Chapter 7

Stokes flow

We have seen in section 6.3 that the dimensionless form of the Navier-Stokes equations for a Newtonian viscous fluid of constant density and constant viscosity is, now dropping the stars,

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p - \frac{1}{Re}\nabla^2 \mathbf{u} = 0, \quad \nabla \cdot \mathbf{u} = 0. \tag{7.1}$$

The Reynolds number Re is the only dimensionless parameter in the equations of motion. In the present chapter we shall investigate the fluid dynamics resulting from the *a priori* assumption that the Reynolds number is very small compared to unity, $Re \ll 1$. Since $Re = UL/\nu$, the smallness of Re can be achieved by considering extremely small length scales, or by dealing with a very viscous liquid, or by treating flows of very small velocity, so-called *creeping flows*.

The choice $Re \ll 1$ is an very interesting and important assumption, for it is relevant to many practical problems, especially in a world where many products of technology, including those manipulating fluids, are shrinking in size. A particularly interesting application is to the swimming of micro-organisms. In all of these areas we shall, with this assumption, unveil a special dynamical regime which is usually referred to as *Stokes flow*, in honor of George Stokes, who initiated investigations into this class of fluid problems. We shall also refer to this general area of fluid dynamics as the *Stokesian realm*, in contrast to the theories of inviscid flow, which might be termed the *Eulerian realm*.

What are the principle characteristics of the Stokesian realm? Since Re is indicative of the ratio of inertial to viscous forces, the assumption of small Re will mean that viscous forces dominate the dynamics. That suggests that we may be able to drop entirely the term $D\mathbf{u}/Dt$ from the Navier-Stokes equations, rendering the system *linear*. This will indeed be the case, with some caveats discussed below. The linearity of the problem will be a major simplification.

Looking at (7.1)in the form

$$Re\Big(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p\Big) = \nabla^2 \mathbf{u}, \quad \nabla \cdot \mathbf{u} = 0, \tag{7.2}$$

It is tempting to say that the smallness of Re means that we can neglect the left-hand side of the first equation, leading to the reduced (linear) system

$$\nabla^2 \mathbf{u} = 0, \quad \nabla \cdot \mathbf{u} = 0. \tag{7.3}$$

Indeed solutions of (7.3) belong to the Stokesian realm and are legitimate.

Example 7.1: Consider the velocity field $\mathbf{u} = \frac{\mathbf{A}\times\mathbf{R}}{R^3}$ in three dimensions with $\mathbf{A}$ a constant vector and $\mathbf{R} = (x, y, z)$. Note that $\mathbf{u} = \nabla \times \frac{\mathbf{A}}{R}$, and so $\nabla \cdot \mathbf{u} = 0$ and also $\nabla^2 \mathbf{u} = 0, R > 0$ since $\frac{1}{R}$ is a harmonic function there. This in fact an interesting example of a Stokes flow. Consider a sphere of radius a rotating in a viscous fluid with angular velocity Ω. The on the surface of the sphere the velocity is $\Omega \times \mathbf{R}$ if the no-slip condition holds. Comparing this with our example we see that if $\mathbf{A} = \Omega a^3$ we satisfy this condition with a Stokes flow. Thus we have solved the Stokes flow problem of a sphere spinning in an infinite expanse of viscous fluid.

It is not difficult to see, however, that (7.3) does not encompass all of the Stokes flows of interest. The reason is that the pressure has been expelled from the system, whereas there is no physical reason for this. If, in the process of writing the dimensionless equations, we had defined the dimensionless pressure as $pL/(\mu U)$ instead of $p/(\rho U^2$, (7.2) would be changed to

$$Re\Big(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}\Big) + \nabla p = \nabla^2 \mathbf{u}, \quad \nabla \cdot \mathbf{u} = 0, \tag{7.4}$$

leading in the limit $\Re \to 0$ to

$$\nabla p - \nabla^2 \mathbf{u} = 0, \quad \nabla \cdot \mathbf{u} = 0. \tag{7.5}$$

We see that any solution of (7.5) will have the form $\mathbf{u} = \nabla\phi + \mathbf{v}$ where $\nabla^2\phi = p$ and $\nabla^2 \mathbf{v} = 0, \nabla \cdot \mathbf{v} = -p$. This larger class of flows, valid for Re small, are called *Stokes flows*. The special family of flows with zero pressure form a small subset of all Stokes flows.

7.0.1 Some caveats

We noted above that the dropping of the inertial terms in Stokes flow might have to be questioned in some cases, and we consider these exceptions now. First, it can happen that there is more than one possible Reynolds number which can be formed, involving one or more distinct lengths, and/or a frequency of oscillation, etc. It can then happen that the time derivative of $\mathbf{u}$ needs to be kept even though the $\mathbf{u} \cdot \nabla \mathbf{u}$ nonlinear term may be dropped. An example is a wall adjacent too a viscous fluid, executing a standing wave with amplitude A, frequency ω and wavelength L. If $\omega L^2/\nu$ is of order unity, and we take $U = \omega L$, then the Reynolds number UL/ν is of order unity and no terms may be dropped. However the actual velocity is of order ωA, and if $A \ll L$ then the nonlinear terms are negligible.

Another unusual situation is associated with the non-uniformity of the Stokes equations in three dimensions near infinity, in steady flow past a finite body.

Even through the Reynolds number is small, the fall off of the velocity at R^{-1} (associated with the fundamental solution of the Stokes equations) means that near infinity the perturbation of the free stream speed U is or order R^{-1}. Thus the $\mathbf{u} \cdot \nabla \mathbf{u}$ term is $O(U^2/R^2))$ while the viscous term is $O(\nu U/R^3)$. The ratio is UR/ν, which means that when $R \sim \nu/U$ the stokes equations cannot govern the perturbational velocity. The momentum equation needed to replace the Stokes equation contains the term $U\frac{\partial \mathbf{u}}{\partial x}$. We shall remark on the need for the Oseen system later in connection with two-dimensional Stokes flow.

7.1 Solution of the Stokes equations

Returning to dimensional equations, the Stokes equations are

$$\nabla p - \mu \nabla^2 \mathbf{u} = 0, \quad \nabla \cdot \mathbf{u} = 0. \tag{7.6}$$

From the divergence of $\nabla p - \nabla^2 \mathbf{u} = 0$, using the solenoidal property of $\mathbf{u}$, we see that $\nabla^2 p = 0$, and hence that $\nabla^4 \mathbf{u} = \nabla^2 \nabla^2 \mathbf{u} = 0$. The curl of this equation gives also $\nabla^2 \nabla \times \mathbf{u} = 0$. The components of $\mathbf{u}$ thus solves the *biharmonic equation* $\nabla^4 \phi = 0$ as well as the solenoidal condition, and the vorticity is a harmonic vector field. We shall combine these constraints now and set up a procedure for constructing solutions from a scalar biharmonic equation.

We first set

$$u_i = \Big(\frac{\partial^2 \chi}{\partial x_i \partial x_j} - \delta_{ij} \chi \Big) a_j, p = \mu \frac{\partial \nabla^2 \chi}{\partial x_j} a_j, \tag{7.7}$$

where $\mathbf{a}$ is a constant vector. Inserting these expressions into (7.6) we see that the equations are satisfied identically provided that

$$\nabla^4 \chi = 0. \tag{7.8}$$

A second class of solution, having zero pressure, has the form

$$\varepsilon_{ijk} \frac{\partial \phi}{\partial j} a_k, \tag{7.9}$$

for a constant vector $\mathbf{a}$, where $\varepsilon_{ijk} = 1$ for subscripts which are an even permutation of 123, and is -1 otherwise. The solutions (7.9) include example 7.1, with $\mathbf{A} = \mathbf{a}$ and $\phi = R^{-1}$.

Example 7.2: The fundamental solution of the Stokes equations in three dimensions corresponds to a point force $\mathbf{F}\delta(\mathbf{x})$ on the right of the momentum equation, $\mathbf{F}$ a constant vector:

$$\nabla p - \mu \nabla^2 \mathbf{u} = \mathbf{F}\delta(\mathbf{x}), \quad \nabla \cdot \mathbf{u} = 0. \tag{7.10}$$

Setting $\mathbf{a} = \mathbf{F}$ in (7.7) we must have

$$\mu \nabla^4 \chi = \delta(\mathbf{x}). \tag{7.11}$$

We know the fundamental solution of $\nabla^2\phi = 0$, satisfying $\nabla^2\phi = \delta(\mathbf{x})$ and vanishing at infinity is $-\frac{1}{4\pi R}$ in three dimensions. Thus

$$\nabla^2\chi = -\frac{1}{4\pi}\frac{1}{R} = \frac{\mu}{R}\frac{d^2 R\chi}{dR^2}, \tag{7.12}$$

and so

$$\chi = -\frac{1}{8\pi\mu}R + A + BR^{-1}. \tag{7.13}$$

The singular component is incompatible with (7.11) and the constant A may be set equal to zero without changing $\mathbf{u}$, and so $\chi = -\frac{1}{8\pi\mu}R$. Then we find

$$u_i = \frac{1}{8\pi\mu}\Big(\frac{x_i x_j}{R^3} + \frac{\delta_{ij}}{R}\Big)F_j, p = \frac{1}{4\pi}\frac{x_j F_j}{R^3}. \tag{7.14}$$

The particular Stokes flow (7.14) is often referred to as a *Stokeslet*.

7.2 Uniqueness of Stokes flows

Consider Stokes flow within a volume V having boundary S. Let the boundary have velocity $\mathbf{u}_S$. By the no-slip condition (which certainly applies when viscous forces are dominant), the fluid velocity $\mathbf{u}$ must equal $\mathbf{u}_S$ on the boundary. Suppose now that there are two solutions $u_{1,2}$ to the problem of solving (7.6) with this boundary condition on S. Then $\mathbf{v} = u_1 - u_2$ will vanish on S while solving (7.6). But then

$$\int_V \mathbf{v}\cdot(\nabla p - \mu\nabla^2\mathbf{v})dV = 0 = \int_V \frac{\partial}{\partial x_j}\Big(v_j p - \mu v_i \frac{\partial v_i}{\partial x_j}\Big)dV + \mu\int_V \Big(\frac{\partial v_i}{\partial x_j}\Big)dV, \tag{7.15}$$

where the solenoidal property of $\mathbf{v}$ has been used. The first integral on the right vanishes under the divergence theorem because of the vanishing of $\mathbf{v}$ on S. The second is non-negative (with understood summation over i, j), and vanishes only if $\mathbf{v} = 0$. We remark that the non-negative term is equal to the rate of dissipation of kinetic energy into heat as a result of viscous stresses, for the velocity field $\mathbf{v}$. This dissipation can vanish only if the velocity is identically zero.

The solution of the Stokes equations is not easy in most geometries, and frequently the coordinate system appropriate to the problem will suggest the best formulation. We illustrate this process in the next section.

7.3 Stokes' solution for uniform flow past a sphere

We now consider the classic solution of the Stokes equations representing the uniform motion of a sphere of radius a in an infinite expanse of fluid. We shall first consider this problem using the natural coordinates for the available symmetry, namely spherical polar coordinates. Then we shall re-derive the solution using (7.7). The velocity field in spherical coordinates has the form $(u_R, u_\theta, u_\phi) = (u_R, u_\theta, 0)$ and the solenoidal condition is

$$\frac{1}{R}\frac{\partial R^2 u_R}{\partial R} + \frac{1}{\sin\theta R}\frac{\partial \sin\theta u_\theta}{\partial\theta} = 0. \tag{7.16}$$

We thus introduce the Stokes steam function Ψ,

$$u_R = \frac{1}{R^2\sin\theta}\frac{\partial\Psi}{\partial\theta}, \quad u_\theta = -\frac{1}{R\sin\theta}\frac{\partial\Psi}{\partial R}. \tag{7.17}$$

Now Stokes' equations in spherical coordinates are

$$\frac{\partial p}{\partial R} = \mu\Big(\nabla^2 u_R - \frac{2u_R}{R^2} - \frac{2}{R^2\sin\theta}\frac{\partial\sin\theta u_\theta}{\partial\theta}\Big), \tag{7.18}$$

$$\frac{1}{R}\frac{\partial p}{\partial\theta} = \mu\Big(\nabla^2 u_\theta + \frac{2}{R^2}\frac{\partial u_R}{\partial\theta} - \frac{u_\theta}{R^2\sin^2\theta}\Big), \tag{7.19}$$

together with (7.16). The vorticity is $(0, 0, \omega_\phi)$, where

$$\omega_\phi = -\frac{1}{R\sin\theta}\mathcal{L}\Psi, \tag{7.20}$$

where

$$\mathcal{L} = \frac{\partial^2}{\partial R^2} + \frac{\sin\theta}{R^2}\frac{\partial}{\partial\theta}\Big(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\Big). \tag{7.21}$$

Now from the form $\nabla p + \mu\nabla\times\nabla\times\mathbf{u} = 0$ of the momentum equation, we have the alternative form

$$\frac{\partial p}{\partial R} = -\frac{\mu}{R\sin\theta}\frac{\partial}{\partial\theta}\omega_\phi\sin\theta, \tag{7.22}$$

$$\frac{1}{R}\frac{\partial p}{\partial\theta} = \frac{\mu}{R}\frac{\partial}{\partial R}R\omega_\phi. \tag{7.23}$$

Eliminating the pressure and using (7.20) we obtain

$$\frac{1}{R^2}\frac{\partial}{\partial\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\mathcal{L}\Psi + \frac{\partial}{\partial R}\frac{1}{\sin\theta}\frac{\partial}{\partial R}\mathcal{L}\Psi = 0. \tag{7.24}$$

We seek to solve (7.24) with the conditions

$$u_R = u_\theta = 0,\ R = a, \quad \Psi \sim \frac{1}{2}R^2\sin^2\theta U,\ R\to\infty. \tag{7.25}$$

We now separate variables in the form

$$\Psi = \sin^2\theta f(R), \tag{7.26}$$

to obtain from (7.24)

$$\Big(\frac{\partial^2}{\partial R^2} - \frac{2}{R^2}\Big)^2 f = 0. \tag{7.27}$$

Trying $f = R^\lambda$ we get $(\lambda^2 - 1)(\lambda - 2)(\lambda - 4) = 0$ and therefore the general solution of (7.27) is

$$f = \frac{A}{R} + BR + CR^2 + DR^4. \tag{7.28}$$

From the behavior needed for large R, $D = 0, C = U/2$. The two conditions at $R = a$ then require that

$$A = \frac{1}{4}Ua^2, \quad B = -\frac{3}{4}Ua. \tag{7.29}$$

Thus

$$\Psi = \frac{1}{4}U\Big(\frac{a^3}{R} - 3aR + 2R^2\Big)\sin^2\theta. \tag{7.30}$$

7.3.1 Drag

To find the drag on the sphere, we need the following stress component evaluated on $R = a$:

$$\sigma_{RR} = -p + 2\mu\frac{\partial u_R}{\partial R}, \quad \sigma_{R\theta} = \mu R\frac{\partial}{\partial R}\Big(\frac{u_\theta}{R}\Big) + \frac{\mu}{R}\frac{\partial u_R}{\partial \theta}. \tag{7.31}$$

Given these functions the drag D is determined by

$$D = a^2\int_0^{2\pi}\int_0^\pi [\sigma_{RR}\cos\theta - \sigma R\theta\sin\theta]\sin\theta d\theta d\phi. \tag{7.32}$$

Now from (7.23) the pressure is determined by

$$\frac{1}{R}\frac{\partial p}{\partial \theta} = -\frac{\mu}{R\sin\theta}\frac{\partial}{\partial R}\sin^2\theta\Big(f_{RR} - \frac{1}{R^2}f\Big), \tag{7.33}$$

or, using (7.30),

$$p = -\frac{3}{2}\mu Ua\frac{\cos\theta}{R^2} + p_\infty. \tag{7.34}$$

Also

$$u_R = \frac{1}{R^2\sin\theta}\frac{\partial\Psi}{\partial\theta} = \frac{U\cos\theta}{2R^2}(a^3/R - 3aR + 2R^2), \tag{7.35}$$

$$u_\theta = -\frac{1}{R\sin\theta}\frac{\partial\Psi}{\partial R} = -\frac{U\sin\theta}{4R}(-a^3/R - 3a + 4R). \tag{7.36}$$

Thus

$$D = 2\pi a^2\int_0^\pi\Bigg[\Big[\frac{3}{2}\mu Ua\frac{\cos\theta}{R^2} - p_\infty + 2\mu\cos\theta\big(\frac{-3a^3}{R^4} - \frac{3a}{R}\big)_{R=a}\Big]\cos\theta$$
$$+\frac{\mu U\sin^2\theta}{4}\big(\frac{3a^2}{R^4} + \frac{3a}{R^2}\big)_{R=a}\Bigg]\sin\theta d\theta. \tag{7.37}$$

Thus

$$D = \underbrace{3\pi\mu aU \int_0^\pi \cos^2\theta \sin\theta d\theta}_{\text{pressure}} + \underbrace{3\pi\mu aU \int_0^\pi \sin^3\theta d\theta}_{\text{viscous}},$$
$$= 2\pi\mu aU + 4\pi\mu aU = 6\pi\mu aU. \tag{7.38}$$

That is, one-third of the drag is due to pressure forces, two-thirds to viscous forces.

7.3.2 An alternative derivation

We can re-derive Stokes' solution for a sphere by realizing that at large distances from the sphere the flow field must consist of a uniform flow plus the fundamental solution for a force $-6\pi\mu Ua\mathbf{i}$. This must be added a term or terms which will account for the finite sphere size. Given the symmetry we try a dipole term proportional to $\nabla(x/R^3)$. We thus postulate

$$\mathbf{u} = U\mathbf{i} - \frac{6\pi\mu aU}{8\pi\mu}\Big(\frac{x\mathbf{R}}{R^3} + \frac{\mathbf{i}}{R}\Big) + C\Big(\frac{\mathbf{i}}{R^3} - 3\frac{x\mathbf{R}}{R^5}\Big), \tag{7.39}$$

where C remains to be determined. By inspection we see that $C = -\frac{1}{4}a^2U$ makes $\mathbf{u} = 0$ on $R = a$, so we are done! The pressure is as given previously $p = -\frac{3}{2}\mu Uax/R^3 + p_\infty$, and is entirely associated with the fundamental part of the solution.

7.4 Two-dimensions: Stokes' paradox

The fundamental solution of the Stokes equations in two dimensions sets up as given in example 7.2, except that the biharmonic equation is to be solved in two dimensions. If Radial symmetry is again assumed, we may try to solve the problem equivalent to flow past a sphere, i.e. Stokes flow past a circular cylinder of radius a. If the pressure is eliminated from the Stokes equations in two dimensions, we get

$$\mu\nabla^2\omega = \mu\nabla^4\psi = 0. \tag{7.40}$$

in terms of the two-dimensional stream function ψ. We the set $\psi = \sin\theta f(r)$ to separate variables in polar coordinates, leading to

$$\Big[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}\Big]^2 f = 0. \tag{7.41}$$

We are now in a position to study from over a circular cylinder of radius a. The no-slip condition at the surface of the cylinder requires that $\psi(a) = \frac{\partial\psi}{\partial r}(a) = 0$, while the attaining of a free stream $\mathbf{u} = (U, 0)$ at infinity requires that $f \sim Ur, r \to \infty$. Now by quadrature we can find the most general solution of (7.41) as

$$f(r) = Ar^3 + Br\ln r + Cr + Dr^{-1}. \tag{7.42}$$

The condition at infinity requires that $A = B = 0$. The no-slip conditions then yield

$$Ca + Da^{-1} = 0, \quad C - Da^{-2} = 0, \tag{7.43}$$

which imply $C = D = 0$. *There is not satisfactory steady solution of the two-dimensional Stokes equations representing flow of an unbounded fluid past a circular cylinder.* This result, known as *Stokes paradox*, underlines the profound effect that dimension can play in fluid dynamics.

What is the reason for this non-existence? We can get some idea of what is going on by introducing a finite circle $r = R$ on which we make $\mathbf{u} = (U, 0)$. Then there *does* exist a function $f(r)$ satisfying $f(a) = f'(a) = 0, f(R) = R, f'(R) = 1$.[1] We shall obtain as asymptotic approximation for large R/a to this solution by setting $A = 0$ in (7.42) and satisfying the conditions at $r = a$ with the remaining terms. Then we obtain

$$f \sim B[r \ln r - (\ln a + 1/2)r + \frac{1}{2}a^2/r]. \tag{7.44}$$

We then make $f(R) \sim UR, R/a \to \infty$ by setting $B = U/\ln(R/a)$. Then also $f'(R) \sim 1 + o(1), R/a \to \infty$, so all conditions are satisfied exactly or asymptotically for large R/a. Thus

$$f \sim \frac{U}{\ln(R/a)}\big[r(ln(r/a) - 1/2) + \frac{1}{2}a^2/r\big]. \tag{7.45}$$

A a fixed value of $r/a > 1$ we see that $f \to 0$ as $R/a \to \infty$. It is only when $\frac{\ln(r/a)}{\ln(R/a)}$ become $O(1)$ that order UR values of f, and hence order U values of velocity, are realized. Thus a cylindrical body in creeping through a viscous fluid will tend to carry with a large stagnant body of fluid, and there is no solution of the boundary-value problem for an infinite domain in Stokes flow.

This paradox results from a failure to properly account for the balance of forces in a viscous fluid at large distances from a translating body, however small the Reynolds number of translation may be. If the velocity of translation is U and the body size L The remedy for this paradox is involves a problem of singular perturbation wherein the regions distant from the cylinder see a disturbance from a point force. Let the velocity at some point a distance $R \gg a$ from the body be q. The the inertial forces at this point sill be approximately $\sim \rho Uq/R$, (since we should linearize $\mathbf{u} \cdot \nabla \mathbf{u}$ about the free stream velocity). Also the viscous forces there are of order $\mu q/R^2$. These two estimates are comparable when $R?L \sin \nu/(UL) = 1/Re$. Thus when $Re < 1$ and we try to apply the Stokes equations, there is always distant points where the neglect of the inertial terms fails to be valid.

In the case of three dimensions, we have Stokes' solution for a sphere and we know that at distances $O(1/Re)$ the perturbation velocity caused by the sphere is small, or order Re. Thus, the Stokes approximation fails in a region

[1] If $\psi_y = U, \psi_x = 0$ on a circle $r = R$, then $f/r + (f/r)'(y^2/r) = U, (f/r)'(xy/r) = 0$ when $r = R$, by differentiation of $\sin \theta f$. Thus $f(R) = R$ and $f'(R) = 1$.

where the free stream velocity is essentially unperturbed, and there is no Stokes paradox. In two dimensions, the perturbation caused by the cylinder persists out to distances of order $1/Re$. Thus the Stokes equations fail to be uniformly valid in a domain large enough to allow necessary conditions at infinity to be satisfied.

The remedy for this paradox in two dimensions involves a proper accounting for the singular nature of the limit $Re \to 0$ in the neighborhood of infinity. At distances $r \sim Re^{-1}$ the appropriate equations are found to be

$$\rho\frac{\partial \mathbf{u}}{\partial x} + \nabla p - \mu\nabla^2\mathbf{u} = 0, \quad \nabla\cdot\mathbf{u} = 0. \tag{7.46}$$

This system is known as *Oseen's equations*. Oseen proposed them as a way of approximately accounting for fluid inertia in problems where there is an ambient free stream $U\mathbf{i}$. Their advantage is of course that they comprise a *linear* system of equations. The fact remains that they arise rigorously to appropriately treat viscous flow in the limit of small Reynolds numbers, in a way that expels any paradox associated with large distances.

To summarize, in creeping flow the Stokes model works well in three dimensions; near the body the equations are exact, and far from the body the non-uniformity, leading to the replacement of the Stokes equations by the Oseen equations, is of no consequence and Stokes' solution for a sphere is valid. In two dimensions the distant effect of a cylinder must be determined from Oseen's model. It is only by looking at that solution, expanded near the position of the cylinder, that we can determine the appropriate solution of Stokes' equations in two dimensions; this solution remains otherwise undetermined by virtue of the Stokes paradox.

7.5 Uniqueness and time-reversibility in Stokes flow

Consider a viscous fluid contained in some finite region V bounded by surface or surfaces ∂V. If Stokes flow prevails, and if the boundary moves, each point of ∂V being assigned a boundary velocity $\mathbf{u}_b$, then we have a boundary-value problem for the Stokes equations, whose solution will provide the instantaneous velocity of every fluid particle in V. We assume the existence of this solution, and verify now that it will be unique. Indeed if $\mathbf{w}$ is the difference of two distinct solutions, then for some pressure p_w we have a solution of the Stokes equations in V which vanishes on ∂V. But then

$$\int_V w_i\Big[\frac{\partial pw}{\partial x_i} - \mu\nabla^2 w_i\Big]dV = 0$$

$$\int_{\partial V}\Big[iw_ip_w - \mu w_in_j\frac{\partial w_i}{\partial x_j}\Big]dS + \mu\int_V\Big(\frac{\partial w_i}{\partial x_j}\Big)^2 dV = \mu\int_V\Big(\frac{\partial w_i}{\partial x_j}\Big)^2 dV \geq 0, \tag{7.47}$$

by use of the divergence theorem, the solenoidal property of $\mathbf{w}$, and the vanishing of $\mathbf{w}$ on ∂V. It follows that $\mathbf{w}$ must be a constant, and therefore zero throughout V.

Thus in Stokes flow the instantaneous velocity of a fluid particle at P is determined by the instantaneous velocities of all points on the boundary of the fluid domain. Let us now assume a motion of the boundary through a sequence of configurations $\mathcal{C}(t)$. Each $\mathcal{C}$ represents a point in *configuration space*, and the motion can be thought of as a path in configuration with time as a parameter. Indeed "time" has no dynamical significance. A path from A to B in configurations space can be taken quickly or slowly. In general, let the configuration at time t be given by $\mathcal{C}(\tau(t))$, where $\tau(0) = 0, \tau(1) = 1$ but is otherwise an arbitrary differentiable function of time. If the point P has velocity $\mathbf{u}_P(t), 0 \leq t \leq 1$ when $\tau(t) = t$, then in general $\mathbf{u}_P(t) = \dot{\tau}(t)\mathbf{u}_P(\tau(t))$. The vector displacement of the point P under this sequence of configurations is

$$\Delta_P = \int_{t=0}^{t=1} \dot{\tau}(t)\mathbf{u}_P(\tau(t))dt = \int_0^1 \mathbf{u}(\tau)d\tau \tag{7.48}$$

and so is independent of the choice of τ. Another way to say this is that the displacement depends upon the ordering of the sequence of configurations but not on the timing of the sequence.

The displacement *does* however depend in general on the *path* taken in configuration space in going from configuration $\mathcal{C}_0$ to $\mathcal{C}_1$. We now give an example of this dependence.

Example 7.3: We must find two paths in configuration space having the same starting and finishing configurations (i.e. the boundary points coincide in each case), but for which the displacement of some fluid particle is not the same. Consider then a two-dimensional geometry with fluid contained in the circular annulus $a < r < b$. Let the inner cylinder of radius a rotate with time so that the angle made by some fixed point on the cylinder is $\theta(t)$ relative to a reference axis. The outer circle $r = b$ sis fixed. The instantaneous velocity of each point of the fluid. Given that $\frac{\partial p}{\partial \theta} = 0$ and that the velocity is $0.u_\theta(r)$, the function $u_\theta(r)$ satisfies (from the Stokes form of (6.15)) $Lu_\theta = 0$. Integrating and applying boundary conditions, the fluid velocity in the annulus is

$$u_\theta = \frac{a\dot{\theta}}{a^2 - b^2}r - \frac{ab^2\dot{\theta}}{a^2 - b^2}r^{-1}. \tag{7.49}$$

Consider not two paths which leave the position of the point of the inner circle unchanged. In the first, θ rotates from 0 to $\pi/4$ in one direction, then from $\pi/4$ back to zero in the other direction. Clearly every fluid particle will return to its original position after these two moves. For the second path, rotate the cylinder through 2π. Again every point of the boundary returns to its starting point, but now every point of fluid in $a < r < b$ moves through an angle θ which is positive and less than 2π. Thus only the points on the two circles $r = a, b$ are in their starting positions at the end of the rotation.

Note that in this example the first path, returning all fluid particles to their starting positions, is special in that *the sequence of configurations in the* *second movement is simply a reversal of the sequence of configurations in the first movement* (a rotation through angle $\pi/4$. A moments reflection show that zero particle displacement is a necessary consequence of this kind path- a sequence followed by the reverse sequence. And note that the timing of each of these sequences may be different.

The second path, a full rotation of the inner circle, involves no such reversal. In fact if the direction of rotation is reversed, the fluids point move in the opposite direction. If we now let these to paths be repeated periodically, say every one unit of time t, then in the first case fluid particles move back and forth periodically with no net displacement, while in the second case particles move on circles with a fixed displacement for each unit of time. Notice now an importance difference in the time symmetry of these two cases. If time is run backwards in the first case, we again see fluid particles moving back and forth with no net displacement. In the second case, reversal of time leads to steady rotation of particles in the opposite direction. We may say that the flow in the first case exhibits *time reversal symmetry*, while in the second case it does not exhibit this symmetry. In general, a periodic boundary motion exhibiting time reversal symmetry cannot lead to net motion of any fluid particle over one period, as determined by the resulting time-periodic Stokes flow. On the other hand, if net motion is observed, the boundary motion cannot be symmetric under time reversal.

However a motion that is not symmetric under time reversal may in fact not produce any displacement of fluid particles.

Example 7.4 In the previous example, let both circles rotate through 2π with $\dot{\theta}_b = \frac{b}{a}\dot{\theta}_a$. The boundary motion does not then exhibit time-reversal symmetry, and in fact the fluid can be seen to be in a solid body rotation. Thus every fluid particle returns to its starting position.

Theorem 7 *Time reversal symmetry of periodic boundary motion is sufficient to insure that all fluid particles return periodically to their starting positions. If particles do not return periodically to their starting position, the boundary motion cannot be time-symmetric.*

7.6 Stokesian locomotion and the scallop theorem

One of the most important and interesting applications of Stokes flow hydrodynamics is to the swimming of micro-organisms. Most micro-organisms move by a periodic or near periodic motion of organelles such as cilia and flagella. The aim of this waving of organelles is usually to move the organisms from point A to point B, a process complementary to a variable boundary which moves the fluid about but does not itself locomote. Indeed time-reversal symmetry plays an key role in the selection of swimming strategies.

Theorem 8 *(The scallop theorem) Suppose that a small swimming body in an infinite expanse of fluid is observed to execute a periodic cycle of configurations,* *relative to a coordinate system moving with constant velocity* $\mathbf{U}$ *relative to the fluid at infinity. Suppose that the fluid dynamics is that of Stokes flow. If the sequence of configurations is indistinguishable from the time reversed sequence, then* $\mathbf{U} = 0$ *and the body does not locomote.*

The reasoning here is that actual time reversal of the swimming motions would lead to locomotion with velocity $-\mathbf{U}$. But if the two motions are indistinguishable then $\mathbf{U} = -\mathbf{U}$ and so $\mathbf{U} = 0$. The name of the theorem derives from the non-locomotion of a scallop in Stokes flow that simple opens and closes its shell periodically. In Stokes flow this would lead to a back and forth motion along a line (assuming suitable symmetry of shape of the shell), with no net locomotion.

In nature the breaking of time-reversal symmetry takes many forms. Flagella tend to propagate waves from head to tail. The wave direction gives the arrow of time, and it reverses, along with the swimming velocity, under time reversal. Cilia also execute complicated forward and return strokes which are not time symmetric.

Problem set 7

1. Consider the uniform slow motion with speed U of a viscous fluid past a spherical bubble of radius a, filled with air. Do this by modifying the Stokes flow analysis for a rigid sphere as follows. The no slip condition is to be replaced on $r = a$ by the condition that both u_r and the tangential stress $\sigma_{r\theta}$ vanish. (This latter condition applies since there is no fluid within the bubble to support this stress.) Show in particular that

$$\Psi = \frac{U}{2}(r^2 - ar)\sin^2\theta$$

and that the drag on the bubble is $D = 4\pi\mu U a$. Note: On page 235 of Batchelor see the analysis for a bubble filled with a second liquid of viscosity $\bar{\mu}$. The present problem is for $\bar{\mu} = 0$.

2. Prove that Stokes flow past a given, rigid body is unique, as follows. Show if $p_1, \mathbf{u}_1$ and $p_2, \mathbf{u}_2$ are two solutions of

$$\nabla p - \mu\nabla^2\mathbf{u} = 0, \nabla\cdot\mathbf{u} = 0,$$

satisfying $u_i = -U_i$ on the body and

$$\mathbf{u} \sim= O(1/r), \frac{\partial u_i}{\partial x_j}, p \sim O(1/r^2)$$

as $r \to \infty$, then the two solutions must agree. (Hint: Consider the integral of $\partial/\partial x_i(w_j \partial w_j/\partial x_i)$ over the region exterior to the body, where $\mathbf{w} = \mathbf{u}_1 - \mathbf{u}_2$.)

3. Two small spheres of radius a and density ρ_s are falling in a viscous fluid with centers at P and Q. The line PQ has length $L \gg a$ and is perpendicular to gravity. Using the Stokeslet approximation to the Stokes solution past a sphere, and assuming that each sphere sees the unperturbed Stokes flow of the other sphere, show that the spheres fall with the same speed

$$U \approx U_s(1 + ka/L + O(a^2/L^2)),$$

and determine the number k. Here $U_s = 2a^2g/9\nu(\rho_s/\rho - 1)$ is the settling speed of a single sphere in Stokes flow.