Examples from Lecure 12: Integration along a branch cut.

1.

$$I = \int_0^\infty \frac{dx}{\sqrt{x}(1+x^2)}$$

Here we can chose a contour either of type A or B. I will work out using A, and you should try it using B. Contours here are all positively oriented, and the subcontours $C+1...C_4$ are numbered as shown. The small cicular arcs have radius ϵ, the large ones radius R.

Now by the residue theorem, with $f(z) = (\sqrt{z}(1+z^2))^{-1}$,

$$\int_{C_1+C_2+C_3+C_4} f(z) = 2\pi i Res_{z=i} f(z)dz = \pi e^{-i\pi/4} = \pi(1-i)/\sqrt{2}.$$

Also

$$\lim_{R\to\infty}\lim_{\epsilon\to 0}\int_{C_1} f(z)dz = I$$

$$\lim_{R\to\infty}\lim_{\epsilon\to 0}\int_{-C_3} f(z)dz = e^{i\pi/2} I$$

$$|\int_{C_2} f(z)dz| \le \pi\frac{R}{\sqrt{R}(R^2-1)} \to 0, R\to\infty.$$

$$|\int_{C_4} f(z)dz| \le \pi\frac{\epsilon}{\sqrt{\epsilon}(1-\epsilon^2)} \to 0, \epsilon\to\infty.$$

Thus, taking the limits of all terms and noticing that we calculated the intgegral over $-C_3$, not C_3, we have

$$(1-i)I = \pi(1-i)/\sqrt{2}$$

or $I = \pi/\sqrt{2}$.

2.

$$I = \int_0^\infty \frac{(\ln x)^2}{1+x^2}dx.$$

Then $f(z) = (\log z)^2/(1+z^2)$ with the branch taken as $\log z = \ln r + i\theta, 0 < \theta < 2\pi$. We choose contour A. You should show that contour B will not work because the integral contributions we are trying to evaluate actually cancel out in the limit if that contour is use.

So with A we have again

$$\lim_{R\to\infty}\lim_{\epsilon\to 0}\int_{C_1} f(z)dz = I$$

$$\lim_{R\to\infty}\lim_{\epsilon\to 0}\int_{-C_3} f(z)dz = -e^{i\pi}\int_0^\infty \frac{(\ln r + i\pi)^2}{1+r^2}dr$$

Also

$$\int_{C_1+C_2+C_3+C_4} f(z) = 2\pi i Res_{z=i} f(z)dz = 2\pi i[(\ln i)^2/(2i)] = -\pi^2/4.$$

$$|\int_{C_2} f(z)dz| \le \pi\frac{R(\ln R)^2}{(R^2-1)} \to 0, R\to\infty.$$

$$|\int_{C_4} f(z)dz| \le \pi\frac{\epsilon[(\ln\epsilon)^2+\pi^2]}{(1-\epsilon^2)} \to 0, \epsilon\to\infty.$$

Finally, we note that

$$\int_0^\infty \frac{dx}{1+x^2} = \pi/2$$

using the antiderivative $\tan^{-1} x$. Thus we have

$$2I + 2\pi i \int_0^\infty \frac{\ln r dr}{1+r^2} - \pi^3/2 = -\pi^3/4,$$

showing that $I = \pi^3/8$ and

$$\int_0^\infty \frac{\ln r dr}{1+r^2} = 0.$$