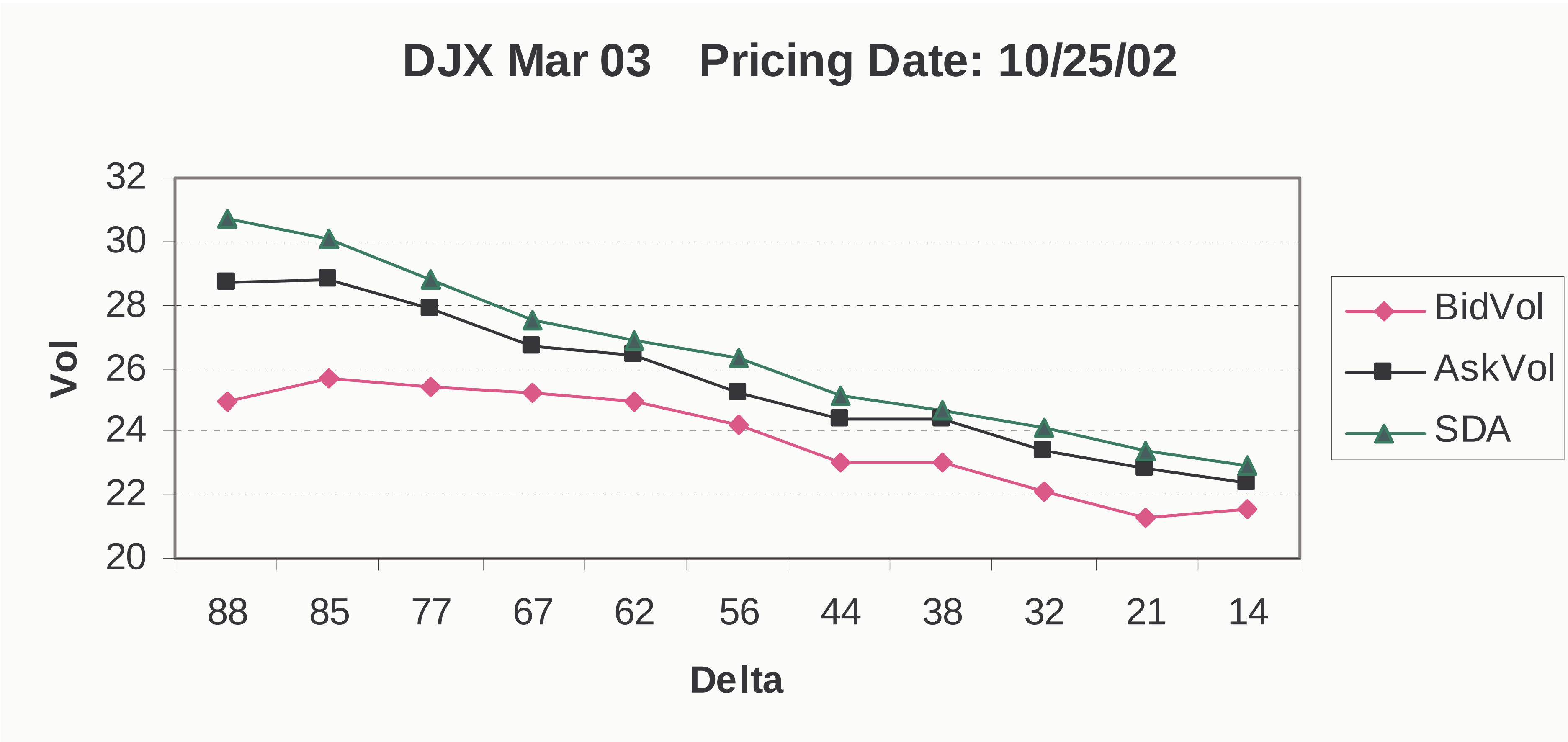
DJX Dec 02 Pricing Date: 10/25/02
Vol
34
32
30
28
26
24
22
20
87
82
80
74
71
67
59
54
50
46
41
37
32
28
21
14
Delta
BidVol
AskVol
SDA

T=3 months



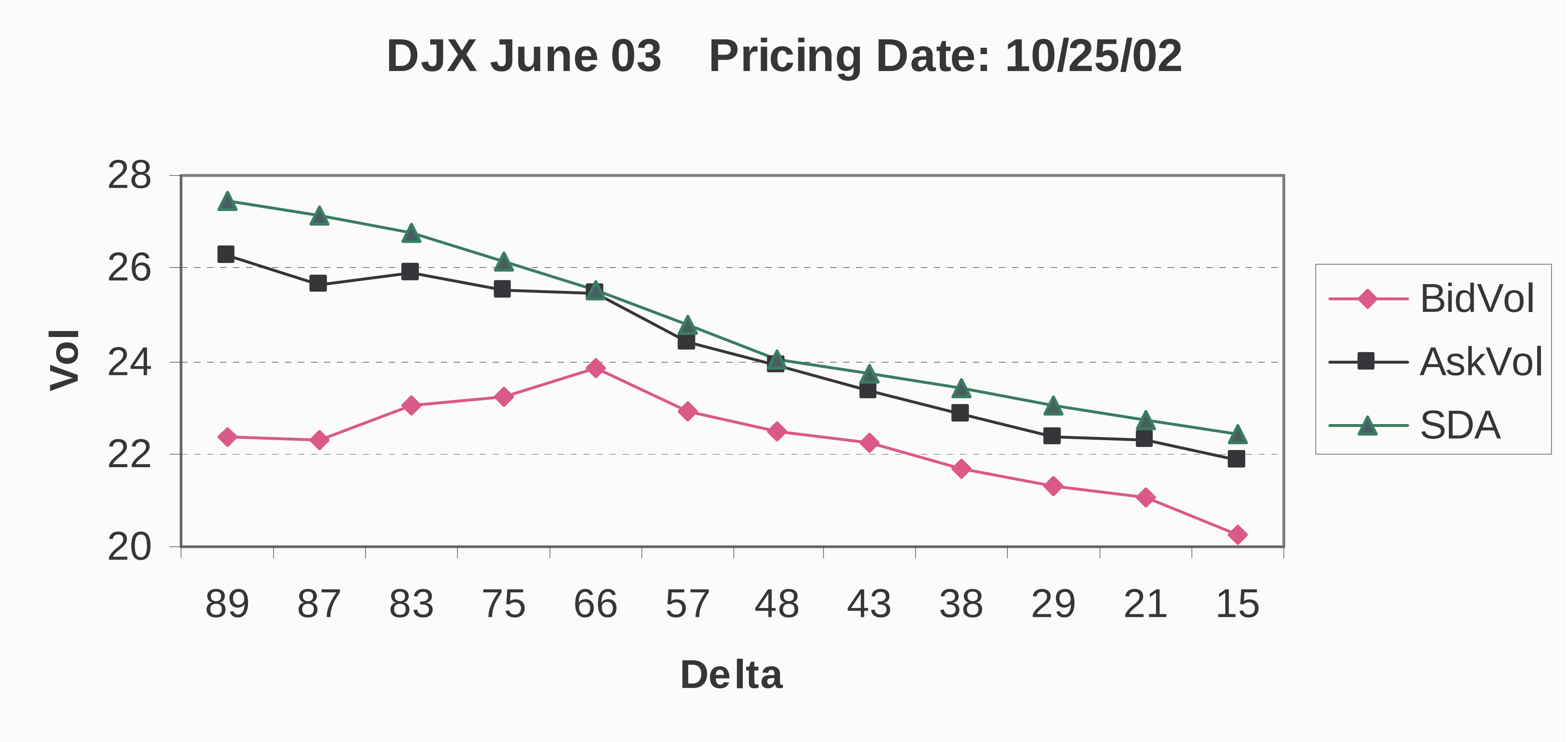
DJX Jan 03 Pricing Date: 10/25/02
36
34
32
30
28
26
24
22
20
Vol
88
84
79
75
69
62
55
48
40
33
26
15
Delta
BidVol
AskVol
SDA

T= 5 months



DJX Mar 03 Pricing Date: 10/25/02
Vol
32
30
28
26
24
22
20
88
85
77
67
62
56
44
38
32
21
14
Delta
BidVol
AskVol
SDA

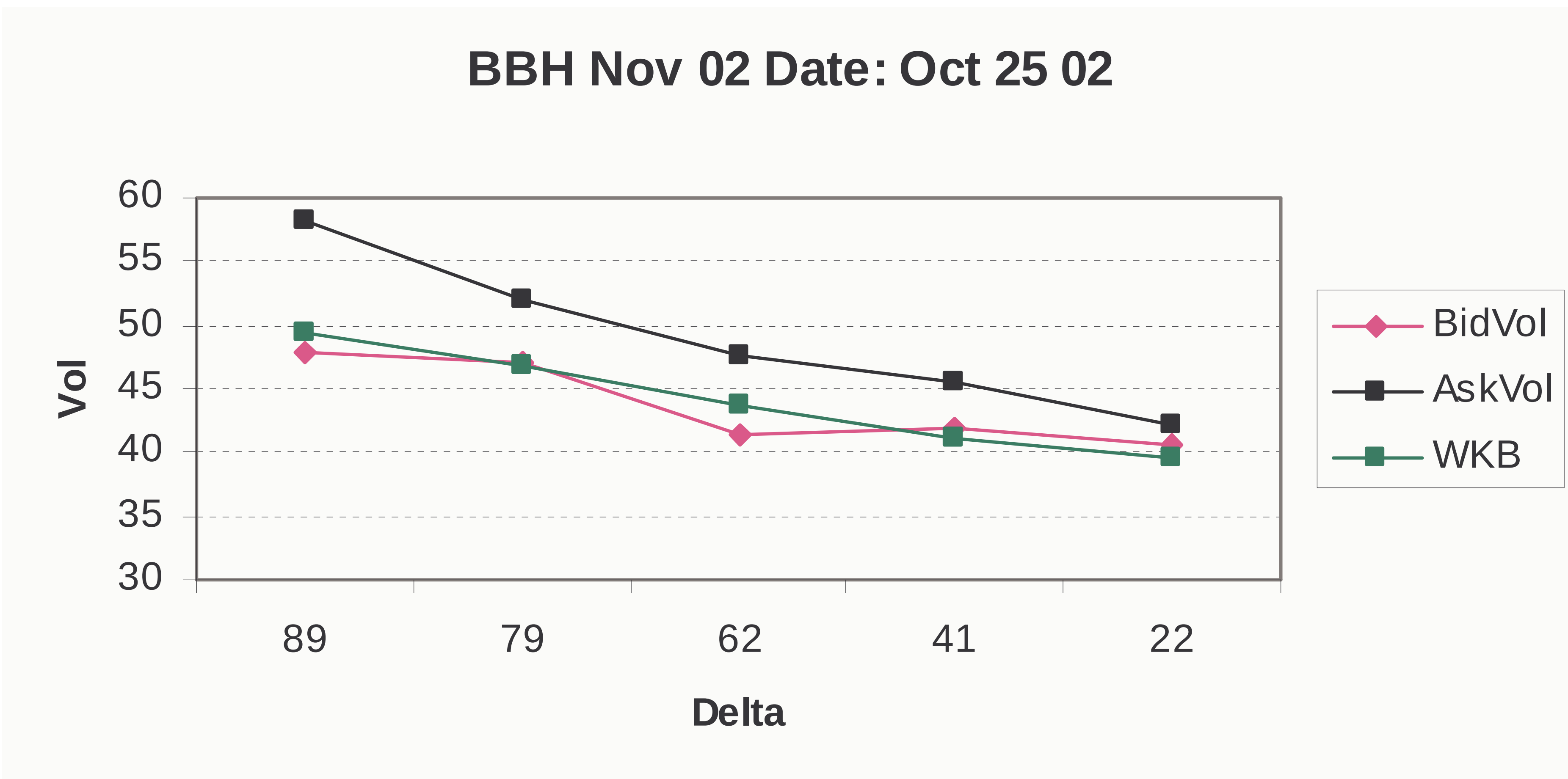
T=7 months



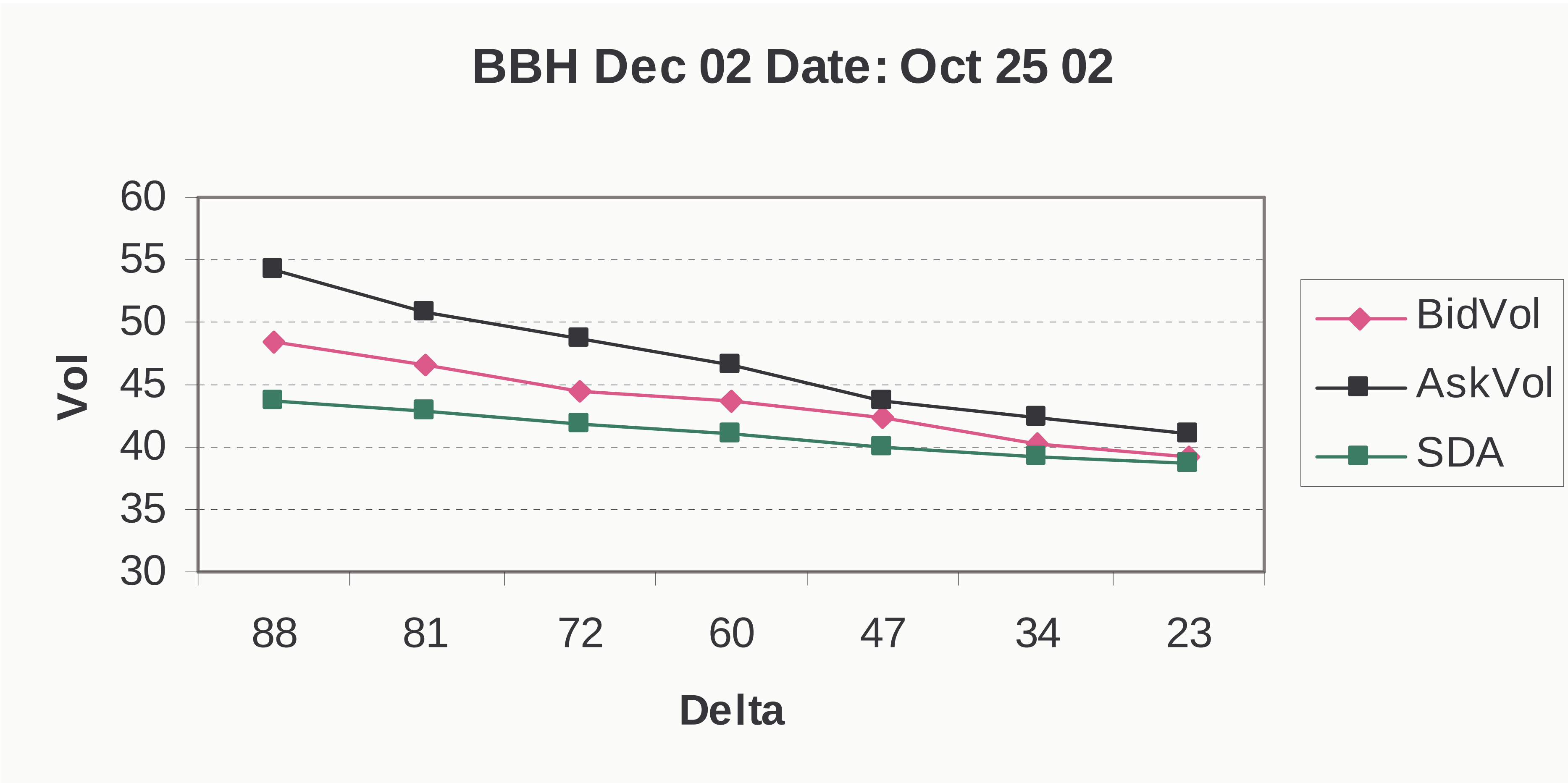
DJX June 03 Pricing Date: 10/25/02
28
26
24
22
20
Vol
89
87
83
75
66
57
48
43
38
29
21
15
Delta
BidVol
AskVol
SDA

BBH: Biotechnology HLDR

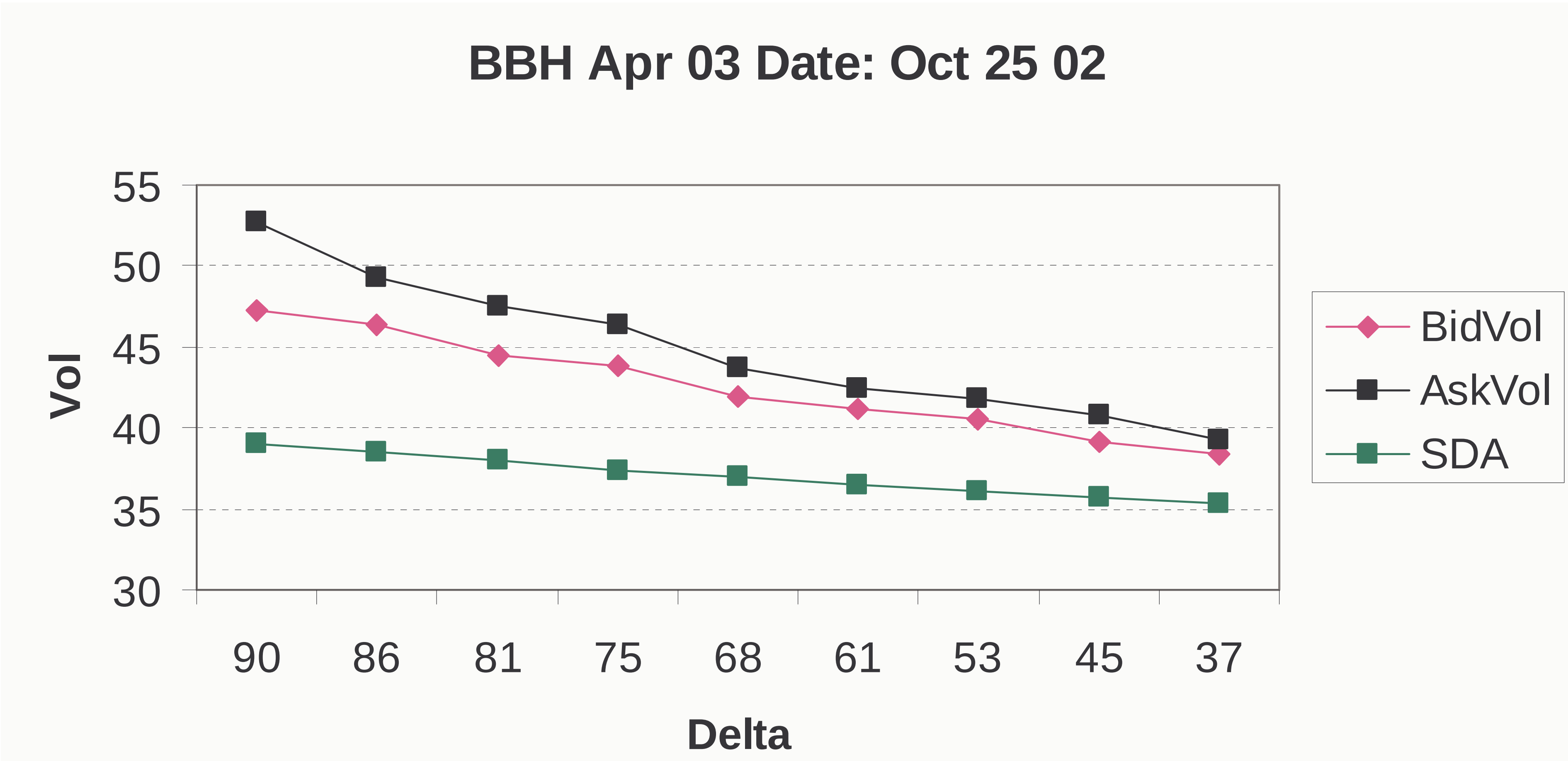
T = 1 month



T = 2 months


BBH Dec 02 Date: Oct 25 02
60
55
50
45
40
35
30
Vol
88
81
72
60
47
34
23
Delta
BidVol
AskVol
SDA

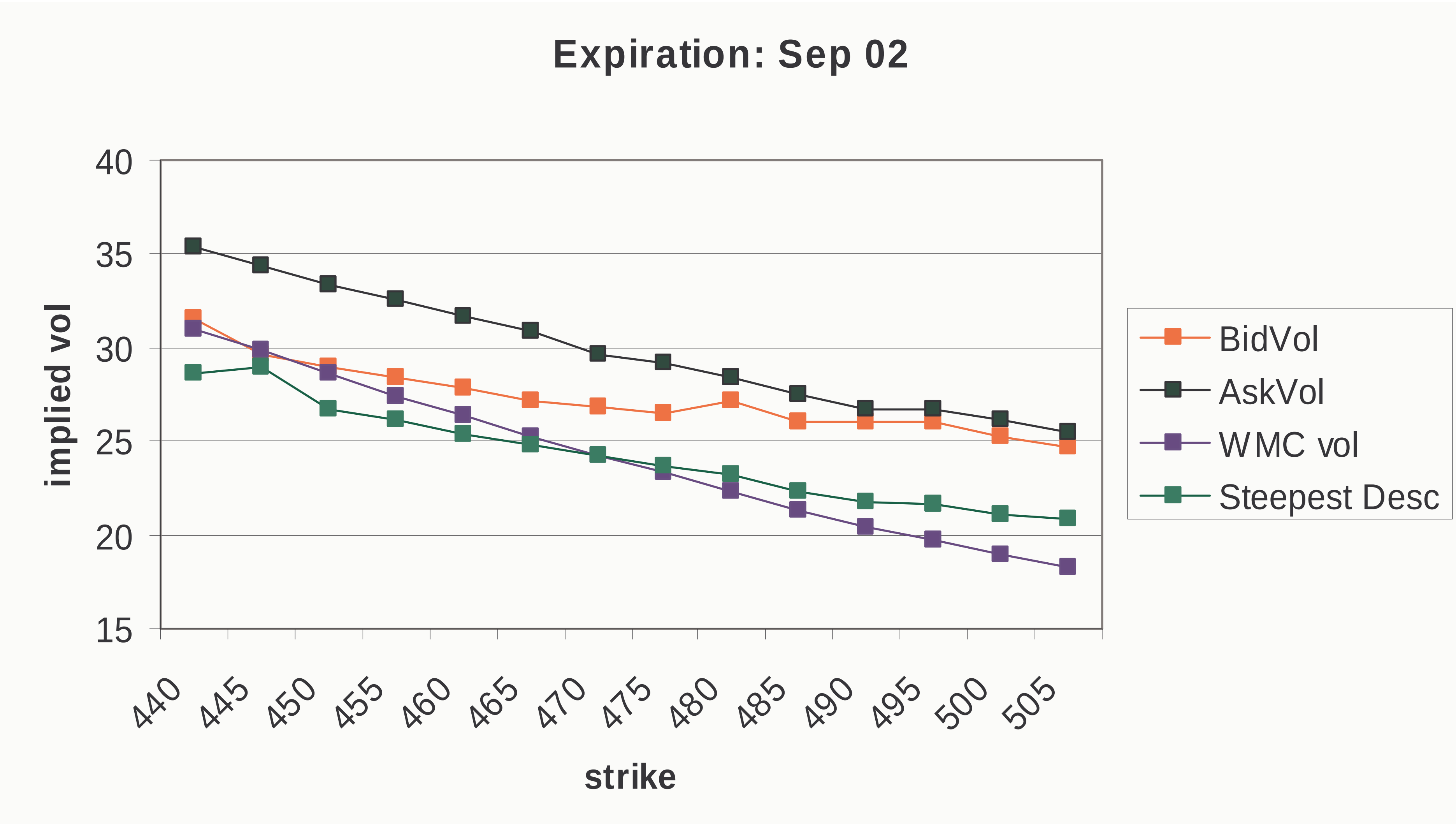
T = 6 months



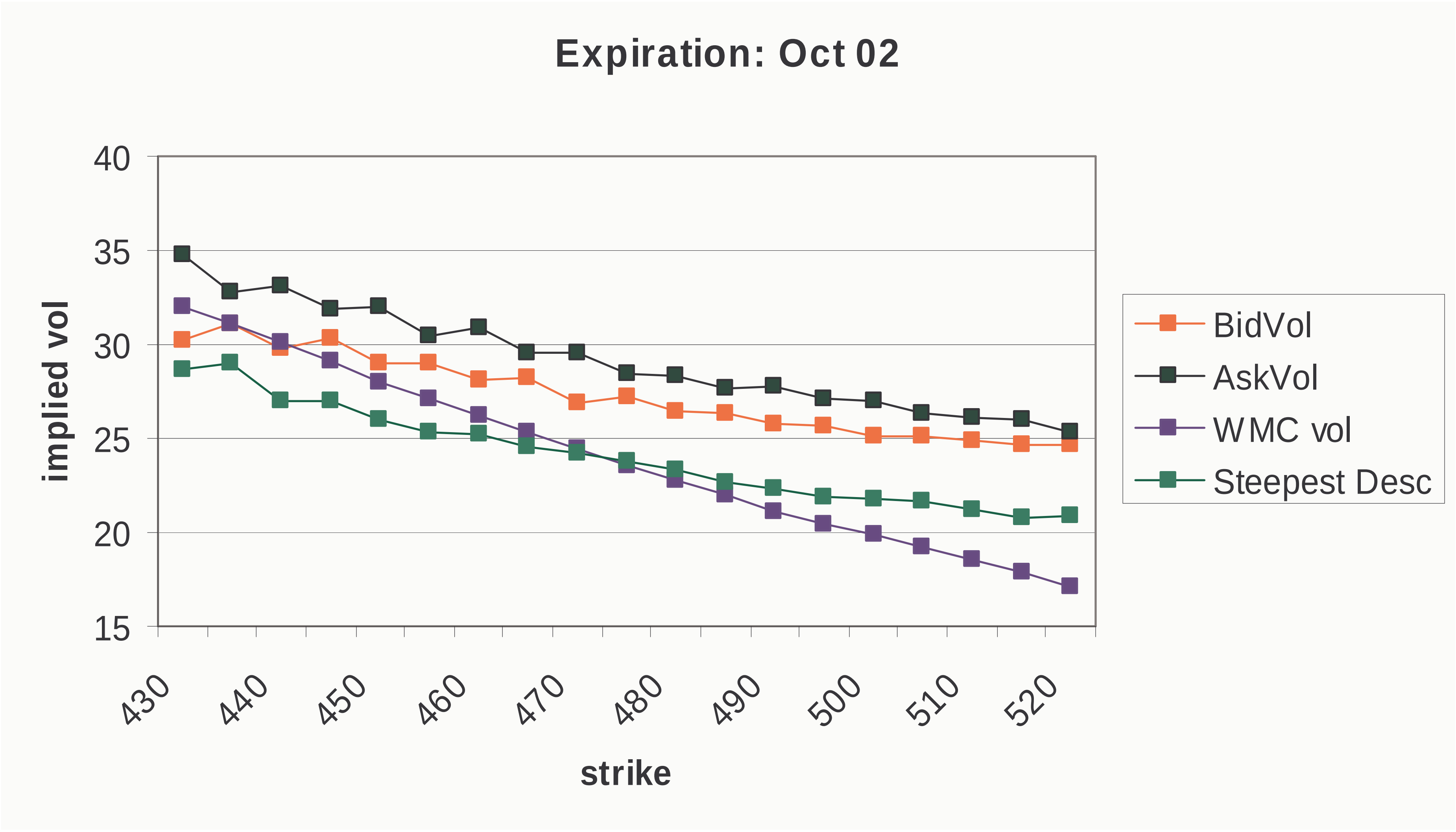
Is dimensionless time is too long? (Error bars: Juyoung Lim)

Is correlation causing the discrepancy?

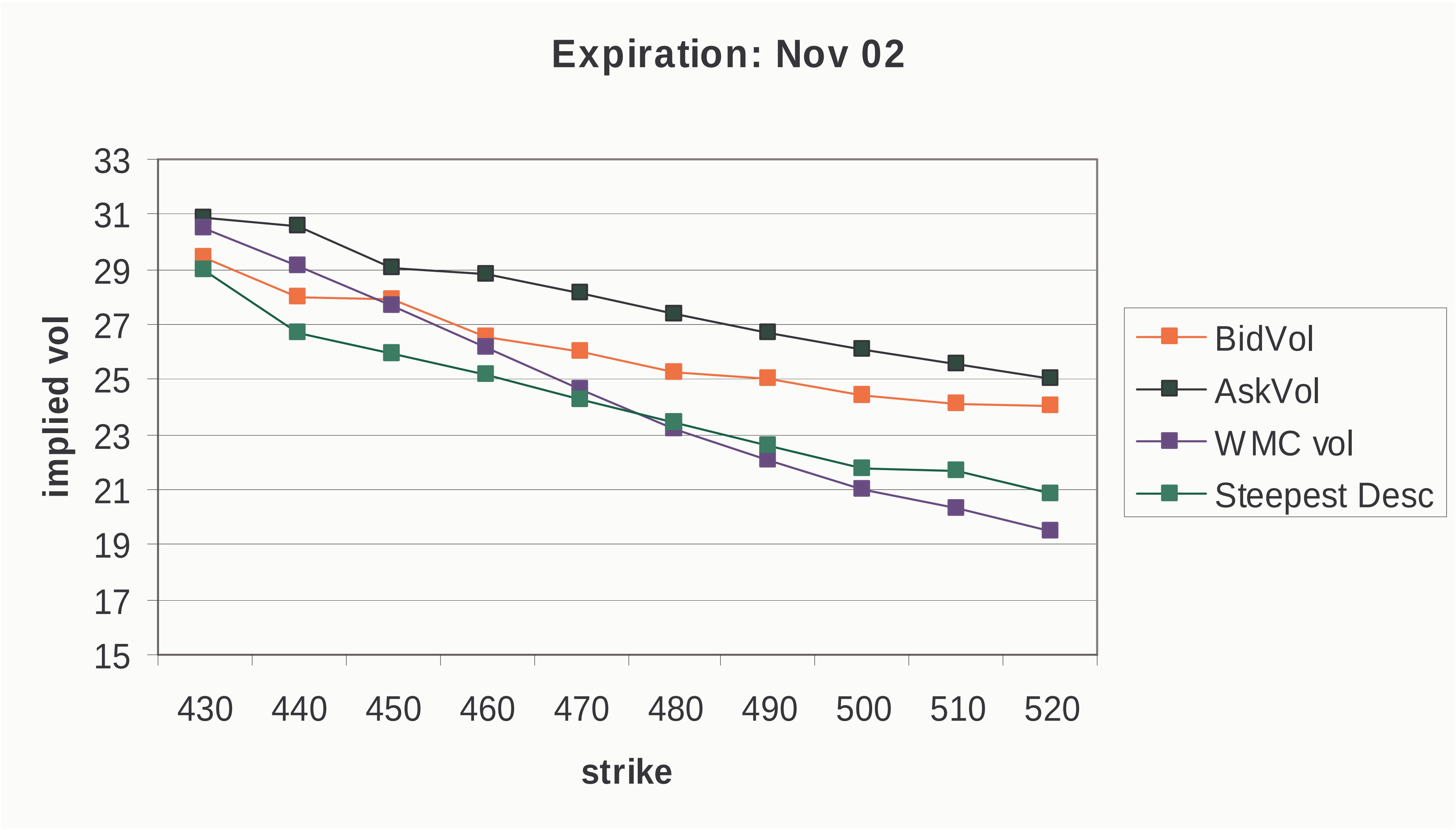
S&P 100 Index Options
(Quote date: Aug 20, 2002)



S&P 100 Index Options
(Quote date: Aug 20, 2002)

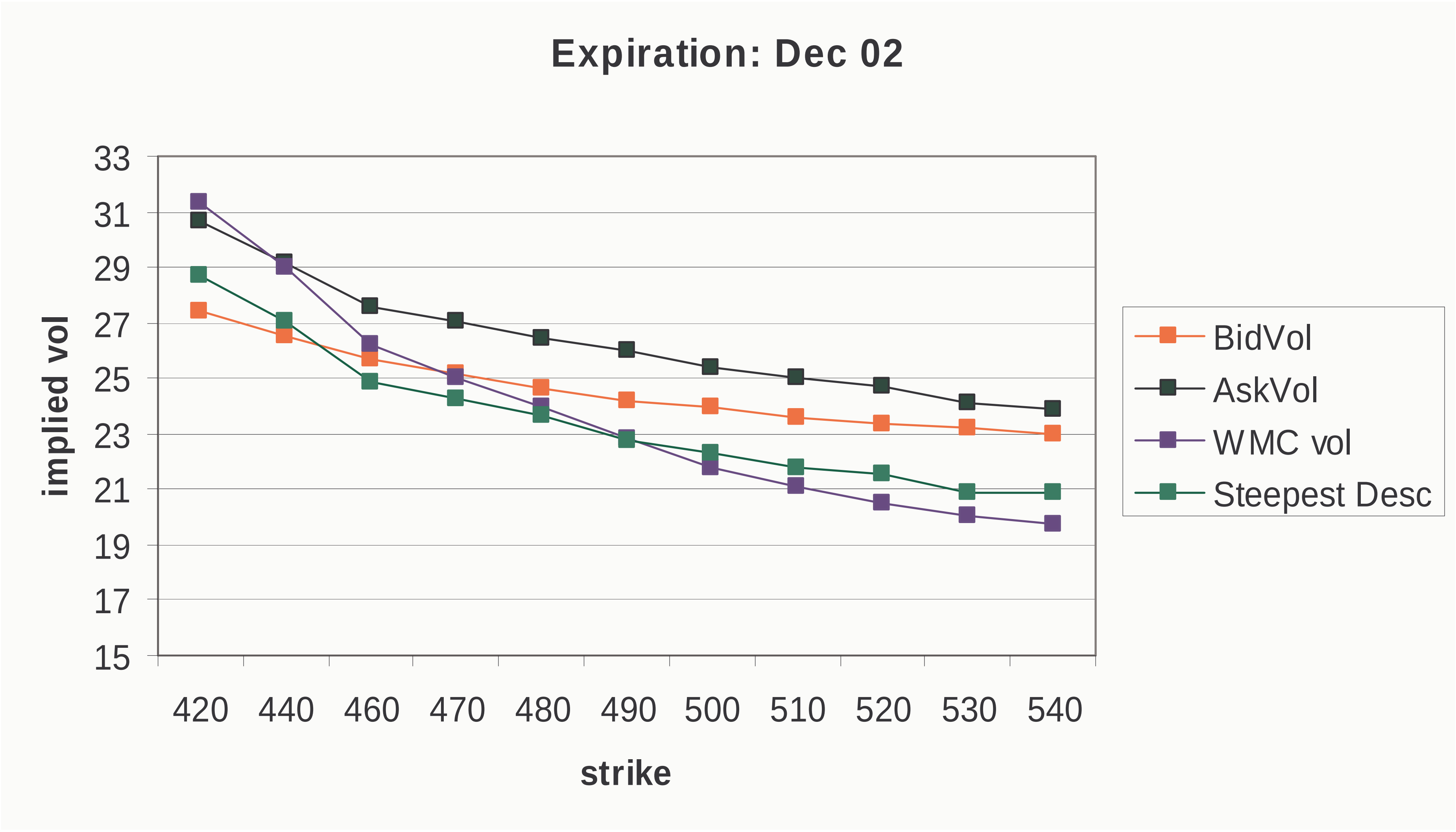


S&P 100 Index Options
(Quote date: Aug 20, 2002)



S&P 100 Index Options

(Quote date: Aug 20, 2002)



Implied Correlation: a single correlation coefficient consistent with index vol

$$\left(\sigma_I^{\text{impl}}\right)^2 = \sum_{i=1}^{N} p_i^2 \left(\sigma_i^{\text{impl}}\right)^2 + \overline{\rho} \sum_{i \neq j}^{N} p_i p_j \sigma_i^{\text{impl}} \sigma_j^{\text{impl}}$$

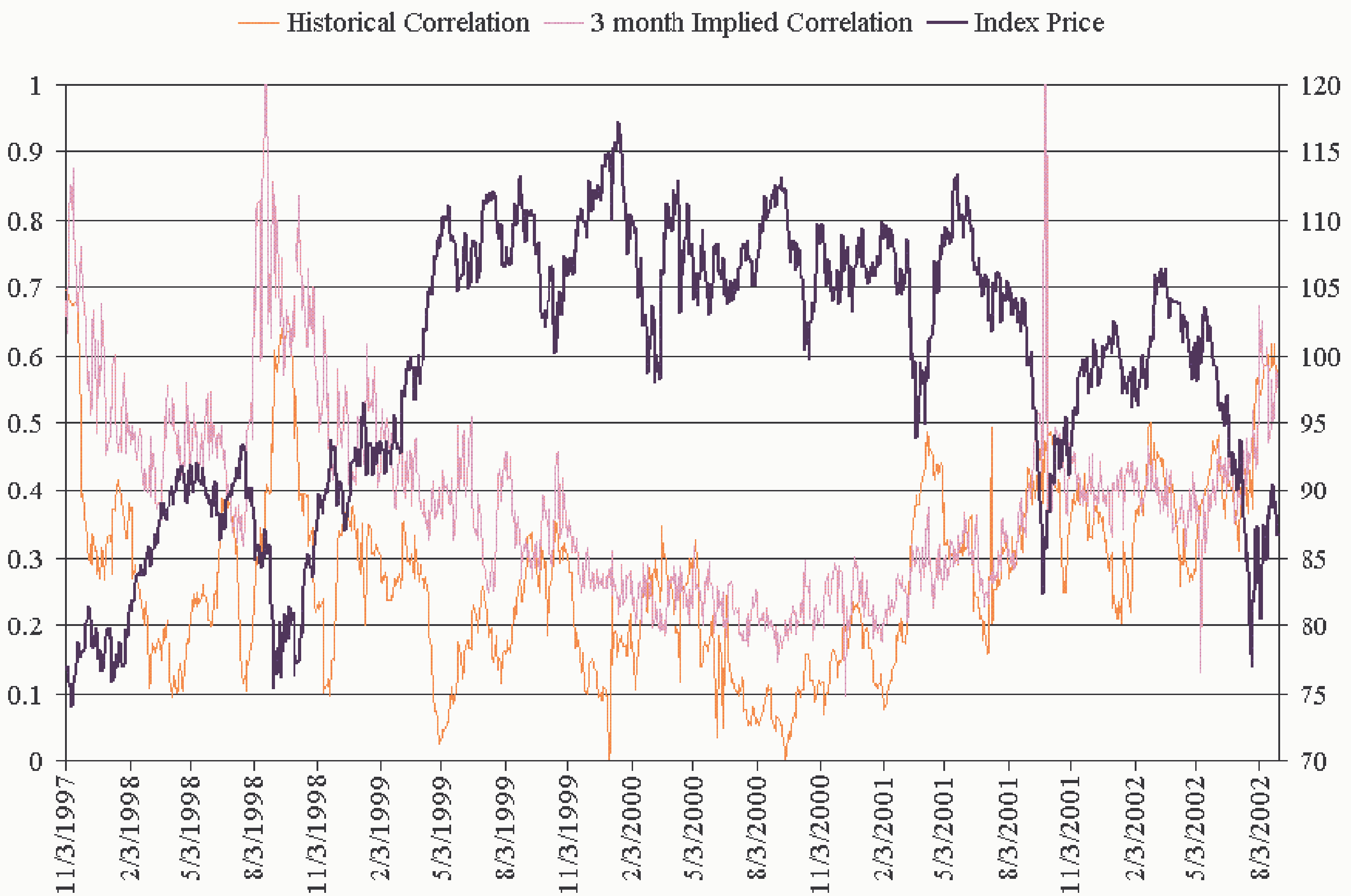
$$\therefore \quad \overline{\rho} = \frac{\left(\sigma_I^{\text{impl}}\right)^2 - \sum_{i=1}^{N} p_i \left(\sigma_i^{\text{impl}}\right)^2}{\sum_{i \neq j}^{N} p_i p_j \sigma_i^{\text{impl}} \sigma_j^{\text{impl}}} = \frac{\left(\sigma_I^{\text{impl}}\right)^2 - \sum_{i=1}^{N} p_i^2 \left(\sigma_i^{\text{impl}}\right)^2}{\left(\sum_{i=1}^{N} p_i \sigma_i^{\text{impl}}\right)^2 - \sum_{i=1}^{N} p_i^2 \left(\sigma_i^{\text{impl}}\right)^2}$$

Approximate formula:

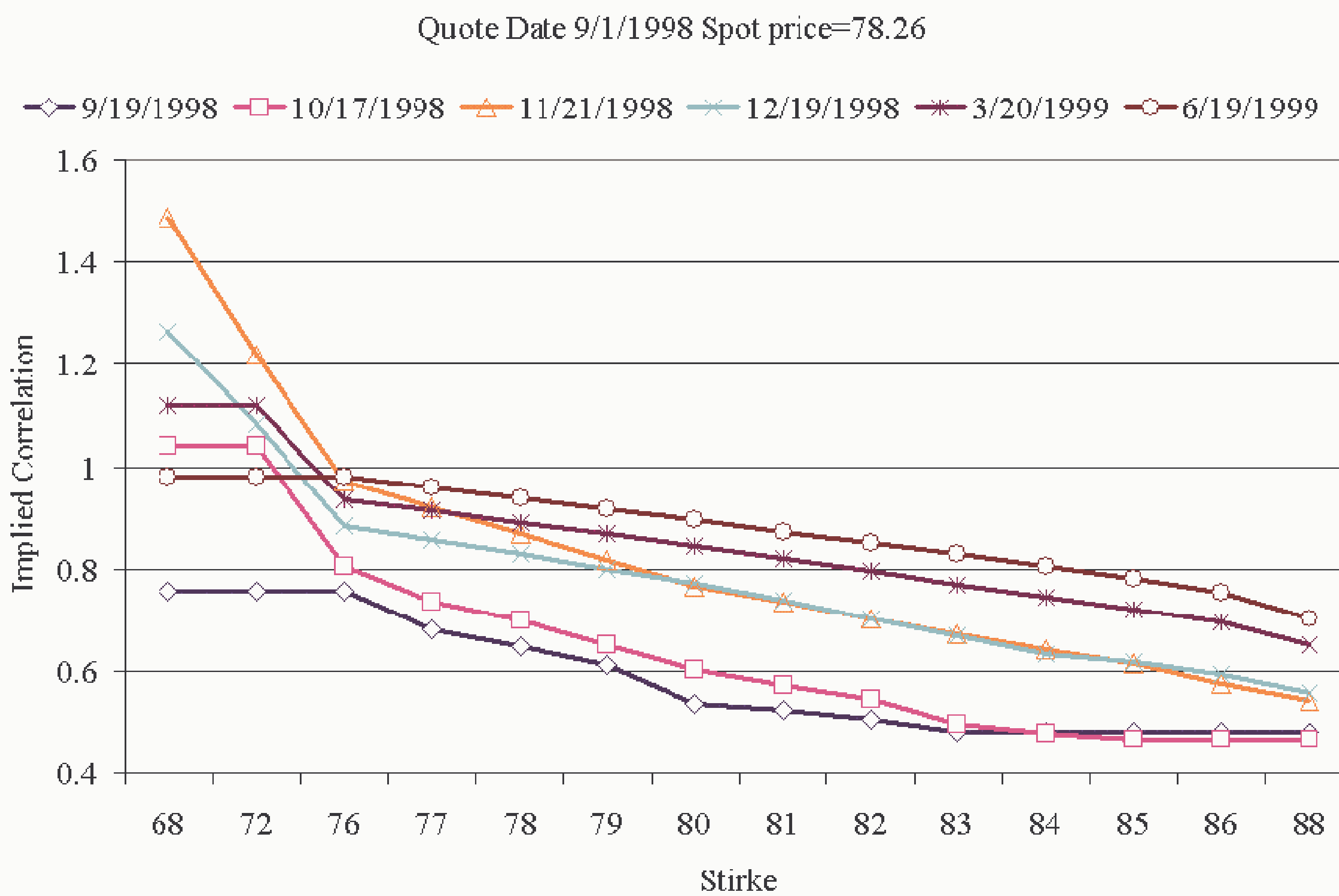
$$\overline{\rho} \approx \left(\frac{\sigma_I^{\text{impl}}}{\sum_{i=1}^{N} p_i \sigma_i^{\text{impl}}} \right)^2$$

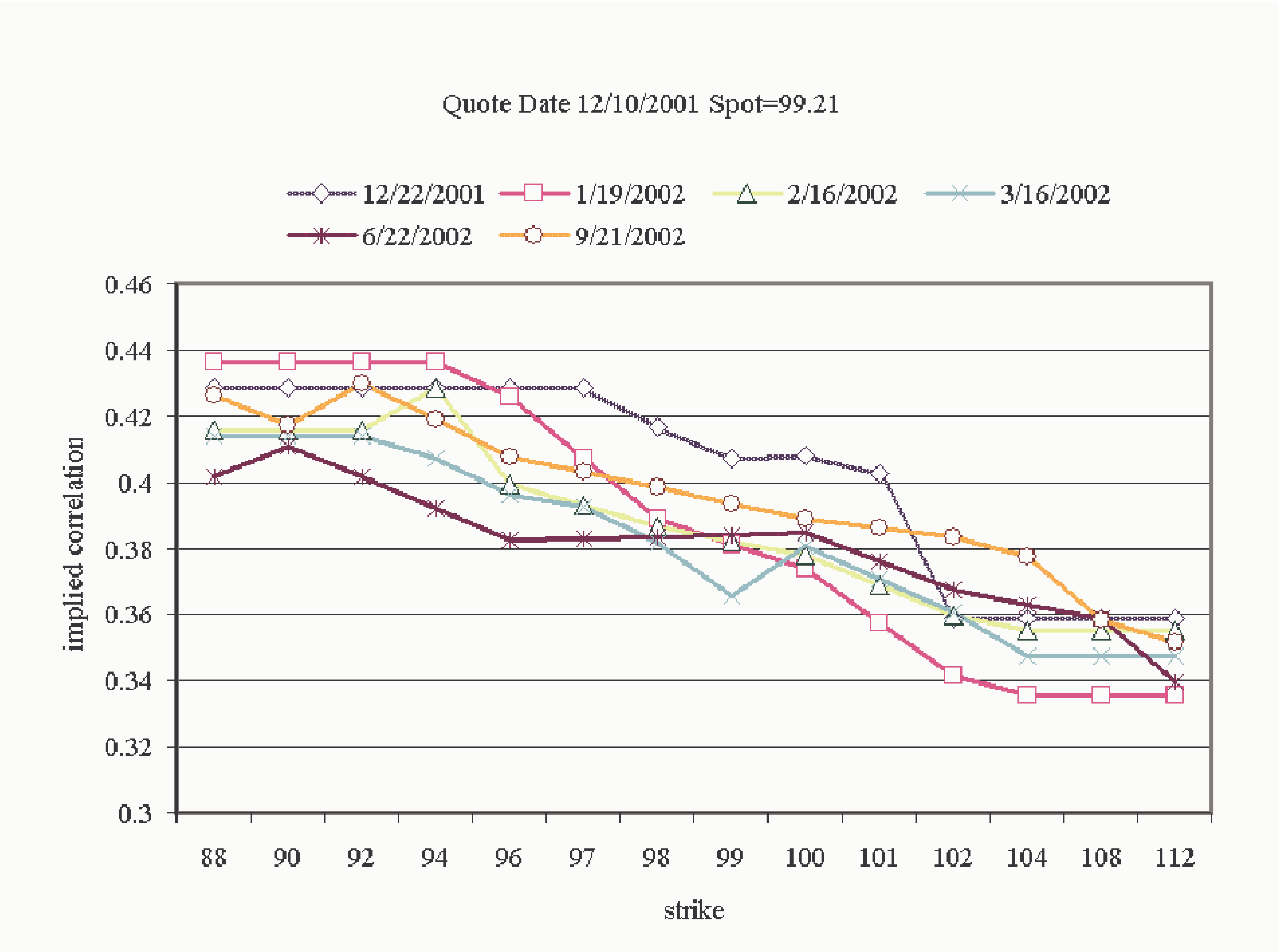
Implied correlation can be defined for different strikes, using SDA

Dow Jones Index

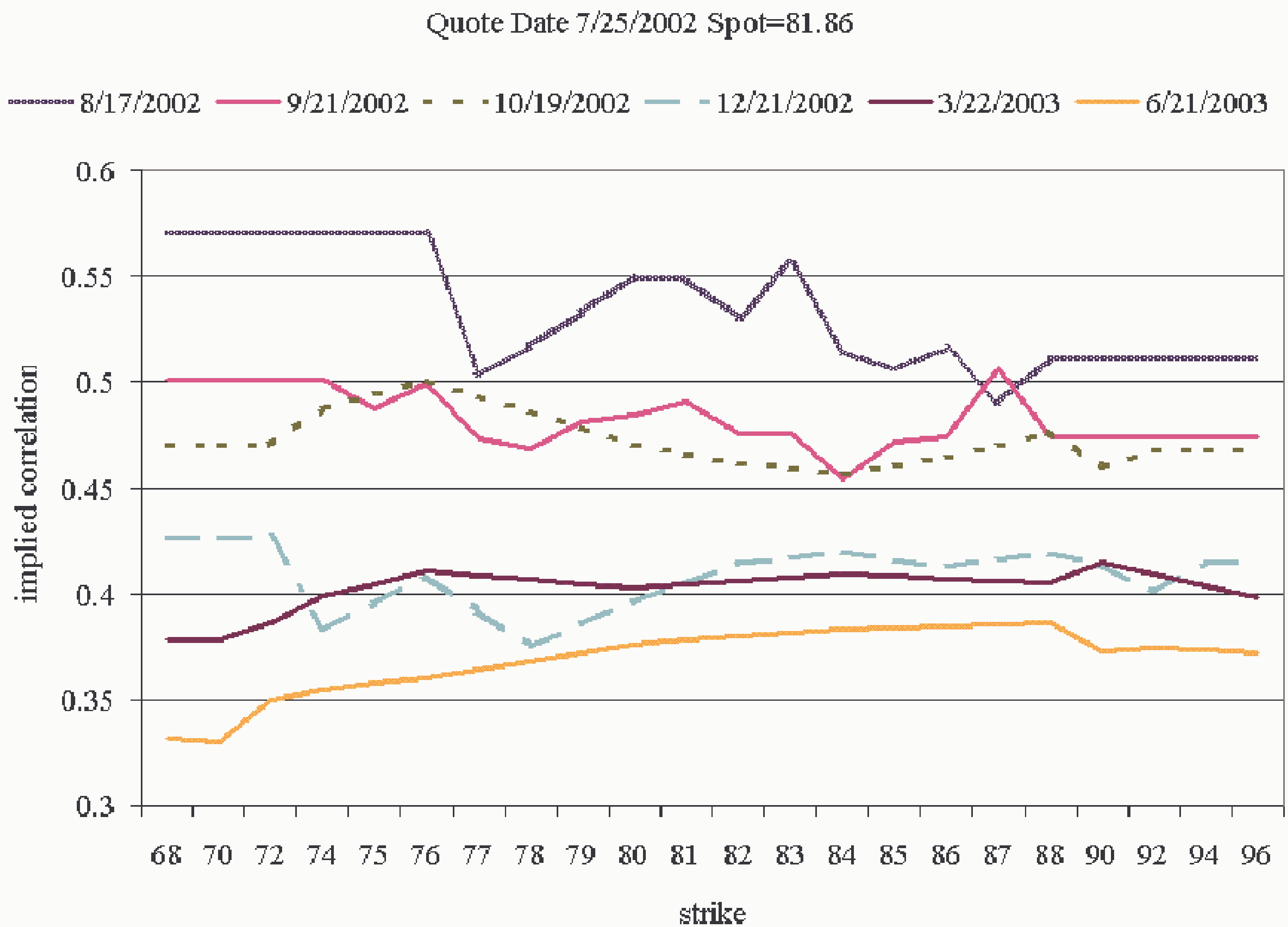


Dow Jones Index: Correlation Skew





Quote Date 12/10/2001 Spot=99.21
12/22/2001
1/19/2002
2/16/2002
3/16/2002
6/22/2002
9/21/2002
implied correlation
0.46
0.44
0.42
0.4
0.38
0.36
0.34
0.32
0.3
88
90
92
94
96
97
98
99
100
101
102
104
108
112
strike


Quote Date 7/25/2002 Spot=81.86
8/17/2002
9/21/2002
10/19/2002
12/21/2002
3/22/2003
6/21/2003
implied correlation
0.6
0.55
0.5
0.45
0.4
0.35
0.3
68 70 72 74 75 76 77 78 79 80 81 82 83 84 85 86 87 88 90 92 94 96
strike

A model for ``Correlation skew'': Stochastic Volatility Systems

$$\frac{dS_i}{S_i} = \sigma_i dW_i \qquad \frac{d\sigma_i}{\sigma_i} = \kappa_i dZ_i$$

$$E\left(dW_i dW_j\right) = \rho_{ij} dt \qquad E\left(dW_i dZ_j\right) = r_{ij} dt$$

$$\bar{x} = \frac{dI}{I}, \qquad x_i = \frac{dS_i}{S_i} \qquad y_i = \frac{d\sigma_i}{\sigma_i}$$

Look for most likely configuration of <u>stocks and vols</u> $\left(x_1,\ldots,x_n, y_1,\ldots,y_n\right)$ corresponding to a given index displacement $\bar{x}$

Most likely configuration for Stochastic Volatility Systems

$$x_i^* = \beta_i \bar{x} \qquad \beta_i = \frac{\sigma_i \rho_{iI}}{\sigma_I}$$

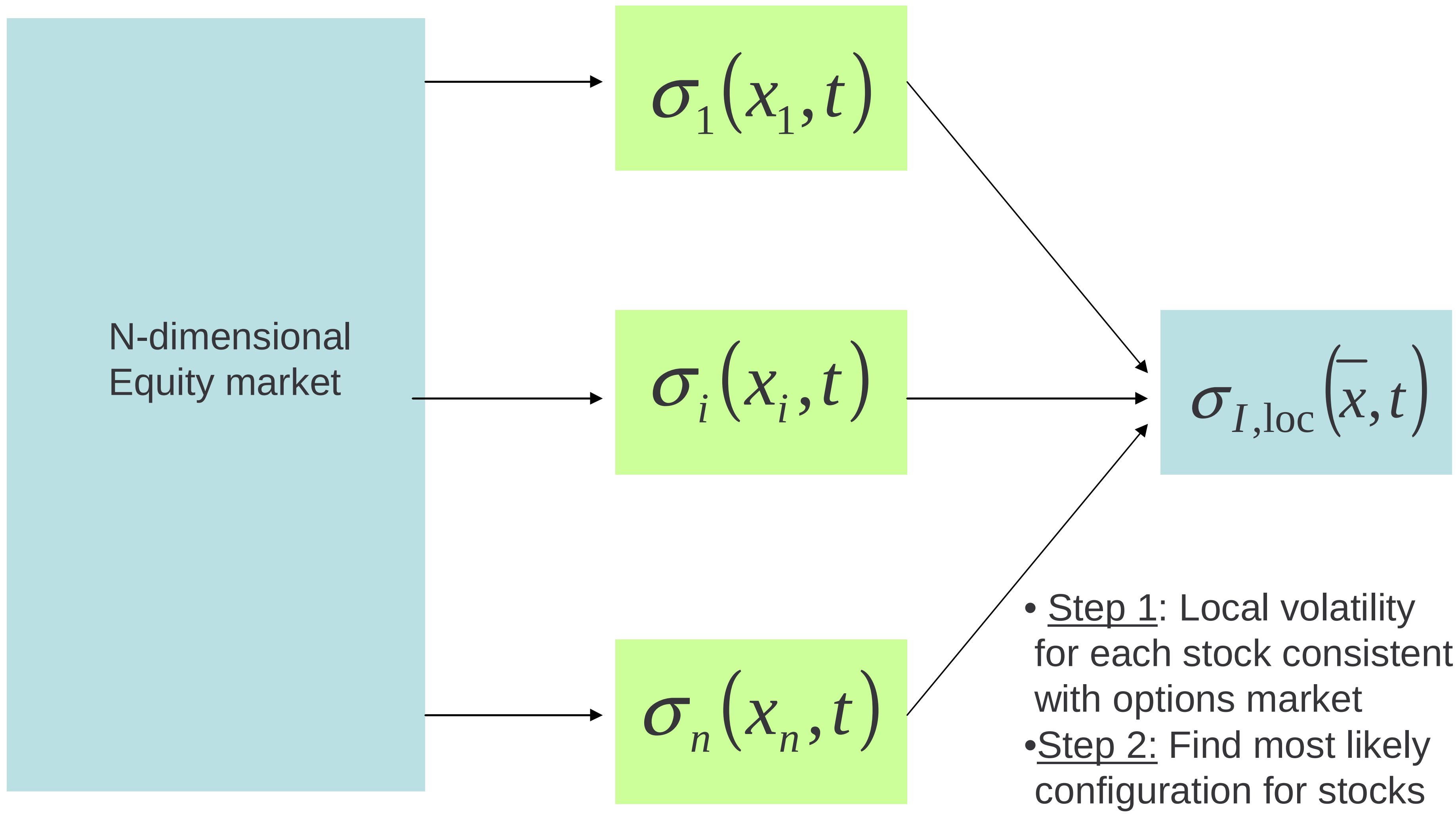
$$y_i^* = \gamma_i \bar{x} \qquad \gamma_i = \frac{\kappa_i r_{iI}}{\sigma_I}$$

Most likely configuration for stocks moves and volatility moves, given the index move

$$\sigma_{I,\mathrm{loc}}^2(\bar{x},t) \cong \sum_{ij=1}^{n} p_i p_j \sigma_i(0,t)\sigma_j(0,t) e^{\gamma_i \bar{x}} e^{\gamma_j \bar{x}} \rho_{ij}$$

SDA

Method I: Dupire & Most Likely Configuration for Stock Moves



- Step 1: Local volatility for each stock consistent with options market
- Step 2: Find most likely configuration for stocks

Method II: Stochastic Volatility System and joint MLC for Stocks and Volatilities

- <u>Only one step</u>: compute the most likely configuration of stocks and volatilities at the same time

Methods I and II are not `equivalent'

Dupire local vol. for single names

$$\sigma_{i,\mathrm{loc}}(x_i,t) \approx \sigma_i(0,t)e^{\varpi_i x_i} \qquad \varpi_i = \frac{\kappa_i r_{ii}}{\sigma_i}$$

Index vol., Method I

$$\sigma^2_{I,\mathrm{loc}}(\bar{x},t) = \sum_{ij} p_i p_j \sigma_i(0,t)\sigma_j(0,t)\rho_{ij} e^{\varpi_i \beta_i \bar{x}} e^{\varpi_j \beta_j \bar{x}}$$

Index vol., Method II

$$\sigma^2_{I,\mathrm{loc}}(\bar{x},t) = \sum_{ij} p_i p_j \sigma_i(0,t)\sigma_j(0,t)\rho_{ij} e^{\gamma_i \bar{x}} e^{\gamma_j \bar{x}}$$

Stochastic Volatility Systems give rise to Index-dependent correlations

Method II

$$\sigma^2_{I,\text{loc}}(\bar{x},t) \approx \sum_{ij} p_i p_j \sigma_i(0,t)\sigma_j(0,t)\rho_{ij} e^{\gamma_i \bar{x}} e^{\gamma_j \bar{x}}$$

$$\approx \sum_{ij} p_i p_j \underbrace{\sigma_i(0,t) e^{\beta_i \varpi_i \bar{x}}} \sigma_j(0,t) e^{\beta_j \varpi_j \bar{x}} \rho_{ij} e^{\gamma_i \bar{x}} e^{\gamma_j \bar{x}} e^{-\beta_i \varpi_i \bar{x}} e^{-\beta_i \varpi_i \bar{x}}$$

$$\approx \sum_{ij} p_i p_j \sigma_{i,\text{loc}}(\beta_i \bar{x},t) \sigma_{i,\text{loc}}(\beta_i \bar{x},t) \rho_{ij}(\bar{x})$$

$$\rho_{ij}(\bar{x}) \equiv \rho_{ij} e^{(\gamma_i + \gamma_j - \beta_i \varpi_i - \beta_j \varpi_j)\bar{x}}$$

Equivalence holds only under additional assumptions on stock-volatility correlations

$$\varpi_i \beta_i = \frac{\kappa_i r_{ii}}{\sigma_i} \frac{\sigma_i \rho_{iI}}{\sigma_I} = \frac{\kappa_i r_{ii} \rho_{iI}}{\sigma_I}$$

Method I

$$\gamma_i = \frac{\kappa_i r_{iI}}{\sigma_I}$$

Method II

$$r_{iI} = r_{ii} \rho_{iI}$$

$$r_{ij} = r_{ii} \rho_{ij}$$

Conditions under which both methods give equivalent valuations

Open (and very doable) problems

- Apply this technology for pricing swaptions based on the volatility skew of LIBOR rates or forward rates
- If we use a Local Volatility model (e.g. BGM with square-root volatility), the answer is <u>identical</u> to the previous formula
- The ``full'' SABR multi-asset model gives rise to a complicated Riemannian metric

$$dL^2 = \sum_{ij=1}^{n} g_{ij} \frac{d\eta_i}{\sigma_i}\frac{d\eta_j}{\sigma_j} + \sum_{i=1}^{n} \frac{(d\sigma_i)^2}{\kappa_i^2 \sigma_i^2}$$

- Credit default models for pricing CDOs are amenable to the same approach, especially copula-type models. I am not aware of any solutions

Epilogue: Structural Credit Model

$$\mathrm{x} = (x_1, \ldots, x_n) \qquad \text{vector of firm values}$$

Firm i defaults before time T if $x_i(T) < \alpha_i$

Equal weighted CDO : loss of m dollars if

$$\mathrm{x}(T) \in \Omega_m = \bigcup_{\mathrm{card}(I) \geq m} \bigcap_{i \in I} \{\mathrm{x} : x_i < \alpha_i\}$$

Solve

$$\inf\{L(0, \mathrm{x}) : \mathrm{x} \in \Omega_m\}$$