On vortex stretching and the global regularity of Euler flows II. A singular Euler flow in a product geometry

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Abstract

Continuing our study of the global regularity of Euler flows, we describe in the present paper the construction of an Euler flow which , under certain assumptions, can be demonstrated to blow up in finite time. The approach we take derives from the kinematic cocoon of constant volume described in paper I. We first consider the motion of a planar Jordan curve moving with the velocity-stretching relationship of that cocoon. We exhibit a class of self-similar solutions of the equations for the curve, which blow up in finite time at a point. We next adopt such a curve as the core vortex line for a "dynamic cocoon" consistent with Euler's equations. Within this framework, the singular solution becomes locally two dimensional. Considered as a 3D flow, the structure can be regarded as a product of local motions in R^2 and the core vortex motion in R^1. Different self-similarity prevails in the two factors of the product. The complete structure stretches only a finite amount during blow-up, and only a finite amount of energy flows into the singularity.

1 Introduction

The purpose of the present note is to suggest an approach to the problem of Euler blow-up utilizing paired vortex structures of the kind studied in [6]. We shall describe a method which appears to answer the issue in the affirmative: there exist initial conditions for the 3D Euler equations in R^3, which are at least bounded and continuous in the vorticity, which lead to blow up in finite time of the corresponding solutions. Our results further suggest that the Navier-Stokes equations do not possess singular solutions, in that the present Euler solutions cannot be approximated by Navier-Stokes flows.

The present discussion is incomplete and preliminary. The approach described may however be useful independent of the validity of some of the claims.

We summarize the basic idea. The key starting point is the existence of a class of singular solutions for motion of a planar curve $C(t)$ in the direction of its normal, at a velocity which increases as the square root of the stretching of the curve (as determined by a Jacobian of the Lagrangian map of the curve). We take such a curve as the core vortex of a cocoon, in the sense of [6]. However in the present paper we allow the cocoon to be dynamic, in the that the vorticity within the cocoon is allowed to evolve according to Euler's equations. This approach is found to be consistent for a family of structures with parameter $\gamma \in (.5, 1)$. Here consistency implies that the support of the vorticity is confined to a cocoon which is under control as the core vortex reaches its singularity. The singularity involves only a finite amount of total stretching by the core vortex line. Also, only a finite amount of energy is delivered to the singularity.

The underlying structural element of the flow is a paired vortex structure equivalent locally to the two dimensional version of a Hill's vortex, here referred to as Batchelor's vortex. This structure generates within the cocoon the flow needed to move the vorticity in such a way that it stays well within the cocoon, the motion of the core of the cocoon being predetermined by the motion of the singular curve $C(t)$. In the vicinity of the singularity both vorticity and the cocoon boundary contract, but the vortical structures remain slender and locally 2D. The negative of the limiting vorticity at the blow-up time is then used as the initial condition for a "flow reversed" evolution back to a non-singular structure, but now with the cocoon boundary removed and the conditions those of an Euler flow in R^3.

2 Geometry of curve stretching

We first summarize well-known results for stretching of a planar curve C(t) by a velocity field defined at each point of the curve. Suppose that a point on C determined by a given value of a Lagrangian parameter ζ_0 moves with velocity

$$\mathbf{u} = u\mathbf{n} + w\mathbf{t}. \tag{1}$$

Note that, if the curve is to represent the core vortex of the constant volume cocoon of paper I, $w$must vanish and u will be negative. Thus

$$\frac{\partial \mathbf{x}}{\partial t}\Big|_{\zeta_0} = u\mathbf{n} + w\mathbf{t}. \tag{2}$$

Taking the ζ_0 derivative of the last equation, we have

$$\frac{\partial}{\partial t}(J\mathbf{t}) = J[u_\zeta \mathbf{n} - u\kappa \mathbf{t} + w_\zeta \mathbf{t} + w\kappa \mathbf{n}]. \tag{3}$$

Here ζ denotes arc length, κ is the curvature, and J is the Jacobian $\frac{\partial \zeta}{\partial \zeta_0}$. Since $\partial \mathbf{t}/\partial t$ is in the normal direction, we see that (3) implies

$$\frac{\partial J}{\partial t}\Big|_{\zeta_0} - w_\zeta J = -Ju\kappa, \quad \frac{\partial \mathbf{t}}{\partial t} = (w\kappa + u_\zeta)\mathbf{n}. \tag{4}$$

Taking the ζ_0 derivative of (3) we obtain

$$\frac{\partial}{\partial t}\Big|_{\zeta_0}[J^2\kappa \mathbf{n} + JJ_\zeta \mathbf{t}] = JJ_\zeta[w\kappa \mathbf{n} + u_\zeta \mathbf{n}]$$

$$+2w_\zeta J^2\kappa\mathbf{n} + w\kappa_\zeta J^2 + u_{\zeta\zeta}J^2\mathbf{n} - \kappa^2 uJ^2\mathbf{n} + [\ldots]\mathbf{t}. \tag{5}$$

Now, using (4),

$$\frac{\partial}{\partial t}\Big|_{\zeta_0}[J^2\kappa\mathbf{n} + JJ_\zeta\mathbf{t}] = [2J\kappa(w_sJ - Ju\kappa) + J^2\kappa_t + JJ_\zeta(w\kappa + u_s)]\mathbf{n} + [\ldots]\mathbf{t}. \tag{6}$$

Since $\mathbf{t}\cdot\mathbf{t} = 1$ and $\mathbf{t}\cdot\mathbf{n} = 0$ we note that $\frac{\partial \mathbf{n}}{\partial t} = -(w\kappa + u_s)\mathbf{t}$.

Thus, the normal components of (5) yield

$$2J\kappa(w_\zeta J - Ju\kappa) + J^2\kappa_t + JJ_\zeta(w\kappa + u_\zeta) =$$

$$JJ_\zeta[w\kappa + u_\zeta] + 2w_\zeta J^2\kappa + w\kappa_\zeta J^2 + u_{\zeta\zeta}J^2 - \kappa^2 uJ^2. \tag{7}$$

Thus

$$\frac{\partial \kappa}{\partial t}\Big|_{\zeta_0} - w\kappa_\zeta - \kappa^2 u - u_{\zeta\zeta} = 0. \tag{8}$$

If $w = 0$, as in our cocoon construction for axisymmetric flow without swirl, we have the two equations

$$\frac{\partial J}{\partial t}\Big|_{\zeta_0} = -Ju\kappa, \tag{9}$$

$$\frac{\partial \kappa}{\partial t}\Big|_{\zeta_0} - \kappa^2 u - u_{\zeta\zeta} = 0. \tag{10}$$

Note that (9) gives the stretching of the curve and (10) determines the evolution of the curvature. We will need to follow both quantities separately as we have broken the constraint linking curvature and stretching implicit in axisymmetric flow without swirl.

3 A motion of the curve leading to a finite time singularity

We now consider the motion of a volume preserving cocoon in the x, y- plane. We know in this case that $J = \alpha'(\zeta_0)(-u)^\beta$ with $\beta = 2$. That is, a given point on the curve , fixed by ζ_0, moves in such a way that the velocity squared is proportional to the stretching rate. For the moment we leave the exponent β arbitrary. An arbitrary Lagrangian parametrization determined by the monotone function α is allowed here. Thus by (9),

$$\kappa = -\beta\frac{u_t}{u^2}. \tag{11}$$

Using this expression in (10) and expressing ζ-derivatives in terms of ζ_0-derivatives, we obtain the following equation for $u(\zeta_0, t)$:

$$u_{tt} + (\beta - 2)\frac{u_t^2}{u} + \frac{u^2}{\beta\alpha'(\zeta_0)(-u)^\beta}\frac{\partial}{\partial\zeta_0}\frac{1}{\alpha'(\zeta_0)(-u)^\beta}\frac{\partial u}{\partial\zeta_0} = 0. \tag{12}$$

We can thus consider u as a function of $\alpha(\zeta_0)$ and t.

We consider a similarity form, still taking $J = \alpha'(\zeta_0)(-u)^\beta$

$$u = -\tau^{-\gamma} A g(\sigma)), \quad \sigma = \alpha(\zeta_0)\tau^{-\mu}, \tag{13}$$

where A is a constant, and

$$\tau = -t, \; t < 0. \tag{14}$$

The time of the singularity is here stipulated to be $t = 0$. Substituting (13) into (12) we obtain a solution if

$$\mu = (\beta - 1)\gamma + 1. \tag{15}$$

The equation for g can then be integrated once. Applying the conditions $g(0) = 1$ (given the arbitrary constant A), and $g'(0) = 0$ (a symmetry condition), we obtain the following equation for g:

$$\mu\gamma\sigma g^{\beta-1} + \sigma^2\mu^2 g^{\beta-2} g' + \frac{1}{\beta A^{2\beta-2}} \frac{g'}{g^\beta} = 0. \tag{16}$$

A second integration gives

$$\mu\beta A^{2\beta-2}\sigma^2 g^{\frac{2\mu}{\gamma}} + g^{\frac{2}{\gamma}} = 1. \tag{17}$$

Let us regard C as oriented to that at $\sigma = 0$, $\mathbf{t}$ points in the direction of the positive x-axis. We define θ as the angle made by $\mathbf{t}$ with the x-axis, so that $\kappa = \frac{\partial\theta}{\partial\zeta}$. Then

$$\frac{\partial\theta}{\partial\sigma} = -A^{\beta-1}[g^\beta\gamma + \mu\sigma g^{\beta-1}g'] = 0. \tag{18}$$

and so, from (16)

$$\theta = -A^{1-\beta}\mu^{-1} \int g^{-\beta}\sigma^{-1} dg. \tag{19}$$

Here, from (17),

$$\sigma = \frac{A^{1-\beta}}{\sqrt{\mu\beta}} g^{-\mu/\gamma}\sqrt{1 - g^{2/\gamma}}. \tag{20}$$

So

$$\theta = \gamma\sqrt{\frac{\beta}{\mu}}\Big[\frac{\pi}{2} - \sin^{-1}(g^{1/\gamma})\Big]. \tag{21}$$

To get the coordinates of C, we have $(\frac{\partial x}{\partial\zeta}, \frac{\partial y}{\partial\zeta}) = (\cos\theta, \sin\theta)$. Carrying out an integration by parts and using (19) and (20) we obtain

$$A^{-1}\tau^{\gamma-1}(x, y) = \frac{1}{\sqrt{\mu\beta}} g\sqrt{g^{-2/\gamma} - 1}[\cos\theta(g), \sin\theta(g)]$$

$$+\sqrt{\frac{\beta}{\mu}} \int_g^1 \sqrt{g^{-2/\gamma} - 1}[\cos\theta(g), \sin\theta(g)]dg + \mu^{-1} \int_g^1 [\sin\theta(g), -\cos\theta(g)]dg. \tag{22}$$

We see from (17) that $g \to 0$ as $\sigma \to \infty$, and from (21) that

$$\theta \to \frac{\gamma\pi}{2}\sqrt{\frac{\beta}{\mu}} \equiv \theta_\infty, \quad \sigma \to \infty. \tag{23}$$

We will be using below the case $\beta = 2$. Taking this value and requiring that $\theta_\infty = \pi/3$ we find $\gamma = \frac{1}{9}(1+\sqrt{19}) = .5954$. As we shall see, it will be important for us that we take $\gamma > .5$. We show in figure 1 the shape of C for $\beta = 2, \gamma = \frac{1}{9}(1+\sqrt{19})$. When $\gamma = 1/2$, $\theta_\infty \approx 52^o$. Since $\theta_\infty = \pi/2$ when $\gamma = 1$, we restrict this parameter to the interval $(.5, 1)$.

To study the distribution of stretching along C with $\beta = 2$, we observe from (17) that

$$g = [2(1+\gamma)A\sigma]^{-\frac{\gamma}{1+\gamma}} + O(\sigma^{-\frac{2}{1+\gamma}}), \quad \sigma \to \infty. \tag{24}$$

Thus the total amount of stretching at time t for the Lagrangian point in the interval $(0, \zeta_0)$ is

$$S(\zeta_0) = \tau^{1-\gamma}\int_0^\sigma g^2(s)ds = \tau^{1-\gamma}\int_0^\sigma \left(g^2(s) - c\sigma^{-\frac{2\gamma}{1+\gamma}}\right)ds + \frac{1+\gamma}{1-\gamma}c\alpha^{\frac{1+\gamma}{1-\gamma}}(\zeta_0),$$

where $c = [2(1+\gamma)A]^{-\frac{\gamma}{1+\gamma}}$,

$$\sim \tau^{1-\gamma}\int_0^\sigma g^2(s)ds = \tau^{1-\gamma}\int_0^\infty \left(g^2(s) - c\sigma^{-\frac{2\gamma}{1+\gamma}}\right)ds + \frac{1+\gamma}{1-\gamma}c\alpha^{\frac{1+\gamma}{1-\gamma}}(\zeta_0) \tag{25}$$

for large σ. We thus see from (25) that between some time $t = t_0 < 0$ and $t = 0$ the total stretching of C is finite.

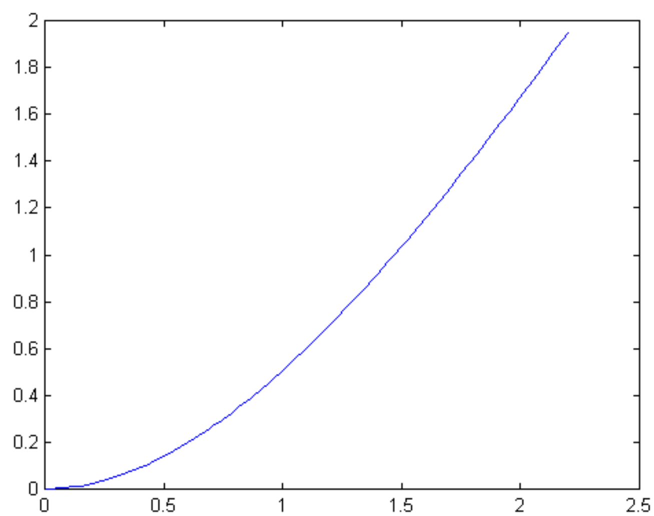

Figure 1. $yA^{-1}\tau^{\gamma-1}$ versus $xA^{-1}\tau^{\gamma-1}$ for the case $\beta = 2, \gamma = \frac{1}{9}(1+\sqrt{19})$.

To understand the movement of C uniformly in ζ_0 it is helpful to consider a specific initial-value problem. Consider the similarity form of C at some given time $\tau = T < 0$. We are free to specify that at $J(\zeta_0, T) = 1$. Since the particular value of γ in the interval $(.5, 1)$ is immaterial, we set $\gamma = 2/3, \mu = 5/3$. Then the last condition gives the parametric equations for $\alpha(\zeta_0)$ in the form

$$\frac{\sqrt{10/3}A}{T^{1/3}}\zeta_0 = G^{-1/2}\sqrt{1-G^{-3}} + 2\int_G^1 \sqrt{G^{-3}-1}, \tag{26}$$

$$\frac{\sqrt{10/3}A}{T^{5/3}} = G^{-5/2}\sqrt{1-G^3}, \tag{27}$$

with $0 < G < 1$. We show this function in figure 2.

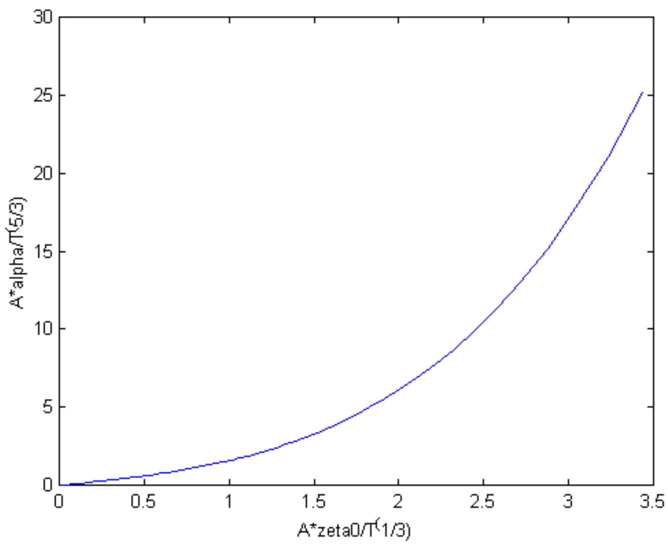

Figure 2. $AT^{-5/3}\alpha$ as a function of $AT^{-1/3}\zeta_0$, for the case $\beta = 2, \gamma = 2/3$.

To see how J varies with τ given this parametrization of C, we may use

$$\left(\frac{\tau}{T}\right)^{2/3}\sqrt{J} = \frac{g(\sigma)}{g(\sigma(\tau/T)^{5/2})}, \tag{28}$$

where $g(\sigma)$ is defined implicitly by $\frac{10}{3}A^2\sigma^2 g^5 + g^3 - 1) = 0$. From (28) we see that J tends to 1 as $\zeta_0 \to \infty$ for any $\tau < T$, but as τ decreases from T the stretching is concentrated toward the developing singularity.

Another important point concerns the ratio of the square root of the local curvature and the square root of the local Jacobian of C. In the construction of the next section, this ratio $\kappa/\sqrt{J}$, determines the ratio of the typical diameter of the cocoon divided by the local radius of curvature of the core vortex. We see that, as a function of τ for fixed σ,

$$\kappa/\sqrt{J} \sim O(\tau^{2\gamma-1}), \quad \tau \to 0. \tag{29}$$

Thus, since $\gamma > 1/2$, the singularity will be associated with a locally 2D cocoon.

Finally, a key issue in the construction of the next section is the setting up of a proper initial condition. It will appear that the lateral extent of vortical structures in the planes orthogonal to C will be inversely proportional to u. Thus the ratio

$$\rho = \kappa/u \tag{30}$$

will be a measure of the ratio of the vorticity extent to the cocoon diameter. It is essential that ρ be bounded as a function of ζ. (The actual value of the ration in the initial condition can be adjusted by multiplying ρ by a constant.) In figure 3 we show this function for the case $\beta = 2, \gamma = 2/3, A = 1$, with α chosen to make $J = 1$ at $\tau = T$.

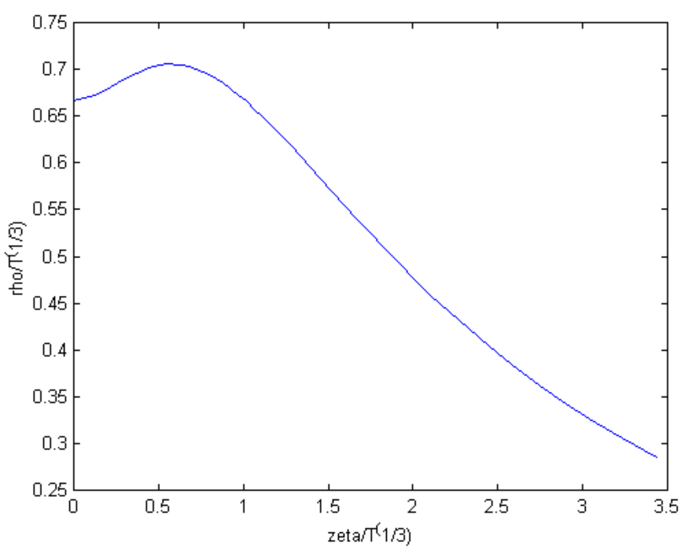

Figure . $\rho T^{-1/3}\alpha$ as a function of $\zeta T^{-1/3}\zeta_0$, for the case $\beta = 2, \gamma = 2/3, A = 1$, and with the condition $J = 1$ at $\tau = T$.

4 The dynamic cocoon

We now consider the geometry of the cocoon we shall build on the core vortex given by the curve $C(t)$ of the preceding section with $\beta = 2, .5 < \gamma < 1$. We place thus curve in the x, z plane. Let the coordinates of C in this plane be $(X(\zeta, t), Z(\zeta, t)$ where ζ remains the arc length along the curve. At each point of C, $(\mathbf{n}, \mathbf{b}, \mathbf{t})$ will form a right-handed triad of orthonormal vectors with corresponding coordinates ξ, η, ζ, with n pointing in the direction of positive ξ. (Here $\mathbf{b}$ is the binormal vector.) Both ξ and η are measured from C. We again refer to the Lagrangian coordinate ζ_0 with $\zeta(0, t) = 0$ and $\frac{\partial \zeta}{\partial \zeta_0} = J$. Each transverse plane $\zeta =$ constant will be referred to as a *section* of the cocoon.

We shall adopt a time-dependent orthogonal curvilinear coordinate system based upon the triad and coordinates defined above. If $\kappa(\zeta, t)$ again denotes the curvature of C, we have a metric $ds^2 = d\xi^2 + d\eta^2 + h^2 d\zeta^2$ where $h - 1\xi\kappa$. To insure $h > 0$ we shall restrict the flow field to the tube defined for all ζ by $\xi^2 + \eta^2 < \kappa^{-2}$. We refer to this tube as the *cocoon boundary* $\mathcal{B}$. We will be considering Euler flows within the region bounded by $\lfloor$. We thus refer to the structure as the *dynamic cocoon*.

The next step is to write down Euler's equations in this coordinate system. For any scalar function f we have

$$\frac{\partial f}{\partial t}\Big|_{\xi,\eta,\zeta_0} = \frac{\partial f}{\partial t}\Big|_{x,y,z} + \frac{\partial \mathbf{x}}{\partial t}\Big|_{\xi,\eta,\zeta_0} \cdot \nabla f. \tag{31}$$

We assume that the curve moves according to (1) but now set the velocity $u = U$ since we want to use u, v, w for fluid velocity in the present coordinate system. Thus $\frac{\partial \mathbf{x}}{\partial t}\Big|_{\xi,\eta,\zeta_0} = U\mathbf{n}$. Then

$$\frac{\partial f}{\partial t}\Big|_{x,y,z} = \frac{\partial f}{\partial t}\Big|_{\xi,\eta,\zeta_0} - U\frac{\partial f}{\partial \xi}. \tag{32}$$

Now

$$\frac{\partial f}{\partial t}\Big|_{\xi,\eta,\zeta_0} = \frac{\partial f}{\partial t}\Big|_{\xi,\eta,\zeta} + \frac{\partial \zeta}{\partial t}\Big|_{\xi,\eta,\zeta_0}\frac{\partial f}{\partial \zeta}. \tag{33}$$

We recall that

$$\frac{\partial \zeta}{\partial t}\Big|_{\xi,\eta,\zeta_0} = -\int JU\kappa d\zeta_0 = -\int U\kappa d\zeta \equiv W. \tag{34}$$

Then

$$\frac{\partial f}{\partial t}\Big|_{x,y,z} = \frac{\partial f}{\partial t}\Big|_{\xi,\eta,\zeta} - U\frac{\partial f}{\partial \xi} + W\frac{\partial f}{\partial \zeta}. \tag{35}$$

We also have

$$\frac{\partial(\mathbf{n},\mathbf{b},\mathbf{t})}{\partial t}\Big|_{x,y,z} = \frac{\partial(\mathbf{n},\mathbf{b},\mathbf{t})}{\partial t}\Big|_{\xi,\eta,\zeta_0} - U\frac{\partial(\mathbf{n},\mathbf{b},\mathbf{t})}{\partial \xi} = \frac{\partial(\mathbf{n},\mathbf{b},\mathbf{t})}{\partial t}\Big|_{\xi,\eta,\zeta_0}, \tag{36}$$

so that, since

$$\frac{\partial(\mathbf{n},\mathbf{b},\mathbf{t})}{\partial t}\Big|_{\xi,\eta,\zeta_0} = (-U_\zeta \mathbf{t}, 0, U_\zeta \mathbf{n}), \tag{37}$$

we have

$$\frac{\partial(u\mathbf{n}+v\mathbf{b}+w\mathbf{t})}{\partial t} = \mathbf{n}\mathcal{D}u + \mathbf{b}\mathcal{D}v + \mathbf{t}\mathcal{D}w, \tag{38}$$

where

$$\mathcal{D} = \frac{\partial}{\partial t}\Big|_{\xi,\eta,\zeta} - U\frac{\partial}{\partial \xi} + W\frac{\partial}{\partial \zeta}. \tag{39}$$

We also have the standard formulas

$$\begin{aligned}\mathbf{u}\cdot\nabla\mathbf{u} = &\Big[uu_\xi + vu_\eta + h^{-1}wu_\zeta - h^{-1}w^2h_\xi\Big]\mathbf{n}\\ &+\Big[uv_\xi + vv_\eta + h^{-1}wv_\zeta - h^{-1}w^2h_\eta\Big]\mathbf{b}\\ &+\Big[uw_\xi + vw_\eta + h^{-1}ww_\zeta + h^{-1}w(uh_\xi + vh_\eta)\Big]\mathbf{t}.\end{aligned} \tag{40}$$

The equation $\nabla\cdot\mathbf{u} = 0$ becomes

$$\frac{\partial hu}{\partial \xi} + \frac{\partial hv}{\partial \eta} + \frac{\partial w}{\partial \zeta} = 0 \tag{41}$$

Finally, the pressure term is $\nabla p = (p_\xi, p_\eta, h^{-1}p_\zeta)$. Taking the density of the fluid to be unity, and setting $u = U + u'$, the equations of motion become

$$u'_t + wU_\zeta + u'u'_\xi + vu'_\eta + h^{-1}wu'_\zeta + Wu'_\zeta - h^{-1}w^2h_\xi + p_\xi + U_t + WU_\zeta = 0, \tag{42}$$

$$v_t + u'v_\xi + vv_\eta + h^{-1}wv_\zeta + Wv_\zeta - h^{-1}w^2h_\eta + p_\eta = 0, \tag{43}$$

$$w_t - u'U_\zeta + u'w_\xi + vw_\eta + h^{-1}ww_\zeta + Ww_\zeta + h^{-1}w(uh_\xi + vh_\eta) + h^{-1}p_\zeta - UU_\zeta = 0, \tag{44}$$

$$\frac{\partial hu'}{\partial \xi} + \frac{\partial hv}{\partial \eta} + \frac{\partial w}{\partial \zeta} + Uh_\xi = 0 \tag{45}$$

We now relegate terms containing w, or containing derivatives with respect to ζ, to secondary status, and regard h as close to 1 except in the term Uh_ξ of (45). We then rewrite (42),(43), and (45) in the form

$$u'_t + u'u'_\xi + vu'_\eta + p_\xi + U_t = F, \tag{46}$$

$$v_t + u'v_\xi + vv_\eta + p_\eta = G, \tag{47}$$

$$\frac{\partial u'}{\partial \xi} + \frac{\partial v}{\partial \eta} - U\kappa = H. \tag{48}$$

We term the two-dimensional system obtained by setting $F = G = H = 0$ the *reduced system*. The reduced equations thus describe 2D Euler flow relative to a moving coordinate system and a dilating metric.

4.1 Batchelor's vortex

We now consider the a solution of the reduced system which will play a key role in the construction of the next section. We note that here ζ be only a parameter. We first carry out a standard elimination of the term $U\kappa$. The ζ-vorticity in the reduced system is $\omega = v_\xi - u'_\eta$. The equation satisfied by ω will be written in the polar coordinates (ρ, ϕ) of the ξ, η plane, with $(u', v) = (v_\rho, v_\phi)$. Set $v_\rho = \frac{1}{2}\rho U\kappa + v'_\rho$. Thus ω satisfies

$$\frac{\partial \omega}{\partial t} + v'_\rho \frac{\partial \omega}{\partial \rho} + \frac{1}{\rho} v_\phi \frac{\partial \omega}{\partial \phi} + \frac{1}{2} U\kappa\rho \frac{\partial \omega}{\partial \rho} + U\kappa\omega = 0. \tag{49}$$

Now set

$$\omega(\rho, \phi, t) = e^{-q(t)} \tilde{\omega}(\tilde{\rho}, \phi, t), \tilde{\rho} = e^{-q/2}\rho, \tag{50}$$

where

$$q(t) = \int^t U(\zeta, t)\kappa(\zeta, t)dt. \tag{51}$$

Also set

$$\tilde{v}'_\rho = e^{q/2} v'_\rho, \tilde{v}_\phi = e^{q/2} v_\phi. \tag{52}$$

Then

$$\frac{\partial \tilde{\omega}}{\partial t} + \tilde{v}'_\rho \frac{\partial \tilde{\omega}}{\tilde{\rho}} + \frac{1}{\tilde{\rho}} \tilde{v}_\phi \frac{\partial \tilde{\omega}}{\partial \phi} = 0. \tag{53}$$

We thus have a purely 2D problem to consider.

We shall introduce the solution of (53) to be used in the next section. We consider *steady* flow in the region $0 \le \tilde{\rho} < e^{-q/2}\kappa^{-1} \equiv \tilde{R}$. The solution has the form

$$\tilde{v}'_\rho = \frac{1}{\tilde{\rho}} \frac{\partial \psi}{\partial \phi}, \tilde{v}_\phi = -\frac{\partial \psi}{\partial \tilde{\rho}}, \tag{54}$$

with, for some $\tilde{a}$, $0 < \tilde{a} < \tilde{R}$

$$\tilde{\nabla}^2 \psi = \begin{cases} -k^2\psi, & \text{if } 0 \le \tilde{\rho} < \tilde{a}, \\ 0, & \text{if } \tilde{a} < \tilde{\rho} < \tilde{R}. \end{cases} \tag{55}$$

We require

$$\tilde{v}'_\rho = 0, \tilde{\rho} = \tilde{a} \tag{56}$$

and

$$\tilde{v}'_\rho = U\cos\phi, \tilde{\rho} = \tilde{R}. \tag{57}$$

The solution is of the form

$$\psi = f(\tilde{\rho})\sin\phi, f(\tilde{a}) = 0, \tilde{R}^{-1} f(\tilde{R}) = U, \tag{58}$$

with f and $f_{\tilde{\rho}}$ continuous on $\tilde{\rho} = a$. The solution is elementary and given by

$$f = \begin{cases} KJ_1(k\tilde{\rho}), & \text{if } \tilde{\rho} < 1, \\ \frac{U}{1 - a^2/\tilde{R}^2}(\tilde{\rho} - \tilde{a}^2/\tilde{\rho}), & \text{if } \tilde{a} < \tilde{\rho} < \tilde{R}^{-1}. \end{cases} \tag{59}$$

Here the velocity U is given by

$$U = \frac{1}{2} k K J_0(\tilde{a}k)(1 - \tilde{a}^2/\tilde{R}^2). \tag{60}$$

We may take $k\tilde{a} = 3.83$ to obtain a single vortical eddy in each half-circle.

Seen by a stationary observer (recall we are in a moving frame), this solution represents locally a circular vortex paidmoving in the **n** direction with velocity U, the cocoon boundary acting like a rigid circular wall moving with velocity U in the ξ-direction. However, we are free to make a as small as we like while maintaining $ka = 3.83$, simultaneously adjusting K to get the desired value of U.

We refer to this solution as Batchelor's vortex the case $\kappa = 0$ is highlighted in his textbook [7]. It is simply a 2D version of Hill's spherical vortex, see [8].

5 Evolution of the dynamic cocoon

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We point out that the vorticity in this solution is continuous but not continuously differentiable. We will discuss elsewhere how to deal with this issue of regularity. Suffice it to say here that since the problem of Euler blow-up involves necessarily unbounded vorticity, lack of continuous differentiability in the initial condition is not a pressing concern, and can in deed be removed by a local smoothing.

The basic idea is to set up at $\tau = T < 0$ an initial condition using a family of Batchelor vortices $\mathcal{F}(\zeta, 0)$ as an analytic structure in ζ. The variations of K, a are chosen to match the velocity and to make $\max_\zeta a\kappa$ as small as we like everywhere on the core vortex curve $C(T)$. Note that this is possible from the boundedness of κ/u noted earlier.[1]

What needs to be shown now is that the essentially two-dimensional structure introduced into the initial condition is maintained as $\tau \to 0$ from the initial time T.

To examine this question we need to estimate the size of the forcing terms F, G, H in (42),(43),(45). The crucial issue here is the size of terms as $\tau \to 0$ holding σ fixed. We have already seen that if $\gamma > 1/2$ the stretching is such as to further slenderize structure, which in the context of the dynamic cocoon can be sated as further reducing $a\kappa$. Similarly, we can concluded that only small errors are made by replacing h by 1 in the forcing terms. Since the secondary terms are rendered negligible by slenderness, we have immediately the validity of the reduced system, subject only to one caveat. Since the w equation has been uncoupled from the 2D flow, we need to verify that the w flow induced by variations with ζ do not adversely affect the 2D structure. The pressure that are developed in the cocoon are of the order of $(u')^2 + v^2$. because these velocities scale inversely with the transverse dimension of the vorticity (i.e. like a^{-1} for

[1]We may now identify the core vortex as the locus of centers of Batchelor's vortices. It happens that the vorticity on the core vortex is in fact zero.

Batchelor's vortex), it follows from (44) w will be or the order of the slenderness times the transverse components, and therefore negligible.

6 Construction of the singular flow

We have indicated the main steps but recapitulate the process here.

1. Choose an initial time $\tau = T < 0$. The singular curve C is constructed at that time for some chosen $\gamma \in (.5, 1)$ and assumes the role of the initial core vortex for the dynamical cocoon. Note that T can be taken as small as desired.

2. Parametrize the initial core vortex, e.g. by setting the initial Jacobian $J = 1$ everywhere.

3. Construct the initial cocoon from the Batchelor's vortex in each transverse plane, insuring that $U(\zeta, T)$ is that of $C(T)$ and that $a \ll \kappa$ in all sections. Except for the lack of continuity of first derivatives at the boundary of the support of the vorticity, the flow can be made arbitrarily smooth. The initial condition can be thought of as an isovortical map of a uniform Batchelor vortex in 2D, extending indefinitely in the ignorable dimension,

4. Let the structure evolve as a solution of the 3D Euler equations up to $\tau = 0$, at which time the section $\zeta = 0$ of the cocoon shrinks to zero diameter and the vorticity blows up. (Note that the $a\kappa$ tends to zero also at this point.)

5. At time $\tau = 0$ define the vorticity field within the cocoon as the *limit vorticity*.

6. To see what initial vortex structure that would evolve to the limit cocoon, reverse the sign of the limit vorticity and use this as the initial condition for Euler's equation *in* R^3. We stress that the almost trivial statement that you can find an Euler singular flow by starting with a singularity and working backwards finesses the fact that to select a suitable starting singularity you have to know something about its evolution slightly before the blow-up. The role of the cocoon is make this possible by relating the 3D flow to a 1D singular system whose time evolution can be thoroughly examined. The only difficulty in visualizing this reversal is at the blow-up point. But there we have already obtained an essentially exact solution in R^3. To see this, consider a contracting ball with center at the blow-up point and radius τ^{γ^*}, where $1-\gamma < \gamma^* < \gamma$. To the values of a within the ball the boundary of the ball is "at infinity" and to the cocoon boundary the ball is tiny as the breakdown is approached. The flow there must agree therefore with an Euler flow vanishing at infinity.

7. The structure obtained under flow reversal at some finite value of time is the desired initial condition for Euler's equations in 3D leading to blow-up in finite time.

This construction shows why it is essentially impossible to write down an initial condition which produces a clear singularity by numerical computation. The process of producing the limit vorticity involves small deformations of the Batchelor vortices at sections removed from the blow-up point. Proper choice of the initial condition in step 3 has insured that the vorticity stays within the cocoon up to and including the blow-up time, *but we do not know exactly what* *the vorticity is at the blow-up time, at points not close to the blow-up point.* When reversed, and run over a finite time in R^3, the small deformations in the limit vorticity lead to a core vortex which is very different from C.

7 Concluding remarks

1. The present construction has precursors in many of the numerical and analytic studies of Kerr, Pelz,, Pumir, and Siggia, see e.g. [4], [3] [9], [5], and the discussion in [1]. The difficulty with studying paired vortex structures numerically has always been the core deformation in the late stages of breakdown. This difficulty necessitates some way of tailoring the initial condition to conform to a pre-existing known singular structure. The use of a $R^1 \times R^2$ product structure relegates this singular structure to one dimension. A similar point of view was taken in [2].

2. Various constraints are known leading to absence of blow-up, besides the essential Beale-Majda-Kato condition [10],[11]. As yet we have not checked the present construction against these constraints. In a recent paper [12], Chae has proved that suitable Euler solutions of self-similar form $\mathbf{u} = \tau^{-\alpha}\mathbf{F}(\tau^{-\beta}), \alpha + \beta = 1$ cannot blow up in finite time. We emphasize that the present product geometry is not self-similar in this way, provided that $\gamma > 1/2$, but *is* self-similar if $\gamma = 1/2$, referring here to the neighborhood of the blow-up point. Recall that the condition $\gamma > 1/2$ insures that the cocoon geometry is consistent. Note also that our solution is consistent with Beale-Majda-Kato for $\gamma \geq 1/2$.

3. It is interesting that the self-similar choice $\alpha = \beta = 1/2$ remark 2 is the only one consistent with the Navier-Stokes equations. In the present construction, of $\gamma > 1/2$ the effect of viscosity is catastrophic in diffusing away the vorticity of the cocoon. There can be no solution of the Navier-Stokes equation that is in any way "close" to the present Euler solution for sufficiently small viscosity. Such an approximating viscous flow is possible in principle when $\alpha = \beta = 1/2$, but this possibility is expelled by Chae's result.

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