Stochastic Calculus: Lecture 3

Brownian Motion (formal definition in terms of statistics)

1. Gaussian process \( (X(t_1), \ldots, X(t_n)) \)
   Gaussian \( N(t_1, \ldots, t_n) \)

2. \( X(t + \Delta t) - X(t), \ X(t) \) independent

3. \( X(0) = 0, \ X(t) \sim N(0, t) \)

B.M. = a continuous-time random walk with Gaussian increments!

\[ \tilde{X}_n = X(n) \]
\[ \tilde{X}_n = \tilde{X}_{n-1} + \sigma \tilde{N}_n \]

How can we construct such function?

If we look on a set of discrete lines, we take a standard R.W.
Construction of BM by interpolation (N. Kolmogorov).

Consider $0 < t < 1$. $X(t) = Bt + 1$.

Step 1:
- $X(1) \sim N(0,1)$.
- $X(0) \uparrow X(1)$.

Step 2:
The next step is to refine this function by considering $(0, \frac{1}{2}, 1)$.

$X(\frac{1}{2}) = \beta X(1) + \xi_1$:
- Regression of $X(\frac{1}{2})$ on $X(1)$ (best prediction).

$E(\xi_1) = 0$.

$E[X(\frac{1}{2})X(1)] = E[X(\frac{1}{2})^2 + E[X(\frac{1}{2})][X(1) - X(\frac{1}{2})]]$.

$= \frac{1}{2}$

$\frac{1}{2} = \beta$.

$X(\frac{1}{2}) = \frac{1}{2} X(1) + \xi_1$.

$E(\xi_1) = 0$.

$E(\xi_1^2 + \beta^2) = \frac{1}{2}$.

Thus: $\frac{E(\xi_1^2)}{1} = \frac{1}{1}$. 

\[\boxed{\xi_1^2 = \frac{1}{1}}\]
Define: \( X_1(t) = X_0(t) + Y_1(t) \)

\( X_1(t) \) is consistent with the statistical properties of \( BH \) on the set of trees \((0, \frac{1}{2}, 1)\).

Step 3: Refine each "lobe" of \( Y(t) \)

\[ X(\frac{1}{4}) = \beta X(\frac{1}{2}) + \xi_{2,1} \]

\[ \frac{1}{4} = \frac{1}{2} + 0 \quad \therefore \beta = \frac{1}{2} \]

\[ \frac{1}{4} = \beta^2 + \xi_{2,1}^2 \quad \therefore \frac{1}{4} = \frac{1}{2} + \sigma_2^2 \]
\[ X_2(t) = X_0(t) + Y_2(t) + Y_3(t) \]

\[ X_2(t) \text{ is restricted to } (6, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1) \]

**Step 4:** Derive the process inductively for all scales!

\[ X_N(t) = X_0(t) + \sum_{j=1}^{N} Y_j(t) \]

\[ Y_j(t) = \sum_{m=1 \mod 2^j}^{2^{j-1}} \sum_{i=0}^{2^j-1} T_{j \left\lfloor \log_2 i \right\rfloor}(t) \in \text{tent function} \]

\[ E \left( \frac{1}{2} \right) \sum_{m=1}^{2^j} \sum_{i=0}^{2^j-1} T_{j \left\lfloor \log_2 i \right\rfloor}(t) = \sigma^2 \]

\[ \sigma_m = \frac{1}{2^j} \]

\[ \sigma = \frac{1}{2} \]

\[ M_1 = M_2 = M_3 = M_4 = 1 \]
Proposition:
Brownian Motion is the limit of \( X_N(t) \) as \( N \to \infty \). This limit exists with probability 1 and defines a continuous path.

In fact, if \( \| Y_j \|_\infty = \max_{t \in [0, 1]} |Y_j(t)| \)

\[
P[\| Y_j \|_\infty > a] = P\left[ \max_{1 \leq m < 2^j} |\sum_{n=1}^{2^j} y_{jm}| > a \right]
\]

\[
\leq 2^j P[|\sum_{n=1}^{2^j} y_{jm}| > a]
\]

\[
\leq 2^j \frac{1}{a^4} 3 \left( \frac{1}{2^j} \right)^2
\]

\[
= \frac{3}{a^4 2^{j+2}} \leq \frac{1}{a^4 2^j}
\]

\[
P[\| Y_j \|_\infty > \frac{1}{2je}] < \frac{2^j e}{2^j} = \frac{1}{2^j(1-\varepsilon)}
\]
If $\varepsilon < \frac{1}{4}$

\[ P \left[ \|Y_j\|_{\infty} > \left( \frac{1}{2\varepsilon} \right)^j \right] \leq \left( \frac{1}{2^{1+\varepsilon}} \right)^j \]

\[ \sum_{j=1}^{\infty} P \left[ \|Y_j\|_{\infty} > \left( \frac{1}{2\varepsilon} \right)^j \right] < \infty \]

The convergence of this series implies, by the Borel-Cantelli Lemma of elementary probability, that

\[ P \left\{ \|Y_j\|_{\infty} \leq \left( \frac{1}{2\varepsilon} \right)^j \quad \text{for} \quad j > \frac{1}{(1/2\varepsilon)} \right\} = 1 \]

where $j^{(1/2)}$ depends on the sequence $\beta_k, \beta_k^{(1/2)}, \beta_k^{(2)}, \beta_k^{(3)}, \beta_k^{(3)}$, $\beta_k^{(3)}, \beta_k^{(3)}, \beta_k^{(3)}$, $\beta_k^{(3)}, \beta_k^{(3)}, \beta_k^{(3)}$

Since, eventually, $\|Y_j\|_{\infty} \leq \left( \frac{1}{2\varepsilon} \right)^j$ we have

\[ \sum_{j=1}^{\infty} \|Y_j\|_{\infty} < \infty \quad \text{for almost all sequences} \; x \]
This convergence of the norms implies that $\|Y_j(t)\|$ is a continuous function.

Note: This is a sketch of the proof of continuity of Brownian paths. It "works" because (notly) because

$$E \xi_{j,m}^4 = \frac{3}{2(j+1)} \ll \frac{1}{2^j} \text{ as } j \to \infty$$

Recalling that $\xi_{j,m} = \frac{\partial^j}{\partial t^j} \xi_0$, we are saying is that

$$E(X(2j) - \frac{1}{2} (X(2j-1) + X(2j+1)))$$

we are saying is that

$$E((X(t+8t) - X(t))^4) = 38t^2 \ll St.$$  

Kolmogorov showed, using a similar construction, that if
a process has a distribution such that

\[ E[X(t + \delta t) - X(t)]^b \leq C ||\delta t||^{\frac{b}{a}} \]

with \( a, b, c > 0 \), then \( X(t) \) can be constructed as a continuous function (just like we constructed BM).

Example: Revisiting AR(1)

\[ X_n = b X_{n-1} + \epsilon_n \]

\[ \epsilon_n = N(0, \sigma^2) \]

\[ b < 1 \]

\[ E X_n = 0 ; \quad E X_n^2 = \frac{\sigma^2}{1 - b^2} = \frac{\sigma^2}{\gamma^2} \]

\[ X_n = b^n X_0 + \sum_{j=0}^{n-1} b^{n-j-1} \epsilon_j \]

\[ \sigma^2(X_n^2) = \sigma^2 (1 + b^2 + b^4 + \ldots + b^{2n}) = \frac{\sigma^2}{1 - b^2} \]
\[ E(X_n, X_m) = \ell^{-|m-n|} E X^2_m \]  
\[ \text{Corr}(X_n, X_m) = \ell^{-|m-n|} \]

This suggests defining a Gaussian process in continuous time, \( X(t) \), such that

\[
\begin{cases} 
E X(t) = 0 \\
& E [X(t + \delta t) \cdot X(t)] = \sigma^2 e^{-k \cdot \delta t} 
\end{cases}
\]

If such a process existed, then

\[
\text{Var}[X(t + \delta t) - X(t)]^2 = E (X(t + \delta t))^2 + E (X(t))^2 - 2E X(t)X(t + \delta t)
\]

\[
= \sigma^2 + \sigma^2 - 2\sigma^2 e^{-k \cdot \delta t} 
= 2\sigma^2 (1 - e^{-k \cdot \delta t})
\]
$X(t+\delta t) - X(t) \sim N(0, 2\sigma^2(1-e^{-k\delta t}))$ \(\text{(10)}\)

\[ E(X(t+\delta t) - X(t))^4 = 3.4 \sigma^4 (-e^{-k\delta t})^2 \]

\[ = 12 \sigma^4 (-e^{-k\delta t})^2 \]

\[ \approx 12 \sigma^2 k^2 (\delta t)^2 \quad \text{as} \; \delta t \ll 1 \]

Thus, the AR(1) process admits a version on continuous time which has continuous paths that look like

![Continuous Paths Diagram]

This process is also called the Ornstein-Uhlenbeck process.

We will study it later in
1. Let $X(t)$ be BM.
\[ \text{Cov}(X(t), X(t+s)) = t \delta_{st} \quad (s \geq 0) \]
\[ \text{Cov}(X(t_1), X(t_2)) = \min(t_1, t_2) \]
Notice that this is not a function of $(2-t)$, BM is not stationary (Ornstein-Uhlenbeck is).

2. For any $t$ and any $
abla t < t_n < t$
\[
(*) \quad E[X(t) | X(t_1), \ldots, X(t_n)] = \bar{X}(t_n)
\]
Proof: $X(t) = X(t_n) + V$
where $V$ is independent of $X(t_n)$
Best linear predictor is $X(t_n)$, but since BM is Gaussian $(*)$ holds.
Conditional expectation relative to the past until time \( t \), \( \mathcal{F}_t^\infty \) \( \forall t \geq 0 \). 

\[
E \left[ X(t) \mid X(s), s \leq t \right] \text{ is the best predictor (in the sense of least squares) of } X(t) \text{ given } X(s), s \leq t.
\]

\[
E \left[ X(t) \mid X(s), s \leq t \right] = E \left[ X(t) \mid \mathcal{F}_t \right]
\]

\[
\iff \quad E \left[ |X(t) - Y|^2 \right] \leq E \left[ |X(t) - \hat{P}(X_1, \ldots, X_n)|^2 \right]
\]

for all \( \hat{P} \) and all \( t_1 < t_2 < \ldots < t_n \leq t \).

We sometimes write

\[
E \left[ X(t) \mid X(s), \ldots, s \leq t \right] = E \left[ X(t) \mid \mathcal{F}_t \right]
\]
The idea of $E(\mathcal{F}_t | \mathcal{F}_0)$ is to predict $\mathcal{F}_t$ based on the path $X(t)$, $t \leq 0$. 

A process with the property that $E(X(t) | \mathcal{F}_0) = X_0$ is called a [Markov](https://en.wikipedia.org/wiki/Markov_process)

\[ E \left[ e^{ikX(t)} \mid \mathcal{F}_s \right] = e^{ikX(s)} E(e^{ik(X(t) - X(s))}) \\
\quad = e^{ikX(s)} - \frac{ik}{2}(t-s) \]
Also, for real exponentials \((s < t)\)

\[
E[e^{\lambda X(s)} \mid F_s] = e^{\lambda X(s) t} e^{\frac{1}{2} \lambda^2 (t-s)}
\]

\[
E(e^{\lambda X(t) - \frac{1}{2} \lambda^2 t} \mid F_s) = e^{\lambda X(s) t - \frac{1}{2} \lambda^2 t}
\]

\[M_\lambda(t) = e^{\lambda X(t) - \frac{1}{2} \lambda^2 t} \text{ is often called an exponential martingale.}\]

\[
E[M_\lambda(t) \mid F_s] = M_\lambda(s)
\]

Interpretation of \(M_\lambda(t)\) as an econometric model.

Suppose an economic variable - price level - is assumed to
be Gaussian with variance $\sigma^2 \Delta T$

$$R_n = \frac{S_{n \Delta t} - S_{(n-1) \Delta t}}{S_{(n-1) \Delta t}}$$

$$S_{n \Delta t} = S_0 \prod_{j=1}^{n} (1 + R_j)$$

The idea of modeling $\sigma^2(R_n) = \sigma^2 \Delta T$ is so that the variance of returns matches a 1-year variance, for example. So, the assumption is that all intervals contribute equally to the variance (think CLT).
Calculate statistics of $S_t$

\[
\ln \frac{S_{n+1}}{S_0} = \sum_j \ln \left( 1 + R_j \right)
\]
\[
= \sum_j \ln \left( 1 + \sigma^2 \frac{Y_j \sqrt{\Delta t}}{2} \right)
\]
\[
= \sum_j \left[ \sigma Y_j \sqrt{\Delta t} - \frac{1}{2} \sigma^2 Y_j^2 \Delta t \right]
\]

if \( t = n \Delta t \)

\[
\sum_{j=1}^{t/\Delta t} \sigma Y_j \sqrt{\Delta t} = \sigma \sum_{j=1}^{t/\Delta t} Y_j \sqrt{\Delta t}
\]
\[
= \sigma X(t) \quad (X = BM)
\]

\[
\sum_{j=1}^{t/\Delta t} \sigma^2 Y_j^2 \Delta t = \sigma^2 \Delta t \sum_{j=1}^{t/\Delta t} Y_j^2
\]
\[
= \sigma^2 t \left( \frac{n \bar{Y}^2}{n} \right)
\]
\[
\approx \sigma^2 t \quad (n \to \infty)
\]
This is sometimes called an exponential geometric Brownian motion because it is based on log-compoundly rather than adding independent random variables. Model \((***)\) is used in the classical Black–Scholes–Merton option pricing model. Another message: When you compound Gaussian r.v.’s the result is not the standard one. There is an extra term.
in the exponential due to the variance.

\[ \text{Markov Property of Brownian Motion} \]

\[
\mathbb{E} [ f(X_t) \mid F_s ] = \mathbb{E} \left[ f(X_t) \mid X_s^3 \right]
\]

\section*{Proof}

\[
f(X) = \int e^{i k X} \hat{f}(k) \, dk
\]

\[
\mathbb{E} \left[ f(X_t) \mid F_s \right] = \mathbb{E} \left[ \int e^{i k X_t} \hat{f}(k) \, dk \mid F_s \right]
\]

\[
= \int \hat{f}(k) \, dk \mathbb{E} (e^{i k X_t} \mid F_s)
\]

\[
= \int \hat{f}(k) \, dk \mathbb{E} (e^{i k X_s} e^{-\frac{k^2}{2}(t-s)} \mid F_s)
\]

\[
= \int \hat{f}(k) \, dk e^{-\frac{k^2}{2}(t-s)}
\]

\[= \text{a function of } X_s.\]
\[ E \left[ f(X_t) \mid X_s \right] = \int_{-\infty}^{+\infty} f(x) e^{-\frac{y^2}{2(t-s)}} \frac{dy}{\sqrt{2\pi(t-s)}} \]

\[ E \left[ f(X_t) \mid X_s = x \right] = \int_{-\infty}^{+\infty} f(y) e^{-\frac{y^2}{2(t-s)}} \frac{dy}{\sqrt{2\pi(t-s)}} = \frac{(x-y)^2}{2(t-s)} \frac{dy}{\sqrt{2\pi(t-s)}} \]

\[ = \psi(x, s) \]

\[ \frac{\partial \psi}{\partial s} + \frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} = 0 \quad \psi(s=t) = f(x) \]

Since the Cauchy process has a connection with Brownian motion, BM is connected with the diffusion.