1. Consider the ODE $zw'' + (2r+1)w' + zw = 0$. Consider solutions represented by a contour integral $w = \int_C e^{z\sigma} v(\sigma) d\sigma$. Show that if C is as shown in figure 1 (beginning an ending at $+\infty$ along the Re(σ)-axis), and Re(z)¡0, then a solution is obtained if

$$v(\sigma) = A(1+\sigma^2)^{r-1/2},$$

for any complex constant A.

2. The hypergeometric equation $zw'' + (a-z)w' - bw = 0$ has solutions given by a contour representation on C, $w = \int_C e^{z\sigma} v(\sigma) d\sigma$. Show that one solution is given by

$$w = A\int_0^1 e^{z\sigma}\sigma^{b-1}(1-\sigma)^{a-b+1} d\sigma,$$

where Re(b) > 0 and Re($a-b$) > 0, and A is a constant.

3. Using the method of steepest descent, show that

$$\int_0^\infty e^{ik(\zeta^4/4+\zeta^3/3)} e^{-\zeta} d\zeta \sim \frac{e^{i\pi/6}\Gamma(1/3)}{3^{2/3}k^{1/3}},\ k \to \infty.$$

What is the order of the next term of the series?

4. Consider

$$I(k) = \int_0^{\pi/4} e^{ik\zeta^2} \tan\zeta d\zeta.$$

(a) Show that the steepest descent paths through $\zeta = \xi + i\eta = 0$ are given by $\xi = \pm\eta$, and that the steepest descent paths through $\pi/4$ are given by $\xi = \pm\sqrt{(\pi/4)^2 + \eta^2}$.

(b) Show, by considering paths to ∞ from both 0 and $\pi/4$, that I(k) has the asymptotic expansion

$$I(k) \sim \frac{i}{2k} - \frac{2i}{k\pi} e^{ik(\pi/4)^2} + o(1/k),\ k \to \infty$$

5. By evaluating the integral along two paths (along the $\xi = 0$ from 0 to $+\infty$ and from $+\infty$ to 1 along the line $\xi = 1$), show that

$$\int_0^1 log(z) e^{ikz} dz \sim \frac{-i\ln k}{k} - \frac{1}{k}(i\gamma + \frac{\pi}{2}) + ie^{ik}\sum_{n=1}^{\infty}\frac{(-1)^n(n-1)!}{k^{n+1}}.$$

Here $\gamma = -\int_0^\infty \ln t e^{-t} dt$ is Euler's constant $\approx .577216$.