The Semiclassical Limit of the Defocusing NLS Hierarchy

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Abstract

We establish the semiclassical limit of the one-dimensional defocusing cubic nonlinear Schrödinger (NLS) equation. Complete integrability is exploited to obtain a global characterization of the weak limits of the entire NLS hierarchy of conserved densities as the field evolves from reflectionless initial data under all the associated commuting flows. Consequently, this also establishes the zero-dispersion limit of the modified Korteweg–de Vries equation that resides in that hierarchy. We have adapted and clarified the strategy introduced by Lax and Levermore to study the zero-dispersion limit of the Korteweg–de Vries equation, expanding it to treat entire integrable hierarchies and strengthening the limits obtained. A crucial role is played by the convexity of the underlying log-determinant with respect to the times associated with the commuting flows.

1 Introduction

1.1 The NLS Equation

The nonlinear Schrödinger (NLS) equation is one of the simplest nonlinear wave equations,

$$i\hbar\partial_t\Psi + \frac{\hbar^2}{2}\Delta\Psi - U'\big(|\Psi|^2\big)\Psi = 0\,. \tag{1.1}$$

Here $\Psi(x,t)$ is a complex-valued field over a spatial domain $\Omega \subset \mathbb{R}^D$, U' is the first derivative of a twice-differentiable, nonlinear, real-valued function over $\mathbb{R}_+$, and $\hbar$ is a positive parameter. This is just the usual Schrödinger equation of quantum mechanics with the potential $V(x)$ replaced by $U'(|\Psi|^2)$. The parameter $\hbar$ is analogous to Planck's constant, which is usually very small in the quantum setting when evaluated in the natural dimensional scales of the equation as determined by its initial and boundary conditions. For the moment, the precise specification of the domain Ω and the nature of the boundary conditions is left vague in order to make some general statements regarding the structure of (1.1). It will be assumed that they are consistent with all formal calculations.

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That the nonlinear function $U\colon \mathbb{R}_+ \to \mathbb{R}$ is the potential energy density of the field is clearly seen when the NLS equation (1.1) is recast as a Hamiltonian system in the form

$$i\hbar\partial_t\Psi = \frac{\delta H}{\delta\overline{\Psi}}\,, \qquad H = \int_\Omega \frac{\hbar^2}{2}\,|\nabla\Psi|^2 + U\big(|\Psi|^2\big)\,d\overset{D}{x}\,. \tag{1.2}$$

The associated Poisson bracket of any two functionals F and G is given by

$$\{F,G\} \equiv \frac{1}{i\hbar}\int_\Omega \left(\frac{\delta F}{\delta\Psi}\frac{\delta G}{\delta\overline{\Psi}} - \frac{\delta F}{\delta\overline{\Psi}}\frac{\delta G}{\delta\Psi}\right) d\overset{D}{x}\,; \tag{1.3}$$

the evolution of any functional F under the NLS flow (1.2) is then

$$\frac{dF}{dt} = \{F,H\}\,. \tag{1.4}$$

This Hamiltonian structure plays a major role in our subsequent analysis.

Also associated with the NLS equation (1.1) are $D+2$ local conservation laws corresponding to mass, momentum, and energy conservation. Their densities, ρ, μ, and ε, respectively, are given by

(1.5)
$$\rho = |\Psi|^2\,, \qquad \mu = -i\frac{\hbar}{2}\big(\overline{\Psi}\nabla\Psi - \Psi\nabla\overline{\Psi}\big)\,, \qquad \varepsilon = \frac{\hbar^2}{2}|\nabla\Psi|^2 + U\big(|\Psi|^2\big).$$

The mass and momentum densities determine the field Ψ up to a constant phase; the energy density can be written in terms of them as

$$\varepsilon = \frac{1}{2}\frac{|\mu|^2}{\rho} + \frac{\hbar^2}{8}\frac{|\nabla\rho|^2}{\rho} + U(\rho)\,. \tag{1.6}$$

The local conservation laws are then

$$\begin{aligned} &\partial_t\rho + \nabla\cdot\mu = 0\,,\\ &\partial_t\mu + \nabla\cdot\left(\frac{\mu\otimes\mu}{\rho}\right) + \nabla P(\rho) = \frac{\hbar^2}{4}\nabla\cdot\Big[\rho\nabla^2\log\rho\Big]\,,\\ &\partial_t\,\varepsilon + \nabla\cdot\left(\frac{\mu}{\rho}\big(\varepsilon + P(\rho)\big)\right) = \frac{\hbar^2}{4}\nabla\cdot\left[\frac{\mu\Delta\rho}{\rho} - \frac{\nabla\cdot\mu\nabla\rho}{\rho}\right], \end{aligned} \tag{1.7}$$

where $P(\rho) \equiv \rho U'(\rho) - U(\rho)$. The first two of these comprise a closed system governing ρ and μ that has the form of a perturbation of the compressible Euler equations of fluid dynamics with the "pressure" given by $P(\rho)$. If the "Euler part" of these equations is to be hyperbolic, then the pressure $P(\rho)$ must be a strictly increasing function of ρ; in that case, $P'(\rho) = \rho U''(\rho) > 0$. This means that U must be a strictly convex function of ρ and corresponds to a "defocusing" NLS equation. In this context a "focusing" NLS equation can be understood as a fluid whose pressure *decreases* when the mass density increases, a phenomenon leading to the development of mass concentrations and, in some cases, finite-time blowup.

1.2 Posing the Semiclassical Limit

The "semiclassical limit" of the NLS equation can be described as follows: Consider the family, parametrized by $\hbar > 0$, of solutions $\Psi^\hbar(x,t)$ to the Cauchy problems

$$i\hbar\partial_t\Psi^\hbar + \frac{\hbar^2}{2}\Delta\Psi^\hbar - U'\big(|\Psi^\hbar|^2\big)\Psi^\hbar = 0\,, \tag{1.8a}$$

$$\Psi^\hbar(x,0) = \Psi^\hbar(x) \equiv A(x)\exp\left(\frac{i}{\hbar}S(x)\right), \tag{1.8b}$$

where the (nonnegative) amplitude $A(x)$ and (real) phase $S(x)$ are assumed to be smooth and independent of $\hbar$. The initial conserved densities are then

$$\rho^\hbar(x,0) = |A(x)|^2\,, \qquad \mu^\hbar(x,0) = |A(x)|^2\nabla S(x)\,, \tag{1.9a}$$

$$\varepsilon^\hbar(x,0) = \frac{1}{2}|A(x)|^2\,|\nabla S(x)|^2 + \frac{\hbar^2}{2}|\nabla A(x)|^2 + U\big(|A(x)|^2\big). \tag{1.9b}$$

The general problem of the semiclassical limit is to determine the limiting behavior of any function of the field $\Psi^\hbar$ as $\hbar \to 0$, in particular, to ascertain the existence (in some sense) of the limits of the conserved densities

$$\rho = \lim_{\hbar\to 0}\rho^\hbar\,, \qquad \mu = \lim_{\hbar\to 0}\mu^\hbar\,, \qquad \varepsilon = \lim_{\hbar\to 0}\varepsilon^\hbar\,,$$

and, if the limits exist, to determine their dynamics.

If we argue formally, it is natural to conjecture for the defocusing case that the $\mathcal{O}(\hbar^2)$ dispersive terms appearing in (1.7) are negligible as $\hbar \to 0$ and that the limiting densities ρ and μ satisfy the hyperbolic system (the Euler system)

$$\begin{aligned} &\partial_t\rho + \nabla\cdot\mu = 0\,,\\ &\partial_t\mu + \nabla\cdot\left(\frac{\mu\otimes\mu}{\rho}\right) + \nabla P(\rho) = 0\,, \end{aligned} \tag{1.10a}$$

with initial conditions inferred from (1.9a) given by

$$\rho(x,0) = |A(x)|^2\,, \qquad \mu(x,0) = |A(x)|^2\nabla S(x)\,. \tag{1.10b}$$

This argument is self-consistent only so long as the solution of the Euler system (1.10) remains classical. In that case the limiting energy density will be given by

$$\varepsilon = \frac{1}{2}\frac{|\mu|^2}{\rho} + U(\rho)\,, \tag{1.11}$$

and will satisfy

$$\partial_t\,\varepsilon + \nabla\cdot\left(\frac{\mu}{\rho}\big(\varepsilon + P(\rho)\big)\right) = 0\,, \tag{1.12}$$

hence playing the role of a Lax entropy for the Euler system (1.10a). In [24] we stated that a fairly general proof of the above conjecture could be carried out in any setting for which the local well-posedness of classical solutions of the Euler system (1.10) is known. Such a proof has recently been given by Grenier [19]. In

Section 5 we relate this result to a global-in-time theorem about a more restricted problem, that of the cubic NLS equation in one spatial dimension.

The genuinely nonlinear nature of the Euler system (1.10) will ensure that its classical solution will develop singular behavior (an infinite derivative) for all but rarefaction initial data. At the instant such a breaking occurs, the formally small dispersive terms on the right side of (1.7) will no longer be negligible, and the above characterization of the semiclassical limit will break down. Since this small regularizing term is dispersive, one expects that the impending singularity in ρ and μ will be regularized by the development of small wavelength oscillations. Therefore, after this breaktime the conserved densities can be expected to have weak limits at best [1, 24].

1.3 The 1-D Cubic Schrödinger Equation

The specific problem we study in this paper is that of the defocusing 1-D cubic Schrödinger equation given by

$$i\hbar\partial_t\Psi + \frac{\hbar^2}{2}\partial_{xx}\Psi + \left(1 - |\Psi|^2\right)\Psi = 0\,, \tag{1.13a}$$

with the far-field boundary conditions

$$\Psi(x,t) \sim \exp\left(\pm\frac{i}{\hbar}S_\infty\right) \quad \text{as } x \to \pm\infty \tag{1.13b}$$

for some $S_\infty \in \mathbb{R}$, and the initial condition

$$\Psi(x,0) = \Psi^\hbar(x) \equiv A(x)\exp\left(\frac{i}{\hbar}S(x)\right) \tag{1.13c}$$

for some smooth $A(x)$ and $S(x)$ that are independent of $\hbar$ and consistent with the far-field boundary conditions (1.13b).

For this one-dimensional cubic case, the Euler system (1.10) that describes the formal semiclassical limit reduces to

$$\begin{aligned} \partial_t\rho + \partial_x\mu &= 0\,, \\ \partial_t\mu + \partial_x\left(\frac{\mu^2}{\rho} + \frac{\rho^2}{2}\right) &= 0\,, \end{aligned} \tag{1.14a}$$

with the initial conditions

$$\rho(x,0) = A^2(x)\,, \qquad \mu(x,0) = A^2(x)\partial_x S(x)\,. \tag{1.14b}$$

The Riemann invariants for this system are given by

$$\hat\lambda_\pm = \frac{\mu}{2\rho} \pm \sqrt{\rho}\,, \tag{1.15}$$

and the system can be placed in the Riemann invariant form

(1.16a)
$$\partial_t\hat\lambda_+ + \tfrac{1}{2}(3\hat\lambda_+ + \hat\lambda_-)\partial_x\hat\lambda_+ = 0\,, \qquad \partial_t\hat\lambda_- + \tfrac{1}{2}(\hat\lambda_+ + 3\hat\lambda_-)\partial_x\hat\lambda_- = 0\,,$$

with the initial conditions

(1.16b) $$\hat{\lambda}_{\pm}(x,0) = r_{\pm}(x) \equiv \tfrac{1}{2}\partial_x S(x) \pm A(x)\,.$$

The normalization of the Riemann invariants (1.15) that has been chosen is the most natural for our study, which is built upon the integrable structure described below.

Zakharov and Shabat [45] have shown that the initial/boundary-value problem (1.13) is completely integrable using the inverse scattering transform associated with the self-adjoint Dirac operator

(1.17) $$\mathcal{L} = \begin{pmatrix} i\hbar\partial_x & \overline{\Psi} \\ \Psi & -i\hbar\,\partial_x \end{pmatrix}.$$

The solution strategy centers on the eigenvalue problem

(1.18) $$\mathcal{L}f = \lambda f \quad \text{where } f = \begin{pmatrix} f^{(1)} \\ f^{(2)} \end{pmatrix}.$$

Given $\Psi = \Psi(x,0)$, the asymptotics of the eigenfunctions $f(\lambda,x,0)$ as $|x| \to \infty$, referred to as the scattering data, can be calculated in principle. The evolution of the scattering data is then determined and the $\Psi(x,t)$ is then obtained from the knowledge of the large $|x|$ asymptotics of $f(\lambda,x,t)$ using the inverse scattering theory.

More specifically, if the initial data satisfy decay conditions that $A(x) \to 1$ and $\partial_x S(x) \to 0$ sufficiently rapidly as $x \to \pm\infty$ (say, faster than any power of x), then the L^2 spectrum of $\mathcal{L}$ consists of the two semi-infinite intervals $(\infty,-1]$ and $[1,\infty)$ comprising the continuous spectrum, along with a finite set of N simple eigenvalues $\lambda_1,\dots,\lambda_N$ in the interval $(-1,1)$. The asymptotic behavior of an eigenfunction $f(\lambda,x)$ corresponding to a λ in the continuous spectrum is given by

(1.19)
$$f(\lambda,x) \sim \begin{cases} T(k)\overline{E}_-(k)\exp\left(\dfrac{-ikx}{\hbar}\right) & \text{for } x \to -\infty, \\ E_-(k)\exp\left(\dfrac{-ikx}{\hbar}\right) + R(k)E_+(k)\exp\left(\dfrac{ikx}{\hbar}\right) & \text{for } x \to +\infty, \end{cases}$$

where the vectors $E_{\pm}(k)$ are defined by

(1.20) $$E_{\pm}(k) \equiv \begin{pmatrix} \exp\left(\dfrac{-iS_\infty}{2\hbar}\right)(\lambda+k)^{\mp\frac{1}{2}} \\ \exp\left(\dfrac{iS_\infty}{2\hbar}\right)(\lambda+k)^{\pm\frac{1}{2}} \end{pmatrix},$$

and $k = \sqrt{\lambda^2-1}$. The complex-valued functions $R(k)$ and $T(k)$ are referred to as the reflection and transmission coefficients, respectively [45]. For $|\lambda| > 1$, two independent generalized eigenfunctions are then

$$\begin{pmatrix} f^{(1)} \\ f^{(2)} \end{pmatrix}, \qquad \begin{pmatrix} \bar{f}^{(2)} \\ \bar{f}^{(1)} \end{pmatrix}.$$

On the other hand, the asymptotic behavior of one eigenfunction $f = f_j(x)$ corresponding to a discrete eigenvalue $\lambda_j \in (-1, 1)$ is given for $x \to +\infty$ by

$$f_j(x) \sim \begin{pmatrix} \exp\left(\frac{-iS_\infty}{2\hbar}\right)(\lambda_j - i\kappa_j)^{\frac{1}{2}} \\ \exp\left(\frac{iS_\infty}{2\hbar}\right)(\lambda_j + i\kappa_j)^{\frac{1}{2}} \end{pmatrix} \exp\left(\frac{-\kappa_j x + \chi_j}{\hbar}\right), \tag{1.21}$$

where $\kappa_j = \sqrt{1 - \lambda_j^2}$ and the so-called norming exponents χ_j, which are real-valued, are determined by the normalization

$$\int_{-\infty}^{\infty} \left| f_j^{(1)} f_j^{(2)} \right| dx = 1\,. \tag{1.22}$$

Because the eigenfunction (1.21) satisfies the symmetry relation $f_j^{(1)} = \bar{f}_j^{(2)}$, the absolute values in (1.22) are redundant.

The inverse theory prescribes that the fundamental scattering data $\mathcal{S}$ consists of the reflection coefficient $R(\lambda)$, the eigenvalues λ_j, and the norming exponents χ_j:

$$\mathcal{S} \equiv \{R(\lambda),\, \lambda_j,\, \chi_j : |\lambda| \geq 1,\, j = 1, \ldots, N\}\,. \tag{1.23}$$

The transmission coefficient $T(\lambda)$, as well as all other asymptotic information, can be computed in terms of this fundamental set.

As Ψ evolves according to the NLS equation (1.13), the eigenvalues λ_j remain unchanged while the time dependence of the other scattering data is

$$\chi_j(t) = \chi_j(0) + \kappa_j \lambda_j t\,, \qquad R(\lambda, t) = R(\lambda, 0) \exp\left(\frac{-i2k\lambda t}{\hbar}\right). \tag{1.24}$$

Hence, given $R(\lambda, 0)$, λ_j, and $\chi_j(0)$ computed from the initial data $\Psi(x, 0)$, the solution $\Psi(x, t)$ of the NLS equation (1.13) is then reconstructed via inverse theory from $\mathcal{S}(t)$, the fundamental scattering data

$$\mathcal{S}(t) = \{R(\lambda, t),\, \lambda_j,\, \chi_j(t) : |\lambda| \geq 1,\, j = 1, \ldots, N\}\,, \tag{1.25}$$

as determined by (1.24).

1.4 The 1-D Cubic Schrödinger Hierarchy

The complete integrability of the cubic Schrödinger equation implies the existence of an infinite family of independent, conserved quantities [13],

$$H_m = \int_{-\infty}^{\infty} \rho_m \, dx \quad \text{for } m = 0, 1, 2, \ldots, \tag{1.26}$$

which are in involution with respect to the Poisson bracket (1.3),

$$0 = \{H_m, H_n\} = \frac{1}{i\hbar} \int_{-\infty}^{\infty} \left(\frac{\delta H_m}{\delta \Psi} \frac{\delta H_n}{\delta \overline{\Psi}} - \frac{\delta H_m}{\delta \overline{\Psi}} \frac{\delta H_n}{\delta \Psi} \right) dx\,. \tag{1.27}$$

The first three of these quantities correspond to the general conserved densities mentioned earlier (1.5) and are given by

$$
\begin{aligned}
H_0 &= \int_{-\infty}^{\infty} \rho_0 \, dx = \int_{-\infty}^{\infty} \left(|\Psi|^2 - 1\right) dx \,, \\
H_1 &= \int_{-\infty}^{\infty} \rho_1 \, dx = -i\frac{\hbar}{2} \int_{-\infty}^{\infty} \left(\overline{\Psi}\partial_x \Psi - \Psi \partial_x \overline{\Psi}\right) dx \,, \\
H_2 &= \int_{-\infty}^{\infty} \rho_2 \, dx = \int_{-\infty}^{\infty} \frac{\hbar^2}{2}|\partial_x \Psi|^2 + \frac{1}{2}\left(|\Psi|^2 - 1\right)^2 dx \,.
\end{aligned}
\tag{1.28}
$$

Henceforth, the problem of the semiclassical limit is understood as the evaluation of the limiting behavior of all the conserved densities,

$$
\rho_m = \lim_{\hbar \to 0} \rho_m^{\hbar} \,. \tag{1.29}
$$

Other limits can then be determined from these.

All of the H_m except H_0 are Hamiltonians that generate flows which commute with the cubic NLS flow (1.13a) and leave the boundary condition (1.13b) invariant, the so-called NLS hierarchy. Letting t_m denote the time variable associated with the m^{th} flow, its evolution is then given by

$$
i\hbar \partial_{t_m} \Psi = \frac{\delta H_m}{\delta \overline{\Psi}} \quad \text{for } m = 1, 2, \dots \,. \tag{1.30}
$$

The t_1-flow is just spatial translation, the t_2-flow is given by the NLS equation (1.13a) with $t = t_2$, while the t_3-flow is that of the complex modified Korteweg–de Vries (mKdV) equation

$$
\partial_{t_3} \Psi - \frac{3}{2}|\Psi|^2 \partial_x \Psi + \frac{\hbar^2}{4} \partial_{xxx} \Psi = 0 \,. \tag{1.31}
$$

With this indexing convention, the highest-order spatial derivative of the t_m-flow is a linear term of order m. All of the quantities H_n are conserved by each flow; their densities (1.26) satisfy the local conservation laws

$$
\partial_{t_m} \rho_{n-1} + \partial_x \mu_{m,n} = 0 \quad \text{for } m, n = 1, 2, \dots \,. \tag{1.32}
$$

Here $\mu_{m,n}$ is the flux for the $(n-1)^{\text{th}}$ conserved density under the t_m-flow.

Because all of these flows commute, they may be solved simultaneously for $\Psi^{\hbar}(x, \mathbf{t})$ satisfying the initial condition (1.13c), where $\mathbf{t} = (t_1, t_2, \dots) \in \mathbb{R}^\infty$, meaning that all but finitely many t_m are zero. Associated with each $\mathbf{t} \in \mathbb{R}^\infty$ is a polynomial $p(\cdot, \mathbf{t})$ defined by

$$
p(\lambda, \mathbf{t}) = \sum_{m=0}^{\infty} t_{m+1} \lambda^m \,. \tag{1.33}
$$

The simultaneous evolution of the scattering data is then given by

$$
\begin{aligned}
\chi_j(\mathbf{t}) &= \chi_j(0) + \kappa_j p(\lambda_j, \mathbf{t}) \,, \\
R(\lambda, \mathbf{t}) &= R(\lambda, 0) \exp\left(\frac{-i2kp(\lambda, \mathbf{t})}{\hbar}\right) ,
\end{aligned}
\tag{1.34}
$$

and the solution $\Psi(x, \mathbf{t})$ of the NLS hierarchy is then reconstructed via inverse theory from $\mathcal{S}(\mathbf{t})$, the fundamental scattering data

$$\mathcal{S}(\mathbf{t}) = \{R(\lambda, \mathbf{t}), \lambda_j, \chi_j(\mathbf{t}) : |\lambda| \geq 1,\ j = 1, \dots, N\}, \tag{1.35}$$

as determined by (1.34).

The scope of the semiclassical limit for the defocusing NLS can then be enlarged to consider the solution $\Psi^\hbar(x, \mathbf{t})$ of the whole hierarchy that satisfies the initial condition

$$\Psi^\hbar(x, 0) = \Psi^\hbar(x) \equiv A(x) \exp\left(\frac{i}{\hbar} S(x)\right) \tag{1.36}$$

for some smooth $A(x)$ and $S(x)$ that are independent of $\hbar$ and consistent the far-field boundary conditions

$$\Psi^\hbar(x, \mathbf{t}) \sim \exp\left(\pm \frac{i}{\hbar} S_\infty\right) \quad \text{as } x \to \pm\infty. \tag{1.37}$$

The goal is then to determine the limiting behavior of all the conserved densities $\rho_n^\hbar$ and fluxes $\mu_{m,n}^\hbar$ associated with the entire NLS hierarchy of flows as $\hbar$ tends to zero.

1.5 The Analogy with the KdV Hierarchy

This problem has many similarities with that of the zero-dispersion limit of the Korteweg–deVries (KdV) equation. There one studies the limit as $\varepsilon \to 0$ of the conserved densities for the scaled KdV equation

$$\partial_t u^\varepsilon - 6u^\varepsilon \partial_x u^\varepsilon + \varepsilon^2 \partial_{xxx} u^\varepsilon = 0\,, \tag{1.38a}$$

$$u^\varepsilon(x, 0) = u(x)\,. \tag{1.38b}$$

The limit is strong and given by the solution of the Hopf equation

$$\partial_t u - 6u \partial_x u = 0\,, \tag{1.39a}$$

$$u(x, 0) = u(x)\,, \tag{1.39b}$$

so long as its solution remains classical. After the breaktime the limit is weak due to the development of regularizing small-wavelength oscillations with an amplitude of order unity; thereafter its evolution is no longer governed by the Hopf equation (1.39a).

In their seminal paper, Gardner, Greene, Kruskal, and Miura [15] showed that the KdV equation is completely integrable using the inverse scattering transform associated with the self-adjoint Schrödinger operator

$$\mathcal{L}_S = -\varepsilon^2\,\partial_{xx} + u\,. \tag{1.40}$$

Lax and Levermore [27, 28] analyzed the limiting behavior of the scattering and inverse scattering transform using a WKB analysis of (1.40) and a kind of steepest-descent argument to obtain a characterization of the (weak) limits in terms of the solution of a variational problem. The solution of this variational problem was then constructed through the solution of a Riemann-Hilbert problem. These results are surveyed in [29].

We have employed and improved the same strategy to analyze the semiclassical limit for the defocusing NLS hierarchy. More recently, a similar analysis has obtained the semiclassical limit for the odd flows of the focusing NLS hierarchy [9].

1.6 Outline of the Results

In Section 2 we present an asymptotic analysis of the semiclassical limit (as $\hbar \to 0$) of the Zakharov-Shabat eigenvalue problem, including the "modification of the initial data," the derivation of a "log-determinant" reconstruction formula, and the analysis of this log-determinant representation as a trigonometric Cauchy matrix. The key distinctions of this asymptotic analysis of the Zakharov-Shabat eigenvalue problem from that of the Schrödinger problem for the KdV equation are (1) a reduction of a determinant of an $2N \times 2N$ degenerate matrix to that of $N \times N$ positive definite matrix and (2) the trigonometric nature of the Cauchy determinant. Also, in Section 2, we use the entire sequence of higher densities and fluxes and their associated "times" t_m and argue that the log-determinant is convex with respect to all of these times. The use of all of the times t_m is new and elegant and simplifies several of the arguments.

In Section 3 we derive a quadratic, constrained maximization problem, the solution of which will govern the semiclassical limit ($\hbar \to 0$). The analysis in this section is very similar to that for KdV; however, here we make use of all of the times t_m and have considerably reorganized and restructured the arguments for additional clarity.

In Section 4 we establish (Theorem 4.7) how the semiclassical limit of all the conserved densities and fluxes of the defocusing NLS hierarchy are given in terms of the maximization problem. Our results are stronger than those originally obtained for the KdV. We also obtain (Theorem 4.5) "variational conditions" for this maximization problem.

In Section 5, we construct the maximizer in both the "prebreaking" and "postbreaking" regions of space-time for the NLS flow and use this maximizer to describe the limiting dynamics. In particular, in the prebreaking region, we establish that the weak limit is actually strong. Finally, we make some remarks about future directions.

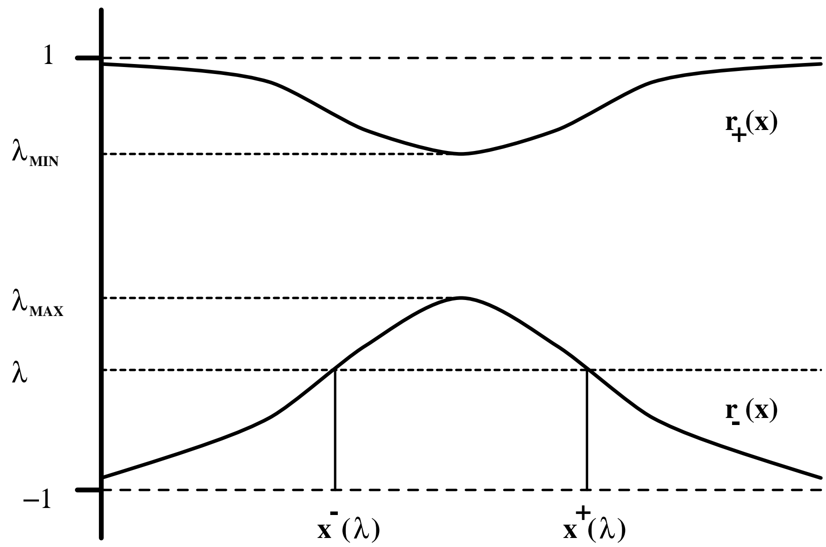

FIGURE 2.1. The initial data $r_{\pm}(x)$. Note their critical values $\lambda_{\min}$ and $\lambda_{\max}$ and the indicated defining relations for the turning points $x^{\pm}(\lambda)$.

2 Analysis of the Scattering Transform

2.1 Asymptotic Analysis of the Initial Scattering Data

In this paper our analysis is restricted to the case when the initial data $A(x)$ is a single upside-down positive bump with a unique minimum and a horizontal asymptote of 1 as $|x| \to \infty$ that is also its upper bound. Moreover, we represent the initial data $A(x)$ and $S(x)$ in terms of the associated initial Riemann invariants $r_{-}(x)$ and $r_{+}(x)$ given by (1.16b) such as they are depicted in Figure 2.1. The values of the two Riemann invariants are assumed to be separated, satisfying the bounds

$$-1 \leq r_{-}(x) \leq \lambda_{\max} < \lambda_{\min} \leq r_{+}(x) \leq 1\,. \tag{2.1}$$

This restriction is the analogue of the restriction to "single well" initial data made in [27, 28] for the case of the KdV zero-dispersion limit in that it, while not necessary, greatly simplifies our analysis.

The scattering data for $A(x)$ and $S(x)$ can then be computed asymptotically for small $\hbar$ in terms of the associated Riemann invariants $r_{\pm}$ using the semiclassical (WKB) method. The WKB turning point analysis yields discrete eigenvalues $\{\lambda_j^{\hbar}\}$ that are distributed within the the intervals $(-1, \lambda_{\max})$ and $(\lambda_{\min}, 1)$. Indeed, Figure 2.1 shows that for values of the transformed spectral variable λ in these intervals, there are exactly two turning points, located at $x = x^{\pm}(\lambda)$. In terms of these turning points, the asymptotic density of eigenvalues is given by the Weyl formula:

$$\text{density of eigenvalues in } (-1,\lambda_{\max})\cup(\lambda_{\min},1) \sim \frac{1}{\pi\hbar}\,\varphi(\lambda)\,, \tag{2.2}$$

where

$$\varphi(\lambda) \equiv \int_{x^-(\lambda)}^{x^+(\lambda)} \frac{\lambda - \frac{1}{2}\big(r_+(x)+r_-(x)\big)}{\sqrt{\big(\lambda - r_+(x)\big)\big(\lambda - r_-(x)\big)}}\,dx \tag{2.3}$$

for $\lambda \in (-1,\lambda_{\max})\cup(\lambda_{\min},1)$, and $\varphi(\lambda)=0$ otherwise. Here the square root is taken to be negative when λ is in $(-1,\lambda_{\max})$ and positive when λ is in $(\lambda_{\min},1)$. Specifically, $\lambda=\lambda_j^\hbar$ is the unique solution of

$$\frac{1}{\pi\hbar}\,\Phi(\lambda) = \begin{cases} j-\frac{1}{2} & \text{for } j=1,\dots,N_+^\hbar\,,\\ j+\frac{1}{2} & \text{for } j=-1,\dots,-N_-^\hbar\,,\end{cases} \tag{2.4}$$

where $\Phi(\lambda)$ is defined in terms of $\varphi(\lambda)$ by

$$\Phi(\lambda) \equiv \int_{\lambda_{\min}}^{\lambda} \varphi(\lambda')\,d\lambda' = \int_{x^-(\lambda)}^{x^+(\lambda)} \sqrt{\big(\lambda - r_+(x)\big)\big(\lambda - r_-(x)\big)}\,dx \tag{2.5}$$

and where $N_+^\hbar$ and $N_-^\hbar$, the number of eigenvalues in $(\lambda_{\min},1)$ and $(-1,\,\lambda_{\max})$, respectively, are defined by

$$\begin{aligned} N_+^\hbar &= \operatorname{Int}\left[\frac{1}{\pi\hbar}\int_{-\infty}^{\infty}\sqrt{\big(1-r_+(x)\big)\big(1-r_-(x)\big)}\,dx\right],\\ N_-^\hbar &= \operatorname{Int}\left[\frac{1}{\pi\hbar}\int_{-\infty}^{\infty}\sqrt{\big(1+r_+(x)\big)\big(1+r_-(x)\big)}\,dx\right]. \end{aligned} \tag{2.6}$$

Here both square roots are taken to be positive and $\operatorname{Int}[\cdot]$ rounds its argument to the closest integer value with half-integers rounded down. Let the index set of the WKB point spectrum be denoted $J^\hbar = \{-N_-^\hbar,\dots,-1,1,\dots,N_+^\hbar\}$. The norming exponent obtained from the WKB analysis is given by

$$\chi_j^\hbar = \chi(\lambda_j^\hbar) \quad \text{for } j\in J^\hbar, \tag{2.7}$$

where

$$\chi(\lambda) \equiv \kappa(\lambda)\,x^+(\lambda) + \int_{x^+(\lambda)}^{\infty}\left(\kappa(\lambda) - \sqrt{\big(r_+(x)-\lambda\big)\big(\lambda - r_-(x)\big)}\right)dx \tag{2.8}$$

for λ in $(-1,\lambda_{\max})\cup(\lambda_{\min},1)$ and, following (1.21), we define

$$\kappa(\lambda) = \sqrt{1-\lambda^2}\,. \tag{2.9}$$

For $|\lambda|>1$ the reflection coefficient $R^\hbar(\lambda)$ is calculated to be zero to all orders of the WKB expansion. Hence, the WKB scattering data is

$$\widetilde{S}^\hbar = \left\{R^\hbar(\lambda)\equiv 0,\ \lambda_j^\hbar,\ \chi_j^\hbar \equiv \chi(\lambda_j^\hbar) : |\lambda|\geq 1,\ j\in J^\hbar\right\}, \tag{2.10}$$

where $\chi(\lambda)$ is defined in (2.8). The derivation of these WKB approximations is completely standard, so here we have merely summarized the results. Details may be found in Jin [23].

Based on the above calculation, we choose to neglect the scattering data related to the continuous spectrum. More precisely, given $A(x)$ and $S(x)$ with Riemann invariants as depicted in Figure 2.1, we replace the initial data $\Psi^\hbar(x)$, given by (1.36), with the reflectionless data

$$\Psi^\hbar(x,0) = \widetilde{\Psi}^\hbar(x) \equiv A^\hbar(x) \exp\left(\frac{i}{\hbar}\, S^\hbar(x)\right) \tag{2.11}$$

whose scattering data $\widetilde{S}^\hbar$ is exactly equal to the approximate WKB scattering data (2.10). Henceforth, we consider only the modified initial data $\widetilde{\Psi}^\hbar(x)$ of (2.11) and will rigorously establish the semiclassical limit for the corresponding solution of the NLS hierarchy, which will still be denoted $\Psi^\hbar(x,\mathbf{t})$. This device sidesteps one important question concerning the limiting behavior of the inverse scattering machinery; however, it is justified a posteriori by the fact that the resulting conserved densities and fluxes have the same strong limits as those associated with the original initial data. More precisely, the following result will be a consequence of Theorem 5.3:

THEOREM 2.1 (Initial Data Recovery) *The modified initial data, $\widetilde{\Psi}^\hbar(x)$, approximates the prescribed initial data $\Psi^\hbar(x)$ in that $A(x)$ and $S(x)$ are recovered through the strong L^1_{loc}-limits*:

$$\begin{aligned} \lim_{\hbar\to 0}\left(|\widetilde{\Psi}^\hbar|^2 - 1\right) &= |A|^2 - 1\,, \\ \lim_{\hbar\to 0} -i\frac{\hbar}{2}\left(\overline{\widetilde{\Psi}^\hbar}\partial_x\widetilde{\Psi}^\hbar - \widetilde{\Psi}^\hbar\partial_x\overline{\widetilde{\Psi}^\hbar}\right) &= |A|^2\partial_x S\,. \end{aligned} \tag{2.12}$$

The agreement of the semiclassical limit for the modified problem with that for the original problem has been established only up to the instant at which the classical solution of the Euler system (1.14) breaks down.

2.2 Reconstruction Formula for Reflectionless Data

Motivated by the previous subsection, we will exploit explicit reconstructions of the conserved densities ρ_n and fluxes $\mu_{n,m}$ in terms of the discrete scattering data when the reflection coefficient $R(\lambda)$ vanishes. We emphasize that such representations are exact provided $R(\lambda) = 0$; they do not rely on the WKB approximation. Our reconstruction formula, although equivalent to an earlier one of Zakharov and Shabat [46], is new and illuminates certain Hermitian positivity and convexity properties that are particularly useful for the evaluation of semiclassical limits. Both our formula and these properties are presented in this subsection. In addition, a formula for an important determinant is derived.

The formula for the N-soliton solution derived by Zakharov and Shabat [46], which involves a $2N\times 2N$ matrix, was not suitable for our analysis, so we modified it into the following more suitable form:

PROPOSITION 2.2 (Reflectionless Reconstruction Formula) *Given any reflectionless scattering data*

$$\mathcal{S} = \{R(\lambda)\equiv 0,\ \lambda_j,\ \chi_j : |\lambda|\geq 1,\ j=1,\dots,N\}\,, \tag{2.13}$$

with distinct λ_j *in* $(-1,1)$ *and* χ_j *real, then the associated conserved densities and fluxes are given by*

$$\begin{aligned}\rho_{n-1}(x,\mathbf{t}) &= \hbar^2\partial_{xt_n}\log\tau(x,\mathbf{t})\,,\\ \mu_{m,n}(x,\mathbf{t}) &= -\hbar^2\partial_{t_mt_n}\log\tau(x,\mathbf{t})\,,\end{aligned} \tag{2.14}$$

where the so-called τ*-function* $\tau(x,\mathbf{t})$ *is the* $N\times N$ *determinant*

$$\tau(x,\mathbf{t}) = \det\big(I+G(x,\mathbf{t})\big)\,. \tag{2.15}$$

Here the $N\times N$ *matrix* G *is written as* $G=\hbar DBD$ *in terms of the trigonometric Cauchy matrix* B *and the diagonal matrix* D *defined by*

$$B = \left(\frac{1}{2\sin\!\big(\frac{1}{2}(\sigma_j+\sigma_k)\big)}\right)\,,\qquad D=\operatorname{diag}\!\left(\exp\!\left(\frac{a_j(x,\mathbf{t})}{\hbar}\right)\right)\,, \tag{2.16}$$

where $\sigma_j\equiv\arccos\lambda_j$ *is in* $(0,\pi)$ *and*

$$a_j(x,\mathbf{t}) \equiv -x\sin\sigma_j + p(\cos\sigma_j,\mathbf{t})\,\sin\sigma_j + \chi_j \tag{2.17}$$

with $p(\lambda,\mathbf{t})$ *defined by* (1.33).

Remark. In particular, the solution $\Psi(x,\mathbf{t})$ of the defocusing NLS hierarchy corresponding to this reflectionless initial data is given in terms of the τ-function $\tau(x,\mathbf{t})$ through the relations

$$\begin{aligned}|\Psi(x,\mathbf{t})|^2-1 &= -\hbar^2\,\partial_{xx}\log\tau(x,\mathbf{t})\,,\\ -i\frac{\hbar}{2}\Big(\overline{\Psi}(x,\mathbf{t})\partial_x\Psi(x,\mathbf{t})-\Psi(x,\mathbf{t})\partial_x\overline{\Psi}(x,\mathbf{t})\Big) &= \ \hbar^2\,\partial_{xt_2}\log\tau(x,\mathbf{t})\,.\end{aligned} \tag{2.18}$$

These relations determine $\Psi(x,\mathbf{t})$ up to a constant phase that is then fixed by the far-field boundary conditions (1.37).

PROOF: The N-soliton solution first derived by Zakharov and Shabat [46] has the same form as (2.14) but with $\tau(x,\mathbf{t})$ given in terms of a $2N\times 2N$ determinant

$$\tau(x,\mathbf{t}) = \det(\mathbf{I}+\mathbf{G}(x,\mathbf{t}))\,. \tag{2.19}$$

Here the $2N\times 2N$ matrix $\mathbf{G}$ is written as

$$\mathbf{G} = \hbar\begin{pmatrix} DCD & \overline{\Theta}^2DCD\\ \Theta^2DCD & DCD\end{pmatrix}, \tag{2.20}$$

where the Cauchy matrix C and the diagonal matrix Θ are defined by

$$C = \frac{1}{2}\left(\frac{1}{\kappa_j + \kappa_k}\right), \qquad \Theta = \operatorname{diag}\big(\exp\big(i\tfrac{1}{2}\sigma_j\big)\big)\,. \tag{2.21}$$

The twofold redundancy in (2.19) becomes evident upon noting that the vector $(z, -z\overline{\Theta}^2)$ is a left null vector of $\mathbf{G}$ for any row vector $z \in \mathbb{C}^N$, whereby $\mathbf{G}$ has 0 as an eigenvalue with multiplicity N. Because 1 is therefore an eigenvalue of $\mathbf{I} + \mathbf{G}$ with multiplicity N, its determinant must reduce to that of an $N \times N$ matrix. Letting $M = \hbar DCD$ and using the fact that D and Θ commute, we get

$$\begin{aligned} \det(\mathbf{I}+\mathbf{G}) &= \det\left(\begin{pmatrix} I & -\overline{\Theta}^2 \\ 0 & \overline{\Theta} \end{pmatrix}\begin{pmatrix} I+M & \overline{\Theta}^2 M \\ \Theta^2 M & I+M \end{pmatrix}\begin{pmatrix} I & \overline{\Theta} \\ 0 & \Theta \end{pmatrix}\right) \\ &= \det\begin{pmatrix} I & 0 \\ \Theta M & I + \overline{\Theta} M \Theta + \Theta M \overline{\Theta} \end{pmatrix} \\ &= \det\Big(I + \overline{\Theta} M \Theta + \Theta M \overline{\Theta}\Big) \\ &= \det(I + \hbar DBD)\,, \end{aligned} \tag{2.22}$$

where

$$B = \overline{\Theta} C \Theta + \Theta C \overline{\Theta} = \left(\frac{\cos\Big(\frac{1}{2}(\sigma_j - \sigma_k)\Big)}{\sin\sigma_j + \sin\sigma_k}\right). \tag{2.23}$$

Employing the trigonometric identity

$$\sin\sigma_j + \sin\sigma_k = 2\cos\Big(\tfrac{1}{2}(\sigma_j - \sigma_k)\Big)\sin\Big(\tfrac{1}{2}(\sigma_j + \sigma_k)\Big) \tag{2.24}$$

reduces formula (2.23) for B to that of (2.16), thus proving the lemma. □

The advantage of the above representation of the N-soliton solutions derives from the following lemma:

LEMMA 2.3 (Trigonometric Cauchy Matrix Properties) *The matrix G of equation* (2.14) *is Hermitian positive and its determinant is given by*

$$\det(G) = \frac{\hbar^N}{2^N}\exp\left(\frac{2}{\hbar}\sum_{j=1}^{N} a_j(x,\mathbf{t})\right)\frac{\displaystyle\prod_{\substack{j,k=1\\ j\neq k}}^{N}\left|\sin\Big(\tfrac{1}{2}(\sigma_j - \sigma_k)\Big)\right|}{\displaystyle\prod_{j,k=1}^{N}\left|\sin\Big(\tfrac{1}{2}(\sigma_j + \sigma_k)\Big)\right|}\,. \tag{2.25}$$

PROOF: Since $G = \hbar DBD$ where D is the positive diagonal matrix given in (2.16), the assertions follow from properties of the symmetric matrix B. By introducing

$$Z_j \equiv \exp\Big(i\tfrac{1}{2}\sigma_j\Big)\,, \tag{2.26}$$

the matrix B is seen to be a trigonometric Cauchy matrix in the form

$$B = \left(\frac{1}{2\sin\left(\frac{1}{2}(\sigma_j + \sigma_k)\right)} \right) = \left(\frac{i}{Z_j Z_k - \overline{Z}_j \overline{Z}_k} \right). \tag{2.27}$$

The determinant of B can then be expressed as

$$\det(B) = \det\left(\frac{i Z_j Z_k}{Z_j^2 Z_k^2 - 1} \right) = i^N \det\left(\frac{1}{Z_j^2 Z_k^2 - 1} \right) \prod_{j=1}^{N} Z_j^2 . \tag{2.28}$$

The classical Cauchy determinant formula (e.g., see [43]) is

$$\det\left(\frac{1}{Z_j^2 Z_k^2 - 1} \right) = \frac{\prod\limits_{\substack{j,k=1 \\ j \le k}}^{N} \left(Z_j^2 - Z_k^2 \right)^2}{\prod\limits_{j,k=1}^{N} \left(Z_j^2 Z_k^2 - 1 \right)} , \tag{2.29}$$

so that (2.28) becomes

$$\det(B) = i^N \frac{\prod\limits_{\substack{j,k=1 \\ j \le k}}^{N} \left(Z_j \overline{Z}_k - \overline{Z}_j Z_k \right)^2}{\prod\limits_{j,k=1}^{N} \left(Z_j Z_k - \overline{Z}_j \overline{Z}_k \right)} = \frac{1}{2^N} \frac{\prod\limits_{\substack{j,k=1 \\ j \neq k}}^{N} \left| \sin\left(\frac{1}{2}(\sigma_j - \sigma_k)\right) \right|}{\prod\limits_{j,k=1}^{N} \left| \sin\left(\frac{1}{2}(\sigma_j + \sigma_k)\right) \right|} . \tag{2.30}$$

Thus $\det(B)$ is manifestly positive. Since all principal minors of B are of the same form, they, too, have positive determinants, but this then implies that B is Hermitian positive. The assertions of the theorem now follow for $G = \hbar DBD$. □

A direct consequence of the Hermitian positivity of G is the following hitherto unnoticed convexity property of the N-soliton solution:

LEMMA 2.4 ($\log\tau$ Convexity) *For each $\hbar > 0$ the function*

$$(x, \mathbf{t}) \mapsto \log\tau(x, \mathbf{t}) \quad \textit{is convex}. \tag{2.31}$$

PROOF: Since x can be identified with $-t_1$ as a translate, it is sufficient to prove

$$\mathbf{t} \mapsto \log\tau(x, \mathbf{t}) \quad \text{is convex;} \tag{2.32}$$

this follows from a direct computation of its Hessian form. Let $\partial_s = \dot{\mathbf{t}} \cdot \partial_{\mathbf{t}}$ denote the directional derivative of $\mathbf{t}$ in a given direction $\dot{\mathbf{t}} \in \mathbb{R}^\infty$. Since $G = \hbar DBD$, it follows from (2.16) that

$$\partial_s G = EG + GE , \tag{2.33}$$

where

$$E = \operatorname{diag}\left(\frac{\partial_s a_j(x,\mathbf{t})}{\hbar}\right) = \operatorname{diag}\left(\frac{\kappa_j p(\lambda_j,\dot{\mathbf{t}})}{\hbar}\right). \tag{2.34}$$

The first derivative of $\log\tau$ is computed directly from (2.15) and is found to be

$$\begin{aligned}\partial_s \log\tau &= \partial_s \log\det(I+G)\\ &= \operatorname{tr}\Big((I+G)^{-1}\partial_s G\Big)\\ &= \operatorname{tr}\Big((I+G)^{-1}(EG+GE)\Big)\\ &= 2\operatorname{tr}\Big(\left(I-(I+G)^{-1}\right)E\Big).\end{aligned} \tag{2.35}$$

The second derivative is then

$$\begin{aligned}\partial_{ss}\log\tau &= 2\operatorname{tr}\Big((I+G)^{-1}\partial_s G(I+G)^{-1}E\Big)\\ &= 2\operatorname{tr}\Big((I+G)^{-1}(EG+GE)(I+G)^{-1}E\Big)\\ &= 4\operatorname{tr}\Big(E(I+G)^{-1}EG(I+G)^{-1}\Big).\end{aligned} \tag{2.36}$$

Because $G(I+G)^{-1}$ is Hermitian positive while $E(I+G)^{-1}E$ is Hermitian nonnegative, and because the product of any Hermitian positive matrix with any nontrivial (nonzero), Hermitian nonnegative matrix has a positive trace, (2.36) shows $\partial_{ss}\log\tau \geq 0$. The lemma then follows from the arbitrariness of the direction $\dot{\mathbf{t}}$. □

Remark. The convexity (2.32) will hold for any τ-function $\tau(x,\mathbf{t})$ that is a linear combination of real exponentials with positive coefficients. The above argument shows more; it shows that $\partial_{ss}\log\tau > 0$ whenever E is nontrivial. However, E will be zero for exactly those directions $\dot{\mathbf{t}}$ for which every eigenvalue λ_j is a root of $p(\lambda,\dot{\mathbf{t}}) = 0$; such a polynomial must have degree at least N.

Remark. The above lemma also applies to the N-soliton formula for the KdV equation. The proof is exactly as above with only a minor change in formula (2.34) for E. The implications of this convexity for the KdV zero-dispersion limit are analogous to those for the defocusing NLS semiclassical limit and will be noted subsequently.

We close this section by evaluating the Hamiltonians H_n for the N-soliton solution (2.14).

LEMMA 2.5 (Hamiltonians) *For each $\hbar > 0$ the Hamiltonians H_n are given by*

$$H_n = 2\hbar\sum_{j=1}^{N}\kappa_j\lambda_j^n. \tag{2.37}$$

PROOF: By choosing $\dot{\mathbf{t}}$ so that $\partial_s = \partial_{t_{n+1}}$, the diagonal matrix E of (2.34) becomes

$$E = E_n \equiv \operatorname{diag}\left(\frac{\kappa_j \lambda_j^n}{\hbar}\right). \tag{2.38}$$

Formula (2.35) then gives

$$\begin{aligned} H_n = \int_{-\infty}^{\infty} \rho_n \, dx &= \hbar^2 \, \partial_{t_{n+1}} \log \tau \Big|_{x=-\infty}^{\infty} \\ &= 2\hbar^2 \operatorname{tr}\Big(\big(I - (I+G)^{-1}\big) E_n\Big) \Big|_{x=-\infty}^{\infty} \\ &= 2\hbar^2 \operatorname{tr}(E_n) = 2\hbar \sum_{j=1}^{N} \kappa_j \lambda_j^n \,. \end{aligned} \tag{2.39}$$

Here we have used that $(I+G)^{-1}$ tends to I as x tends to infinity and to 0 as x tends to minus infinity. □

3 Establishing the Semiclassical Limit

3.1 The Limit Considered

We are now ready to state precisely the central object of study in the remainder of the paper. Given $A(x)$ and $S(x)$, consider the modified initial data (2.9) corresponding to the approximate WKB scattering data (2.13). The resulting solution $\Psi^\hbar(x, \mathbf{t})$ of the NLS hierarchy is an $N^\hbar$-soliton solution where $N^\hbar = N_+^\hbar + N_-^\hbar$. The associated conserved densities and fluxes are then given by the reflectionless reconstruction formula (2.17) as

$$\begin{aligned} \rho_{n-1}^\hbar(x, \mathbf{t}) &= \hbar^2 \partial_{x t_n} \log \tau^\hbar(x, \mathbf{t}) \,, \\ \mu_{m,n}^\hbar(x, \mathbf{t}) &= -\hbar^2 \partial_{t_m t_n} \log \tau^\hbar(x, \mathbf{t}) \,, \end{aligned} \tag{3.1}$$

where $\tau^\hbar(x, \mathbf{t})$ is the $N^\hbar \times N^\hbar$ determinant

$$\tau^\hbar(x, \mathbf{t}) = \det\Big(I + G^\hbar(x, \mathbf{t})\Big) \,. \tag{3.2}$$

Here the matrix $G^\hbar$ is written as $G^\hbar = \hbar D^\hbar B^\hbar D^\hbar$ in terms of the trigonometric Cauchy matrix $B^\hbar$ and diagonal matrix $D^\hbar$ defined by

$$\begin{aligned} B^\hbar &= \left(\frac{1}{2\sin(\frac{1}{2}(\sigma_j + \sigma_k))}\right)_{j,k \in J^\hbar} , \\ D^\hbar &= \operatorname{diag}\left(\exp\left(\frac{a(\sigma_j, x, \mathbf{t})}{\hbar}\right)\right)_{j \in J^\hbar} , \end{aligned} \tag{3.3}$$

where $\sigma_j \equiv \arccos \lambda_j^\hbar$ is in $(0, \pi)$ for each $\lambda_j^\hbar$ determined by (2.11), and

$$a(\sigma, x, \mathbf{t}) \equiv -x \sin\sigma + p(\cos\sigma, \mathbf{t}) \sin\sigma + \chi(\cos\sigma) \tag{3.4}$$

with $p(\lambda, \mathbf{t})$ defined by (1.33) and $\chi(\lambda)$ by (2.8).

3.2 The Limiting Behavior of the τ-Function

In this subsection we establish the limit of $\hbar^2 \log \tau^\hbar(x, \mathbf{t})$ as $\hbar$ tends to zero in the topology of uniform convergence over compact subsets of $(x, \mathbf{t})$, characterizing the limit in terms the solution of a maximization problem.

For every nonempty subset $S \subset J^\hbar$, let $G_S^\hbar$ denote the $|S| \times |S|$ principal minor matrix of $G^\hbar$ obtained by retaining only those elements whose indices belong to S. The determinant in (3.2) is just the characteristic polynomial of $G^\hbar$ evaluated at -1 and can be expanded in terms of its minor determinants as

$$\det\Big(I + G^\hbar\Big) = \sum_{S \subset J^\hbar} \det G_S^\hbar \,, \tag{3.5}$$

where S ranges over all subsets of $J^\hbar$, and we adopt the convention that $\det G_S^\hbar = 1$ when S is the null set. Since every principal minor of $G^\hbar$ has the same form as $G^\hbar$, Proposition 2.2 shows that each $G_S^\hbar$ is Hermitian positive, hence has a positive determinant. The point of this expansion is that the logarithm of the sum is dominated by the logarithm of its largest term as $\hbar$ tends to zero.

THEOREM 3.1 ($\hbar$-Maximum) *Let*

$$q^\hbar(x, \mathbf{t}) \equiv \max \Big\{ \hbar^2 \log \det G_S^\hbar(x, \mathbf{t}) : S \subset J^\hbar \Big\} \,. \tag{3.6}$$

Then

$$\lim_{\hbar \to 0} \Big| \hbar^2 \log \tau^\hbar(x, \mathbf{t}) - q^\hbar(x, \mathbf{t}) \Big| = 0 \tag{3.7}$$

uniformly over $(x, \mathbf{t}) \in \mathbb{R} \times \mathbb{R}^\infty$.

Remark. The maximum of (3.6) will be attained because it is taken over a finite set—namely, all subsets of $J^\hbar$—but is not necessarily attained at a unique $S \subset J^\hbar$.

PROOF: Because the sum of (3.5) contains $2^{N^\hbar}$ positive terms, it can be crudely bounded above and below by

$$\exp\left(\frac{q^\hbar(x, \mathbf{t})}{\hbar^2}\right) \le \det\big(I + G^\hbar\big) = \sum_{S \subset J^\hbar} \det G_S^\hbar \le 2^{N^\hbar} \exp\left(\frac{q^\hbar(x, \mathbf{t})}{\hbar^2}\right) . \tag{3.8}$$

Upon taking the logarithm of (3.8), multiplying by $\hbar^2$, and subtracting $q^\hbar$, one arrives at

$$0 \le \hbar^2 \log \tau^\hbar(x, \mathbf{t}) - q^\hbar(x, \mathbf{t}) \le \hbar^2 N^\hbar \log 2 \,. \tag{3.9}$$

Because $N^\hbar = N_+^\hbar + N_-^\hbar = O(1/\hbar)$ by (2.6), the theorem follows. □

It remains to be shown that the limit of $q^\hbar(x, \mathbf{t})$ exists and can be characterized, just as each $q^\hbar(x, \mathbf{t})$, by a maximization problem. This limiting maximization problem can be inferred by recasting $\hbar^2 \log\det G_S^\hbar(x,\mathbf{t})$ as a Stieltjes integral. For each $S \subset J^\hbar$, introduce the atomic distributions

$$\eta_S^\hbar(\sigma) = \pi\hbar \sum_{j\in S} \delta(\sigma-\sigma_j)\,, \tag{3.10a}$$

$$\eta_S^\hbar(\sigma) \times' \eta_S^\hbar(\theta) = \pi^2\hbar^2 \sum_{\substack{j,k\in S\\ j\neq k}} \delta(\sigma-\sigma_j)\,\delta(\theta-\sigma_k)\,. \tag{3.10b}$$

Denote the class of all distributions of the form (3.10a) by $\mathcal{A}^\hbar$. Define the functional $Q^\hbar$ over these distributions by

(3.11)

$$\begin{aligned} Q^\hbar\left(\eta_S^\hbar; x,\mathbf{t}\right) &\equiv \hbar^2 \log\det G_S^\hbar(x,\mathbf{t}) \\ &= \frac{2}{\pi}\int_0^\pi a^\hbar(\sigma,x,\mathbf{t})\,\eta_S^\hbar(\sigma)\,d\sigma \\ &\quad + \frac{1}{\pi^2}\int_0^\pi\!\!\int_0^\pi \log\left|\frac{\sin\left(\frac{1}{2}(\sigma-\theta)\right)}{\sin\left(\frac{1}{2}(\sigma+\theta)\right)}\right| \eta_S^\hbar(\sigma) \times' \eta_S^\hbar(\theta)\,d\sigma\,d\theta\,, \end{aligned}$$

where

$$a^\hbar(\sigma,x,\mathbf{t}) = a(\sigma,x,\mathbf{t}) - \frac{\hbar}{2}\log(\sin\sigma) + \frac{\hbar}{2}\log\left(\frac{\hbar}{2}\right). \tag{3.12}$$

Clearly (3.6) can now be recast as

$$q^\hbar(x,\mathbf{t}) = \max\left\{Q^\hbar(\eta_S^\hbar; x,\mathbf{t}) : \eta_S^\hbar \in \mathcal{A}^\hbar\right\}. \tag{3.13}$$

Notice that each of the distributions in $\mathcal{A}^\hbar$ satisfies the bounds

$$0 \le \eta_S^\hbar(\sigma) \le \eta_J^\hbar(\sigma)\,, \tag{3.14a}$$

$$0 \le \eta_S^\hbar(\sigma) \times' \eta_S^\hbar(\theta) \le \eta_J^\hbar(\sigma) \times' \eta_J^\hbar(\theta)\,, \tag{3.14b}$$

where $\eta_J^\hbar$ denotes $\eta_S^\hbar$ with $S = J^\hbar$. Moreover, as $\hbar \to 0$,

$$\eta_J^\hbar(\sigma)\,d\sigma \to \phi(\cos\sigma)\sin\sigma\,d\sigma\,, \tag{3.15a}$$

$$\eta_J^\hbar(\sigma) \times' \eta_J^\hbar(\theta)\,d\sigma d\theta \to \phi(\cos\sigma)\phi(\cos\theta)\sin\sigma\sin\theta\,d\sigma\,d\theta\,, \tag{3.15b}$$

in the sense of weak convergence of measures. Therefore, any such limit of distributions in $\mathcal{A}^\hbar$ must lie in the class of measures with L^1 densities in the admissible set

$$\mathcal{A} = \{\eta \in L^1\left((0,\pi)\right) : 0 \le \eta(\sigma) \le \phi(\cos\sigma)\sin\sigma\}. \tag{3.16}$$

It appears that if $\eta^\hbar \in \mathcal{A}^\hbar$ such that $\eta^\hbar \to \eta \in \mathcal{A}$, then $Q^\hbar(\eta^\hbar; x, \mathbf{t}) \to Q(\eta; x, \mathbf{t})$ where

$$Q(\eta; x, \mathbf{t}) = \frac{2}{\pi}\int_0^\pi a(\sigma, x, \mathbf{t})\eta(\sigma)\, d\sigma + \frac{1}{\pi^2}\int_0^\pi\int_0^\pi \log\left|\frac{\sin(\frac{1}{2}(\sigma-\theta))}{\sin(\frac{1}{2}(\sigma+\theta))}\right| \eta(\sigma)\eta(\theta)\, d\sigma\, d\theta\,, \tag{3.17}$$

with $a(\sigma, x, \mathbf{t})$ as defined in (3.4). These observations suggest that the limit of the maxima $q^\hbar(x, \mathbf{t})$ as $\hbar$ tends to zero may itself be characterized by a maximization problem. Indeed, we prove the following theorem:

THEOREM 3.2 (Limiting Maximum)

$$\lim_{\hbar\to 0} q^\hbar(x, \mathbf{t}) = q(x, \mathbf{t})\,, \tag{3.18}$$

uniformly over compact subsets of $(x, \mathbf{t}) \in \mathbb{R} \times \mathbb{R}^\infty$*, where*

$$q(x, \mathbf{t}) = \max\{Q(\eta; x, \mathbf{t}) : \eta \in \mathcal{A}\} \tag{3.19}$$

with $Q(\eta; x, \mathbf{t})$ *and* $\mathcal{A}$ *defined in* (3.16) *and* (3.15)*, respectively.*

The proof of this theorem will follow a sequence of three lemmas. The basic compactness properties are provided by the first lemma.

LEMMA 3.3 ($\hbar$-Compactness)

(a) *The family of measures* $\{\eta^\hbar(\sigma)\, d\sigma : \eta^\hbar \in \mathcal{A}^\hbar\}$ *has bounded total variation and hence is relatively compact in the weak-$*$ topology of measures.*
(b) *The family of functions* $\{Q^\hbar(\eta^\hbar; x, \mathbf{t}) : \eta^\hbar \in \mathcal{A}^\hbar\}$ *is both equicontinuous and equibounded over compact subsets of* $(x, \mathbf{t})$*-space and hence is relatively compact in* $C(\mathbb{R} \times \mathbb{R}^\infty)$.

The proofs of these results follow closely those of Lemma 2.3 and Lemma 2.9 of [28], respectively, and can be found in [23], and so will not be given here.

The next lemma establishes limiting behaviors as $\hbar$ tends to zero and has the flavor of a continuity result.

LEMMA 3.4 ($\hbar$-Continuity) *Let* $\hbar_n \to 0$*. If* $\eta_n \in \mathcal{A}^{\hbar_n}$ *such that* $\eta_n \to \eta$ *in the weak-$*$ topology of measures, then*

(a) $\eta \in \mathcal{A}$ *and*
(b) $Q^{\hbar_n}(\eta_n; x, \mathbf{t}) \to Q(\eta; x, \mathbf{t})$.

The proof of part (a) was indicated above the statement of the limiting maximization Theorem 3.2, while that of assertion (b) closely follows that of Theorem 2.5 of [28] and can be found in [23].

The last lemma completes the characterization of the admissible set $\mathcal{A}$ that was begun in part (a) of the continuity Lemma 3.4.

LEMMA 3.5 (Approximation of $\mathcal{A}$) *Let $\eta \in \mathcal{A}$ and $\hbar_n \to 0$. There exists a sequence of $\eta_n \in \mathcal{A}^{\hbar_n}$ such that $\eta_n \to \eta$ in the weak-$*$ topology of measures.*

The proof of this is straightforward and will be omitted.

PROOF OF THEOREM 3.2 (LIMITING MAXIMUM): The fact that the limit of (3.18) holds pointwise for each $(x, \mathbf{t}) \in \mathbb{R} \times \mathbb{R}^\infty$ is a consequence of the following two facts, which are proved below: First,

$$\limsup_{\hbar \to 0} q^\hbar(x, \mathbf{t}) = Q(\eta^*; x, \mathbf{t}) \tag{3.20}$$

for some η^* in the admissible set $\mathcal{A}$; second,

$$q(x, \mathbf{t}) \equiv \sup \{Q(\eta; x, \mathbf{t}) : \eta \in \mathcal{A}\} \leq \liminf_{\hbar \to 0} q^\hbar(x, \mathbf{t}) \,. \tag{3.21}$$

Comparing these two facts shows that

$$Q(\eta^*; x, \mathbf{t}) = q(x, \mathbf{t}) \equiv \sup \{Q(\eta; x, \mathbf{t}) : \eta \in \mathcal{A}\} \,. \tag{3.22}$$

Hence, the maximum asserted in (3.19), the characterization of $q(x, \mathbf{t})$, is attained. That the limit (3.18) holds uniformly over compact subsets of $(x, \mathbf{t}) \in \mathbb{R} \times \mathbb{R}^\infty$ then follows easily from assertion (b) of Lemma 3.3 ($\hbar$-compactness).

All that remains is to verify the facts (3.20) and (3.21) for an arbitrary fixed $(x, \mathbf{t}) \in \mathbb{R} \times \mathbb{R}^\infty$. To prove (3.20), let $\hbar_n \to 0$ such that

$$\limsup_{\hbar \to 0} q^\hbar(x, \mathbf{t}) = \lim_{n \to \infty} q^{\hbar_n}(x, \mathbf{t}) \,. \tag{3.23}$$

Let $\{\eta_n^*\}$ be any corresponding sequence of maximizing densities with $\eta_n^* \in \mathcal{A}^{\hbar_n}$ such that $q^{\hbar_n}(x, \mathbf{t}) = Q^{\hbar_n}(\eta_n^*; x, \mathbf{t})$. Assertion (a) of Lemma 3.3 ($\hbar$-compactness) implies the existence of a subsequence of $\{\eta_n^*\}$ whose limit point must, by assertion (a) of Lemma 3.4 ($\hbar$-continuity), be realized by a density η^* in $\mathcal{A}$. After passing to this subsequence, assertion (b) of Lemma 3.4 then states that

$$\lim_{n \to \infty} q^{\hbar_n}(x, \mathbf{t}) = \lim_{n \to \infty} Q^{\hbar_n}(\eta_n^*; x, \mathbf{t}) = Q(\eta^*; x, \mathbf{t}) \,. \tag{3.24}$$

Then (3.20) follows by combining (3.23) with (3.24).

Now turning to the proof of (3.21), let $\delta > 0$ be arbitrary and choose $\eta_\delta \in \mathcal{A}$ such that

$$q(x, \mathbf{t}) - \delta \leq Q(\eta_\delta; x, \mathbf{t}) \,. \tag{3.25}$$

Let $\hbar_n \to 0$ be arbitrary. Lemma 3.5 (approximation of $\mathcal{A}$) guarantees the existence of a corresponding sequence $\{\eta_n\}$ of densities with $\eta_n \in \mathcal{A}^{\hbar_n}$ such that $\eta_n \to \eta_\delta$ in the sense of measures. Assertion (b) of Lemma 3.4 ($\hbar$-continuity) together with the fact that $Q^{\hbar_n}(\eta_n; x, \mathbf{t}) \leq q^{\hbar_n}(x, \mathbf{t})$ leads to

$$Q(\eta_\delta; x, \mathbf{t}) = \lim_{n \to \infty} Q^{\hbar_n}(\eta_n; x, \mathbf{t}) \leq \liminf_{n \to \infty} q^{\hbar_n}(x, \mathbf{t}) \,. \tag{3.26}$$

Thus, by the arbitrariness of $\{\hbar_n\}$, the combination of (3.25) and (3.26) gives

$$q(x, \mathbf{t}) - \delta \leq \liminf_{\hbar \to 0} q^\hbar(x, \mathbf{t}) \,. \tag{3.27}$$

Fact (3.21) then follows from the arbitrariness of δ, proving the theorem. □

By combining Theorem 3.1 ($\hbar$-maximum) with Theorem 3.2 (limiting maximum), one obtains the following:

THEOREM 3.6 (τ-Function Limit)

$$\lim_{\hbar \to 0} \hbar^2 \log \tau^{\hbar}(x, \mathbf{t}) = q(x, \mathbf{t}) \tag{3.28}$$

uniformly over compact subsets of $(x, \mathbf{t}) \in \mathbb{R} \times \mathbb{R}^{\infty}$*, where*

$$q(x, \mathbf{t}) = \max \{ Q(\eta; x, \mathbf{t}) : \eta \in \mathcal{A} \}, \tag{3.29}$$

with $Q(\eta; x, \mathbf{t})$ *and* $\mathcal{A}$ *defined in* (3.17) *and* (3.16)*, respectively. Moreover, the map* $(x, \mathbf{t}) \mapsto q(x, \mathbf{t})$ *is*

(a) *nonnegative,*
(b) *continuous,*
(c) *convex, and*
(d) *increasing in* x.

PROOF: All that remains is to prove (a) through (d). Assertion (a) follows from the fact $0 \in \mathcal{A}$, so that $0 = Q(0; x, \mathbf{t}) \leq q(x, \mathbf{t})$. Assertion (b) follows from the uniformity of the limit (3.28) and the continuity of $\hbar^2 \log \tau^{\hbar}(x, \mathbf{t})$. Assertions (c) and (d) follow from (3.29) and the fact that each $Q(\eta; x, \mathbf{t})$ is a linear function of $(x, \mathbf{t})$ that is increasing in x through a. □

4 Strengthening the Semiclassical Limit

4.1 Properties of the Maximization Problem

Before we can characterize the semiclassical limit of the conserved densities and fluxes, more basic facts concerning the maximization problem in (3.29) must be established, including the uniqueness of the density at which the maximum is attained. By introducing the scalar product

$$(\alpha \mid \eta) \equiv \frac{2}{\pi} \int_0^{\pi} \alpha(\sigma) \eta(\sigma)\, d\sigma\,, \tag{4.1}$$

and the integral operator

$$\mathrm{L}\eta(\sigma) = \frac{1}{\pi} \int_0^{\pi} \log \left| \frac{\sin\left(\frac{1}{2}(\sigma - \theta)\right)}{\sin\left(\frac{1}{2}(\sigma + \theta)\right)} \right| \eta(\theta)\, d\theta\,, \tag{4.2}$$

the quadratic functional $Q(\eta; x, \mathbf{t})$ can be recast in a more abstract form as

$$Q(\eta; x, \mathbf{t}) = \frac{1}{2}(\eta \mid \mathrm{L}\eta) + (a(x, \mathbf{t}) \mid \eta)\,, \tag{4.3}$$

where a is again given by (3.4) but here we suppress the dependence on σ. The properties of these objects that we will need are presented in the following three lemmas, the first two of which parallel Lemma 3.3 and 3.4 of the last subsection.

The maximization problem (3.29) is posed over the set $\mathcal{A}$ of admissible L^1 densities (3.16) that naturally inherits a topology from the weak-$*$ topology of measures. However, the densities in $\mathcal{A}$, being all bounded above by $\varphi(\cos\sigma)\sin\sigma$, are equi-integrable and hence comprise a relatively compact set in the weak topology of $L^1([0,\pi])$. This means that the weak-$*$ topology of measures and the weak topology of L^1 coincide on $\mathcal{A}$. We will always consider $\mathcal{A}$ equipped with this topology. The key fact we will need is stated in the first lemma, the proof of which is omitted.

LEMMA 4.1 ($\mathcal{A}$-Compactness) *The set $\mathcal{A}$ is compact.*

The operator L defined by (4.2) and the functional Q defined by (4.3) are well behaved over the set $\mathcal{A}$. Indeed, given that η is in $\mathcal{A}$, it can be shown [23] that $\mathrm{L}\eta$ that is in the class of continuous, odd, 2π-periodic functions of σ, which we denote as $C_{\mathrm{odd}}(\mathbb{S})$ where $\mathbb{S} = \mathbb{R}/2\pi\mathbb{Z}$. Moreover, it is clear from (3.4) that for every $(x,\mathbf{t}) \in \mathbb{R}\times\mathbb{R}^\infty$, the function $\sigma \mapsto a(\sigma,x,\mathbf{t})$ is also in $C_{\mathrm{odd}}(\mathbb{S})$. Consequently, Q defined by (4.3) takes values in $\mathbb{R}$. Moreover, L and Q possess continuity properties that are stated in the next lemma.

LEMMA 4.2 ($\mathcal{A}$-Continuity)

(a) *The operator* $\mathrm{L} : \mathcal{A} \to C_{\mathrm{odd}}(\mathbb{S})$ *is continuous.*
(b) *The functional* $Q : \mathcal{A}\times\mathbb{R}\times\mathbb{R}^\infty \to \mathbb{R}$ *is continuous.*

PROOF: The hardest part is the proof of (a), which resembles that of theorem 3.4 in [28] and may be found in [23]. The proof of (b) follows from (a) and the continuity of a. □

Remark. By combining Lemma 4.1 with Lemma 4.2, it is seen that $\mathrm{L}\mathcal{A}$ is a compact subset of $C_{\mathrm{odd}}(\mathbb{S})$ and that for every $(x,\mathbf{t}) \in \mathbb{R}\times\mathbb{R}^\infty$, the map $\eta \mapsto Q(\eta;x,\mathbf{t})$ has a compact range and hence attains a maximum over $\mathcal{A}$. The existence of the maximum in (3.29), which has already been established through the limiting procedure of Theorem 3.2, is thereby re-established intrinsically.

The uniqueness of the maximum in (3.29) follows from the third lemma.

LEMMA 4.3 (Strict Concavity)

(a) *The quadratic form* $(\eta \mid \mathrm{L}\eta)$ *is negative definite over* $\mathrm{span}(\mathcal{A})$.
(b) *For every* $(x,\mathbf{t}) \in \mathbb{R}\times\mathbb{R}^\infty$*, the map* $\eta \mapsto Q(\eta;x,\mathbf{t})$ *is strictly concave over* $\mathcal{A}$.

PROOF: The hardest part is again the proof of (a), which here resembles that of theorem 3.7 in [28] and may be found in [23]. The proof of (b) follows directly from (a). □

With the above pieces in place, the main result of this subsection can now be presented.

THEOREM 4.4 (Uniqueness and Regularity) *For each* $(x, \mathbf{t}) \in \mathbb{R}\times\mathbb{R}^\infty$ *there exists a unique* $\eta^*(x, \mathbf{t})$ *in the admissible class* $\mathcal{A}$ *such that*

$$q(x, \mathbf{t}) = Q(\eta^*(x, \mathbf{t}); x, \mathbf{t})\,. \tag{4.4}$$

Moreover, this maximizing density satisfies the following:

(a) *The map* $(x, \mathbf{t}) \mapsto \eta^*(x, \mathbf{t})$ *is continuous from* $\mathbb{R}\times\mathbb{R}^\infty$ *into* $\mathcal{A}$ *equipped with the weak-*L^1 *topology.*
(b) *The map* $(x, \mathbf{t}) \mapsto \mathrm{L}\eta^*(x, \mathbf{t})$ *is continuous from* $\mathbb{R}\times\mathbb{R}^\infty$ *into* $C_{\mathrm{odd}}(\mathbb{S})$ *equipped with the uniform topology.*
(c) *The map* $(x, \mathbf{t}) \mapsto q(x, \mathbf{t})$ *is differentiable from* $\mathbb{R}\times\mathbb{R}^\infty$ *into* $\mathbb{R}$ *with its continuous partial derivatives given by*

$$\partial_x q(x, \mathbf{t}) = (\sin\sigma \mid \eta^*(x, \mathbf{t}))\,, \tag{4.5a}$$

$$\partial_{t_n} q(x, \mathbf{t}) = (\sin\sigma\cos^{n-1}\sigma \mid \eta^*(x, \mathbf{t})). \tag{4.5b}$$

PROOF: Lemma 4.3 states that the functional $Q(\,\cdot\,; x, \mathbf{t})$ is strictly concave over $\mathcal{A}$ for every $(x, \mathbf{t})$. This fact insures the uniqueness of the point in $\mathcal{A}$ at which the maximum in (3.29) is attained. Denote this point as $\eta^*(x, \mathbf{t})$.

To prove the continuity asserted in (a), let $(x, \mathbf{t})$ be any point and $\{(x_k, \mathbf{t}_k)\}$ any sequence converging to that point in $\mathbb{R}\times\mathbb{R}^\infty$. Because, by Lemma 4.1, $\mathcal{A}$ is compact, the sequence $\{\eta^*(x_k, \mathbf{t}_k)\}$ has cluster points, all of which lie in $\mathcal{A}$. Let η_* denote one such cluster point. Upon passing to a subsequence if necessary, part (b) of Lemma 4.2 implies

$$\lim_{k\to\infty} q(x_k, \mathbf{t}_k) = \lim_{k\to\infty} Q(\eta^*(x_k, \mathbf{t}_k); x_k, \mathbf{t}_k) = Q(\eta_*; x, \mathbf{t})\,. \tag{4.6}$$

On the other hand, the continuity of q implies

$$\lim_{k\to\infty} q(x_k, \mathbf{t}_k) = q(x, \mathbf{t})\,. \tag{4.7}$$

Comparing (4.6) and (4.7) shows that $Q(\eta_*; x, \mathbf{t}) = q(x, \mathbf{t})$, whereby we conclude that $\eta_* = \eta^*(x, \mathbf{t})$. Hence, the original sequence $\{\eta^*(x_k, \mathbf{t}_k)\}$, having $\eta^*(x, \mathbf{t})$ as the only cluster point, must converge to $\eta^*(x, \mathbf{t})$. The continuity asserted in (a) follows immediately, while that of (b) does so after invoking part (a) of Lemma 4.2.

Now turn to the differentiability (c). It suffices to establish (4.5b). Let $\mathbf{t}$ and $\mathbf{t}'$ differ only in the coordinate t_n. For every η in $\mathcal{A}$, a direct calculation then yields

$$Q(\eta; x, \mathbf{t}') - Q(\eta; x, \mathbf{t}) = (\sin\sigma\cos^{n-1}\sigma \mid \eta)(t_n' - t_n)\,. \tag{4.8}$$

However, because η^* maximizes Q, one has the general two-sided inequality

$$\begin{aligned} &Q(\eta^*(x, \mathbf{t}); x, \mathbf{t}') - Q(\eta^*(x, \mathbf{t}); x, \mathbf{t}) \\ &\quad\le q(x, \mathbf{t}') - q(x, \mathbf{t}) \le Q(\eta^*(x, \mathbf{t}'); x, \mathbf{t}') - Q(\eta^*(x, \mathbf{t}'); x, \mathbf{t})\,, \end{aligned} \tag{4.9}$$

which, when combined with (4.8), gives

$$\begin{aligned}(\sin\sigma\cos^{n-1}\sigma \mid \eta^*(x,\mathbf{t}))(t_n'-t_n)\\ \le q(x,\mathbf{t}')-q(x,\mathbf{t}) \le (\sin\sigma\cos^{n-1}\sigma \mid \eta^*(x,\mathbf{t}'))(t_n'-t_n)\,.\end{aligned} \tag{4.10}$$

The result now follows by the continuity of η^*. □

It will prove useful that the solution $\eta^*(x,\mathbf{t})$ of the maximization problem can be characterized in terms of variational conditions.

THEOREM 4.5 (Variational Conditions) *If $\eta \in \mathcal{A}$, then $\eta = \eta^*(x,\mathbf{t})$ if and only if η satisfies the variational conditions*

$$\eta = \begin{cases} 0 & \textit{where } a(\sigma,x,\mathbf{t}) + \mathrm{L}\eta < 0\,,\\ \varphi(\cos\sigma)\sin\sigma & \textit{where } a(\sigma,x,\mathbf{t}) + \mathrm{L}\eta > 0\,. \end{cases} \tag{4.11}$$

PROOF: The proof is fairly standard and follows closely that of theorem 3.12 in [28]. It uses the continuity of $a+\mathrm{L}\eta$ in σ to assert that the conditionals in (4.11) define open sets. □

4.2 The Limit of the Densities and Fluxes

Combining the differentiability result of Theorem 4.4 with the following elementary but nontrivial lemma will yield a strengthening in the sense of the convergence for the τ-functions in (3.28).

LEMMA 4.6 (Converging Derivatives) *Let $\{h_n\}$ be a sequence of differentiable convex functions over $\mathbb{R}^\infty$ such that $h_n(\mathbf{x}) \to h(\mathbf{x})$ uniformly over compact subsets of $\mathbf{x} \in \mathbb{R}^\infty$ and h is differentiable. Let $\partial_s = \dot{\mathbf{x}} \cdot \partial_{\mathbf{x}}$ denote the directional derivative of $\mathbf{x}$ in a given direction $\dot{\mathbf{x}} \in \mathbb{R}^\infty$. Then*

$$\partial_s h_n(\mathbf{x}) \to \partial_s h(\mathbf{x}) \tag{4.12}$$

uniformly over compact subsets of $\mathbf{x} \in \mathbb{R}^\infty$.

In particular, we can now give the main result of this section.

THEOREM 4.7 (Limit of Densities and Fluxes) *The limit of the τ-function $\tau^\hbar$ is given by*

$$\lim_{\hbar\to 0} \hbar^2 \partial_{t_n} \log\tau^\hbar(x,\mathbf{t}) = \partial_{t_n} q(x,\mathbf{t}) = (\sin\sigma\cos^{n-1}\sigma \mid \eta^*(x,\mathbf{t})) \tag{4.13}$$

uniformly over compact subsets of $(x,\mathbf{t}) \in \mathbb{R}\times\mathbb{R}^\infty$. The densities $\rho_n^\hbar$ and fluxes $\mu_{m,n}^\hbar$ have the distributional limits

$$\mathcal{D}'(dx)\text{–}\lim_{\hbar\to 0} \rho_{n-1}^\hbar = \partial_x(\sin\sigma\cos^{n-1}\sigma \mid \eta^*)\,, \tag{4.14a}$$

$$\mathcal{D}'(dt_m)\text{–}\lim_{\hbar\to 0} \mu_{m,n}^\hbar = -\partial_{t_m}(\sin\sigma\cos^{n-1}\sigma \mid \eta^*)\,. \tag{4.14b}$$

PROOF: Assertion (4.13) follows directly from equation (3.28) of Theorem 3.6 (τ-function limit) and Theorem 4.5 (uniqueness and regularity) upon applying Lemma 4.6 (converging derivatives). Assertion (4.14) then follows by the usual integration-by-parts argument. □

Remark. The significance of (4.14) is that one can now pass to the limit in the local conservation laws

$$\partial_{t_m} \rho^{\hbar}_{n-1} + \partial_x \mu^{\hbar}_{m,n} = 0 \quad \text{for } m, n = 1, 2, \ldots . \tag{4.15}$$

The convergence in (4.14a) can be strengthened to at least $w\text{-}L^1_{\text{loc}}(dx)$ for the mass and momentum densities

$$\rho^{\hbar}_0 = |\Psi^{\hbar}|^2 - 1\,, \qquad \rho^{\hbar}_1 = -i\frac{\hbar}{2}(\overline{\Psi^{\hbar}}\partial_x\, \Psi^{\hbar} - \Psi^{\hbar}\partial_x\, \overline{\Psi^{\hbar}})\,. \tag{4.16}$$

By convergence in $w\text{-}L^1_{\text{loc}}(dx)$ we mean convergence upon integration against test functions that are bounded, Lebesgue-measurable functions with compact support.

THEOREM 4.8 (Mass and Momentum Densities Limit) *There are $w\text{-}L^1_{\text{loc}}(dx)$ limits to the densities $\rho^{\hbar}_0$ and $\rho^{\hbar}_1$*

$$w\text{-}L^1_{\text{loc}}(dx)\text{–}\lim_{\hbar\to 0} \rho^{\hbar}_0 = \partial_x\left(\sin\sigma \mid \eta^*\right), \tag{4.17a}$$

$$w\text{-}L^1_{\text{loc}}(dx)\text{–}\lim_{\hbar\to 0} \rho^{\hbar}_1 = \partial_x\left(\sin\sigma\cos\sigma \mid \eta^*\right). \tag{4.17b}$$

PROOF: All that needs to be done is to show the approximating densities are locally equi-integrable. The density $\rho^{\hbar}_0$ satisfies the bounds

$$-1 < \left|\Psi^{\hbar}\right|^2 - 1 < 0\,, \tag{4.18}$$

so local equi-integrability follows directly. By using the Schwarz inequality, the bound (4.18), and definition (1.28) of $H^{\hbar}_2$ for an arbitrary measurable $A \subset \mathbb{R}$, the densities $\rho^{\hbar}_1$ satisfy

$$\begin{aligned} \int_A |\rho^{\hbar}_1| dx &\le \left(\int_A |\Psi^{\hbar}|^2 dx\right)^{\frac{1}{2}} \left(\int_A \hbar^2 |\partial_x \Psi^{\hbar}|^2 dx\right)^{\frac{1}{2}} \\ &\le \left(\int_A dx\right)^{\frac{1}{2}} \left(2H^{\hbar}_2\right)^{\frac{1}{2}}. \end{aligned} \tag{4.19}$$

By construction (2.11) and by (2.37) of Lemma 2.5, the value of $H^{\hbar}_2$ satisfies

$$H^{\hbar}_2 = 2\hbar \sum_{j\in J^{\hbar}} \sin\sigma_j \cos^2\sigma_j \le \hbar N^{\hbar} \tag{4.20}$$

and is therefore, by (2.6), uniformly bounded from above. The local equi-integrability of $\rho^{\hbar}_1$ is then implied by (4.19), thereby completing the proof of the theorem. □

5 The Limiting Dynamics

5.1 Construction of the Maximizer

In this section, we construct the maximizer locally, and use it to describe the "pre-" and "postbreaking" dynamics. The construction starts from the characterization of the maximizer η^* as the solution of the variational conditions (4.11), which we reformulate as follows: Given $\eta^*(x,\mathbf{t})$, we define the sets $I^0(x,\mathbf{t})$, $I^+(x,\mathbf{t})$, and $I^-(x,\mathbf{t})$ by

$$
\begin{aligned}
I^0(x,\mathbf{t}) &\equiv \{\sigma : a(\sigma,x,\mathbf{t}) + \mathrm{L}\eta^*(\sigma,x,\mathbf{t}) = 0\}\,,\\
I^+(x,\mathbf{t}) &\equiv \{\sigma : a(\sigma,x,\mathbf{t}) + \mathrm{L}\eta^*(\sigma,x,\mathbf{t}) > 0\}\,,\\
I^-(x,\mathbf{t}) &\equiv \{\sigma : a(\sigma,x,\mathbf{t}) + \mathrm{L}\eta^*(\sigma,x,\mathbf{t}) < 0\}\,.
\end{aligned}
\tag{5.1}
$$

By the continuity of $\mathrm{L}\eta^*$ asserted in Theorem 4.4(b), the sets $I^+(x,\mathbf{t})$ and $I^-(x,\mathbf{t})$ are open and are separated by $I^0(x,\mathbf{t})$. The variational conditions (4.11) are then equivalent to:

Property 1.

$$
\begin{aligned}
\mathrm{L}\eta^* &= -a(\sigma,x,\mathbf{t}) &&\text{on } I^0(x,\mathbf{t})\,,\\
\eta^* &= 0, &&\text{on } I^-(x,\mathbf{t})\,,\\
\eta^* &= \varphi(\cos\sigma)\,\sin\sigma &&\text{on } I^+(x,\mathbf{t})\,.
\end{aligned}
\tag{5.2}
$$

Property 1 depends on the initial data implicitly through the sets I^0, I^+, and I^- and explicitly through $\chi(\cos\sigma)$, which enters through definition (3.4) of $a(\sigma,x,\mathbf{t})$.

From this property, we derive two additional properties satisfied by η^*. Let $I(x,\mathbf{t})$ denote the interior of $I^0(x,\mathbf{t})$. We will subsequently suppose that $I(x,\mathbf{t})$ is a finite union of disjoint open intervals. Now by formally differentiating (5.2) with respect to t_n, we obtain the following:

Property 2.

$$
\begin{aligned}
\mathrm{L}\eta^*_{t_n} &= -\cos^{n-1}\sigma\sin\sigma &&\text{on } I(x,\mathbf{t})\,,\\
\eta^*_{t_n} &= 0 &&\text{off } I(x,\mathbf{t})\,.
\end{aligned}
\tag{5.3}
$$

By recalling that $\partial_{t_1}\eta^* + \partial_x\eta^* = 0$, we can identify η^*_x with $-\eta^*_{t_1}$. Property 2 now depends on the initial data only implicitly through the boundary points of $I(x,\mathbf{t})$.

At this point we notice that if η^* is extended as an odd function of σ, then the operator L is related to the Hilbert transform H on the unit disk through

$$\mathrm{L}\eta(\sigma) = \int_0^\sigma \mathrm{H}\eta(\sigma')\,d\sigma'\,, \tag{5.4a}$$

where H is given by

$$\mathrm{H}\eta(\sigma) = \frac{1}{2\pi}\,\mathrm{PV}\int_{-\pi}^{\pi}\cot\left(\frac{\sigma-\sigma'}{2}\right)\eta(\sigma')\,d\sigma'\,. \tag{5.4b}$$

By formally differentiating (5.3) with respect to σ, we obtain the following Riemann-Hilbert problem:

Property 3.

$$\begin{aligned} \mathrm{H}\eta^*_{t_n} &= -\partial_\sigma(\cos^{n-1}\sigma\ \sin\sigma) \quad &\text{on } I(x,\mathbf{t})\,, \\ \eta^*_{t_n} &= 0 &\text{off } I(x,\mathbf{t})\,, \end{aligned} \tag{5.5}$$

which again depends on the initial data only implicitly through the boundary points of $I(x,\mathbf{t})$.

The construction of the minimizer $\eta^*(x,\mathbf{t})$ proceeds from the above three properties as follows: Property 3 enables us to initiate the construction of η^*_x and each $\eta^*_{t_n}$ in terms of the real and imaginary parts of Hardy functions. Property 2 is then used to determine certain free constants in that construction, after which the only parameters are the boundary points of the set $I(x,\mathbf{t})$. The local dynamics of these boundary points under the n^{th} flow is then recovered by imposing the compatibility conditions

$$\partial_{t_n}\,\eta^*_x = \partial_x\,\eta^*_{t_n}\,. \tag{5.6}$$

We remark that once the local dynamics has been so imposed, the integrable structure of the underlying NLS hierarchy then ensures that

$$\partial_{t_n}\,\eta^*_{t_m} = \partial_{t_m}\,\eta^*_{t_n}\,.$$

The evolution equations (5.6), properly initialized using Property 1, uniquely determine the boundary points of $I(x,\mathbf{t})$. Then, because $\mathrm{L}\eta^*$ can be bounded uniformly in $(x,\mathbf{t})$ and because $a(\sigma,x,\mathbf{t}) \to -\infty$ as $x\to\infty$, we have

$$a(\sigma,x,\mathbf{t}) + \mathrm{L}\eta^*(\sigma) \to -\infty \quad \text{as } x\to\infty.$$

Thus, for sufficiently positive x, equation (5.1) shows that $\sigma\in I^-(x,\mathbf{t})$, in which case (5.2) of Property 1 shows $\eta^*(\sigma,x,\mathbf{t})=0$. We then construct η^* from η^*_x as

$$\eta^*(\sigma,x,\mathbf{t}) = -\int_x^\infty \eta^*_x(\sigma,y,\mathbf{t})\,dy\,. \tag{5.7}$$

Finally, we check that this candidate for η^* satisfies the variational conditions (4.11), whereby it is indeed the maximizer. With this plan in mind, we will illustrate the construction for the special case of the NLS flow, which is associated with $t=t_2$.

5.2 Prebreaking NLS Dynamics

We turn first to the construction of the maximizer $\eta^*(x,t)$ for sufficiently short times $0\le t<t_b$, where we anticipate t_b will be the time at which the classical solution of the Euler initial-value problem (1.14) breaks down. Because (1.14) has two unknowns, we further anticipate that, over these times, the set I consists of two disjoint intervals of the form

$$I = (0,\widehat{\sigma}_+)\cup(\widehat{\sigma}_-,\pi)\,, \tag{5.8}$$

where the endpoints $\widehat{\sigma}_+$ and $\widehat{\sigma}_-$ satisfy $0 < \widehat{\sigma}_+ < \widehat{\sigma}_- < \pi$. We extend I symmetrically as

$$I = (-\pi, -\widehat{\sigma}_-) \cup (-\widehat{\sigma}_+, \widehat{\sigma}_+) \cup (\widehat{\sigma}_-, \pi)\,.$$

We also anticipate that $\widehat{\sigma}_+$ and $\widehat{\sigma}_-$ are continuous functions of (x,t) over $\mathbb{R}\times[0,t_b]$ and are differentiable over $\mathbb{R}\times[0,t_b)$.

For a given n, solutions of the Riemann-Hilbert problem (5.5) of Property 3 are found by seeking Hardy functions $f(z)$ that are analytic over the complex disk $\{z : |z| < 1\}$ and satisfy

$$\begin{aligned} &\lim_{r\to 1^-} \operatorname{Re} f(re^{i\sigma}) = 0 && \text{off } I\,, \\ &\lim_{r\to 1^-} \operatorname{Im} f(re^{i\sigma}) = \partial_\sigma(\cos^{n-1}\sigma \sin\sigma) && \text{on } I\,. \end{aligned} \tag{5.9}$$

Then, for every such f,

$$\eta^*_{t_n}(\sigma, x, t) = -\lim_{r\to 1^-} \operatorname{Re} f(re^{i\sigma})\,. \tag{5.10}$$

We now use this procedure to construct $\eta^*_{t_1}$. For every $D \in \mathbb{C}$ the function

$$f(z) = -\frac{z^2-1}{2iz}\,\frac{\dfrac{z^2+1}{2z} - D}{\sqrt{\left(\dfrac{z^2+1}{2z} - \cos\widehat{\sigma}_+\right)\left(\dfrac{z^2+1}{2z} - \cos\widehat{\sigma}_-\right)}} + i\frac{z^2+1}{2z}\,, \tag{5.11}$$

is analytic in the unit disk and limits to

$$= \lim_{r\to 1^-} f(re^{i\sigma}) = -\sin\sigma\,\frac{\cos\sigma - D}{\mathcal{R}(\sigma)} + i\cos\sigma\,,$$

where

$$\mathcal{R}(\sigma) = \sqrt{(\cos\sigma - \cos\widehat{\sigma}_+)(\cos\sigma - \cos\widehat{\sigma}_-)} \tag{5.12}$$

with the branch determined by $\mathcal{R}(\sigma) > 0$ for $0 < \sigma < \widehat{\sigma}_+$. By inspection, this $f(z)$ solves (5.9) provided D is real. Because $\eta^*_x = -\eta^*_{t_1}$, by (5.10) we obtain

$$\eta^*_x = \begin{cases} -\dfrac{\cos\sigma - D}{\mathcal{R}(\sigma)}\sin\sigma & \text{on } I, \\ 0 & \text{off } I; \end{cases} \tag{5.13a}$$

$$\mathrm{H}\eta^*_x = \begin{cases} \cos\sigma & \text{on } I, \\ \dfrac{\cos\sigma - D}{\sqrt{(\cos\sigma - \cos\widehat{\sigma}_+)(\cos\widehat{\sigma}_- - \cos\sigma)}}\sin\sigma + \cos\sigma & \text{off } I. \end{cases} \tag{5.13b}$$

Indeed, the last term on the right side of (5.11) was chosen to recover the $\cos\sigma$ behavior of $\mathrm{H}\eta^*_x$ on I, while the first term was chosen to cancel the singularity at $z = 0$ and recover the proper behavior of $\mathrm{H}\eta^*_x$ off I.

Upon integrating (5.13b) with respect to σ, one sees that (5.3) of Property 2 will be satisfied if and only if

$$\int_{\widehat{\sigma}_+}^{\widehat{\sigma}_-} \frac{\cos\sigma - D}{\sqrt{(\cos\sigma - \cos\widehat{\sigma}_+)(\cos\widehat{\sigma}_- - \cos\sigma)}} \sin\sigma \, d\sigma = 0 \,. \tag{5.14}$$

This condition uniquely determines that D is given by

$$D = \frac{\cos\widehat{\sigma}_+ + \cos\widehat{\sigma}_-}{2} \,.$$

One may construct η_t^* in a similar fashion, obtaining from Property 1

$$\eta_t^* = \begin{cases} -\dfrac{2\cos^2\sigma - 2E_1\cos\sigma - E_2}{\mathcal{R}(\sigma)} \sin\sigma & \text{on } I, \\ 0 & \text{off } I, \end{cases} \tag{5.15}$$

where E_1 and E_2 are determined by Property 2 to be

$$E_1 = \frac{\cos\widehat{\sigma}_+ + \cos\widehat{\sigma}_-}{2} \,, \qquad E_2 = \left(\frac{\cos\widehat{\sigma}_+ - \cos\widehat{\sigma}_-}{2}\right)^2 \,. \tag{5.16}$$

Thus, η_x^* and η_t^* are given by (5.13a) and (5.15) in terms of $\widehat{\sigma}_\pm$, the boundary points of I. As yet, we have no information about the (x, t) dependence of these points.

Upon applying the compatibility conditions (5.6), we find the local dynamics is governed by

$$\partial_t\ell_+ + \tfrac{1}{2}(3\ell_+ + \ell_-)\partial_x\ell_+ = 0 \,, \qquad \partial_t\ell_- + \tfrac{1}{2}(3\ell_- + \ell_+)\partial_x\ell_- = 0 \,, \tag{5.17}$$

where $\ell_\pm \equiv \cos\widehat{\sigma}_\pm$. We have encountered equations (5.17) earlier as the Riemann invariant form (1.16a) of the conjectured semiclassical limit dynamics (1.14a). It is therefore natural to guess that the $\{\ell_\pm\}$ of (5.17) should be exactly the Riemann invariants $\{\hat{\lambda}_\pm\}$ of (1.16a) and should therefore, by (1.16b), be initialized as

$$\ell_\pm(x, 0) = \hat{\lambda}_\pm(x, 0) = r_\pm(x) \equiv \tfrac{1}{2}\partial_x S(x) \pm A(x) \,, \tag{5.18}$$

where $A(x)$ and $S(x)$ give the amplitude and phase of the unmodified initial data of (1.13c), which had the form

$$A(x) \exp\left(\frac{i}{\hbar} S(x)\right) . \tag{5.19}$$

This initialization uniquely determines the set I, whose form was anticipated in (5.8). Specifically, one has

$$\widehat{\sigma}_\pm = \cos^{-1}(\hat{\lambda}_\pm) \,, \tag{5.20}$$

where $\{\hat{\lambda}_\pm\}$ is the classical solution of (1.16) extended by continuity up to time t_b.

We next construct a candidate for the maximizer $\eta^*(\sigma, x, t)$ from $\{\hat{\lambda}_\pm(x, t)\}$. To do this, we introduce the functions $x^+(\sigma, t)$ and $x^-(\sigma, t)$ by

$$\begin{aligned} \hat{\lambda}_+[x^-(\sigma, t)] = \hat{\lambda}_+[x^+(\sigma, t)] = \cos\sigma \,, &\quad \lambda_{\min} < \cos\sigma < 1 \,, \\ \hat{\lambda}_-[x^-(\sigma, t)] = \hat{\lambda}_-[x^+(\sigma, t)] = \cos\sigma, &\quad -1 < \cos\sigma < \lambda_{\max} \,, \end{aligned} \tag{5.21}$$

where $\lambda_{\min}$ and $\lambda_{\max}$ were defined in (2.1). Now we consider η_x^* and η_t^* to be given by equations (5.13a) and (5.15) with $\widehat{\sigma}_\pm$ given by (5.20). Since $\partial_t \eta_x^* = \partial_x \eta_t^*$, we can integrate (5.7) to obtain

$$\eta^*(\sigma; x,t) = -\int_x^\infty \eta_x^*(y,t)\,dy$$

(5.22)
$$= \begin{cases} 0 & \text{for } x^+(\sigma,t) \le x, \\ \displaystyle\int_x^{x^+} \frac{\mathcal{P}(\sigma)}{\mathcal{R}(\sigma)}\,dy & \text{for } x^-(\sigma,t) < x < x^+(\sigma,t), \\ \varphi(\sigma)\sin\sigma & \text{for } x \le x^-(\sigma,t), \end{cases}$$

where $\mathcal{P}(\sigma) \equiv \sin\sigma(\cos\sigma - D)$ and $\varphi(\sigma)$ was defined in (2.3). In writing (5.22), we have used the fact that

(5.23)
$$\varphi(\sigma)\sin\sigma = \int_{x^-}^{x^+} \frac{\mathcal{P}(\sigma)}{\mathcal{R}(\sigma)}\,dy\,,$$

which may be checked directly using (5.21).

In order to check that this η^* is indeed the unique maximizer, we use an explicit calculation similar to those indicated above to show that

$$a + \mathrm{L}\eta^* = \begin{cases} -\displaystyle\int_{x^+}^{x}\int_0^\sigma \operatorname{Im}\frac{\mathcal{P}(\sigma')}{\mathcal{R}(\sigma')}\,d\sigma' dy < 0 & \text{for } x^+(\sigma,t) \le x, \\ 0 & \text{for } x^-(\sigma,t) < x < x^+(\sigma,t), \\ \displaystyle\int_{x}^{x^-}\int_0^\sigma \operatorname{Im}\frac{\mathcal{P}(\sigma')}{\mathcal{R}(\sigma')}\,d\sigma' dy > 0 & \text{for } x \le x^-(\sigma,t). \end{cases}$$

This calculation, together with (5.20), establishes that the variational condition (4.11) is indeed satisfied; thus, η^* as given by (5.20) is the unique maximizer, and assumption (5.8) about the structure of the set I has been verified. In this manner we have established the following:

THEOREM 5.1 *For* $0 \le t \le t_b$, *define* $\{\hat\lambda_\pm(x,t)\}$ *as the unique solution of the initial value problem* (1.16):

$$\partial_t\hat\lambda_+ + \tfrac{1}{2}(3\hat\lambda_+ + \hat\lambda_-)\partial_x\hat\lambda_+ = 0\,, \qquad \partial_t\hat\lambda_- + \tfrac{1}{2}(\hat\lambda_+ + 3\hat\lambda_-)\partial_x\hat\lambda_- = 0\,,$$

with the initial conditions

$$\hat\lambda_\pm(x,0) = r_\pm(x) \equiv \tfrac{1}{2}\partial_x S(x) \pm A(x)\,.$$

Let $\Psi^\hbar(x,t)$ *be the solution of the NLS equation for the modified initial data* (2.11). *Then the weak limits of all the conserved densities are given by*

(5.24)
$$\mathcal{D}'(dx)\text{–}\lim_{\hbar\to 0}\rho_{n-1}^\hbar = (\sin\sigma\cos^{n-1}\sigma \mid \eta_x^*)$$

for $n = 1, 2, \dots$, uniformly in $t \in [0, t_b]$ with η_x^* given in terms of $\{\hat\lambda_\pm\}$ by (5.20). *Moreover, the limits for $n = 1$ and 2 hold in the weak-L^1_{loc} topology uniformly over $[0, t_b]$.*

Remark. The expressions for the limiting densities given for $t \le t_b$ by the right side of (5.24) can be computed explicitly. These integrals may be expressed in terms of the Hardy function $f(z)$ of equation (5.11) as contour integrals around the unit circle $\mathbb{S}^1$ and evaluated in terms of the residue at $z = 0$. To do this, use the fact that the expansion of $f(z)$ about $z = 0$ is

$$f(z) = i\left[\left(1 - \left(\frac{\hat\lambda_+ - \hat\lambda_-}{2}\right)^2\right) z - 2(\hat\lambda_+ + \hat\lambda_-)\left(\frac{\hat\lambda_+ - \hat\lambda_-}{2}\right)^2 z^2 + O(z^3)\right].$$

We obtain

$$\begin{aligned}
w\text{-}L^1_{\mathrm{loc}}(dx)\text{–}\lim_{\hbar\to 0}\rho_0^\hbar &= (\sin\sigma \mid \eta_x^*)\\
&= \frac{2}{\pi}\int_I \frac{\sin^2\sigma(\cos\sigma - D)}{\mathcal{R}(\sigma)}\,d\sigma\\
&= \mathrm{Re}\,\frac{1}{\pi}\int_{\mathbb{S}^1}\frac{1-z^2}{2z^2}\,f(z)dz\\
&= \left(\frac{\hat\lambda_+ - \hat\lambda_-}{2}\right)^2 - 1\,,\\
w\text{-}L^1_{\mathrm{loc}}(dx)\text{–}\lim_{\hbar\to 0}\rho_1^\hbar &= (\sin\sigma\cos\sigma \mid \eta_x^*)\\
&= \frac{2}{\pi}\int_I \frac{\sin^2\sigma\ \cos\sigma(\cos\sigma - D)}{\mathcal{R}(\sigma)}\,d\sigma\\
&= \mathrm{Re}\,\frac{1}{\pi}\int_{\mathbb{S}^1}\frac{1-z^4}{4z^3}\,f(z)\,dz\\
&= (\hat\lambda_+ + \hat\lambda_-)\left(\frac{\hat\lambda_+ - \hat\lambda_-}{2}\right)^2.
\end{aligned}$$

Similarly, one can show that

$$\mathcal{D}'(dx)\text{–}\lim_{\hbar\to 0}\rho_2^\hbar = \left[\frac{1-(\hat\lambda_+ - \hat\lambda_-)^2}{4}\right]^2 + \frac{(\hat\lambda_+ + \hat\lambda_-)^2(\hat\lambda_+ - \hat\lambda_-)^2}{4}\,.$$

Remark. Recall that the above limits are established for the densities corresponding to the modified initial data (2.11). For this prebreaking regime, Grenier [19] has used a clever decomposition of the NLS solution together with classical PDE methods to show that the limiting dynamics corresponding to the unmodified initial data (1.8b) is governed by (1.14) and hence (1.16). The above explicit calculation shows that the same limiting dynamics emerges from the modified initial data.

In the prebreaking regime, the weak-$L^1_{\rm loc}$ convergence of Theorem 4.8 can be strengthened to strong-L^1 convergence. A typical way to prove that a weakly convergent sequence is actually strongly convergent is to establish that, in addition to weak convergence, the norms converge. Here we replace the norm with a convex functional constructed from the conserved densities. Then, using the conservation of the densities, we show that convergence at the initial time $t = 0$, weak convergence at time t, and convexity together imply that certain convex functionals converge. Strong convergence at time t follows immediately. We begin with an elementary technical lemma.

LEMMA 5.2 *Let $\{V_n\}$ be a sequence of L^1 functions, $V_n : \mathbb{R} \to \mathbb{R}^D$, which converge weakly in $L^1_{\rm loc}$ to a limit V:*

$$w\text{-}L^1_{\rm loc}(dx)\text{–}\lim_{n\to\infty} V_n = V\,.$$

Let $s : \mathbb{R}^D \to \mathbb{R}_+$ denote a positive, strictly convex function for which $s(V)$ and each $s(V_n)$ belongs to L^1 and such that

$$\lim_{n\to\infty}\int s(V_n)\,dx = \int s(V)\,dx\,.$$

Then one has the strong-$L^1_{\rm loc}$ limit

$$L^1_{\rm loc}(dx)\text{–}\lim_{n\to\infty} V_n = V\,.$$

With this lemma, we can strengthen the convergence of the semiclassical limit for $0 \le t \le t_b$ and establish the following:

THEOREM 5.3 *For $0 \le t \le t_b$, the semiclassical limits of the first two conserved densities are strong limits in $L^1_{\rm loc}$:*

$$L^1_{\rm loc}(dx)\text{–}\lim_{\hbar\to 0}\rho_k^\hbar = (\eta_x^* \mid \sin\sigma\cos^k\sigma) \quad \textit{for } k = 0, 1, \tag{5.25}$$

uniformly over $[0, t_b]$.

To prove this theorem, we use $\rho^\hbar$ and $\mu^\hbar$ defined by

$$\rho^\hbar \equiv |\Psi^\hbar|^2 = \rho_0^\hbar + 1\,, \qquad \mu^\hbar \equiv -i\frac{\hbar}{2}(\overline{\Psi^\hbar}\partial_x\Psi^\hbar - \Psi^\hbar\partial_x\overline{\Psi^\hbar}) = \rho_1^\hbar\,.$$

We will apply Lemma 5.2 to the function

$$s(\rho,\mu) \equiv \frac{\mu^2}{2\rho} + \frac{1}{2}(\rho - 1)^2\,,$$

which is a convex function of $V = (\rho,\mu)^{\mathsf{T}}$. We then define the functional S by

$$S(\rho(t),\mu(t)) \equiv \int s(\rho(x,t),\mu(x,t))\,dx\,.$$

By (1.28) it is related to the energy H_2 by

$$S(\rho,\mu) \le H_2(\rho,\mu) = \int\left(s(\rho,\mu) + \frac{\hbar^2}{8}\frac{|\partial_x\rho|^2}{\rho}\right)dx\,. \tag{5.26}$$

The proof proceeds in three steps.

First, one recalls the general fact that if $(\rho^\hbar(t), \mu^\hbar(t))$ converges weakly to (ρ, μ) and if $s(\rho, \mu)$ is convex, then

$$S(\rho(t), \mu(t)) \leq \liminf_{\hbar \to 0} S(\rho^\hbar(t), \mu^\hbar(t)) \,. \tag{5.27}$$

Second, by the fact H_2 is conserved and by (2.37) of Lemma 2.5, from (5.26) we have

$$\begin{aligned} S(\rho^\hbar(t), \mu^\hbar(t)) &\leq H_2(\rho^\hbar(t), \mu^\hbar(t)) \\ &= H_2(\rho^\hbar(0), \mu^\hbar(0)) \\ &= 2\hbar \sum_{j \in J^\hbar} \sin\sigma_j \cos^2\sigma_j \equiv H_2^\hbar \,. \end{aligned} \tag{5.28}$$

The limiting density (2.4) of the $\lambda_j = \cos\sigma_j$ allows us to compute

$$\begin{aligned} \lim_{\hbar \to 0} H_2^\hbar &= \frac{2}{\pi} \int_0^\pi \sin\sigma \cos^2\sigma \varphi(\cos\sigma) \sin\sigma \, d\sigma \\ &= \frac{2}{\pi} \int_{-1}^{\lambda_{\max}} \lambda^2 \sqrt{1-\lambda^2}\, \varphi(\lambda)\, d\lambda + \frac{2}{\pi} \int_{\lambda_{\min}}^1 \lambda^2 \sqrt{1-\lambda^2}\, \varphi(\lambda)\, d\lambda \\ &= \frac{2}{\pi} \int_{-\infty}^{\infty} \left[\int_{-1}^{r_-(x)} \sqrt{\frac{1-\lambda^2}{(\lambda - r_+)(\lambda - r_-)}} \left(\lambda - \frac{1}{2}(r_+ + r_-) \right) \lambda^2 \, d\lambda \right. \\ &\qquad \left. + \int_{r_+(x)}^1 \sqrt{\frac{1-\lambda^2}{(\lambda - r_+)(\lambda - r_-)}} \left(\lambda - \frac{1}{2}(r_+ + r_-) \right) \lambda^2 \, d\lambda \right] dx \\ &= \frac{2}{\pi} \int_{-\infty}^{\infty} \int_{-1}^1 \mathrm{Re} \left[\sqrt{\frac{1-\lambda^2}{(\lambda - r_+)(\lambda - r_-)}} \left(\lambda - \frac{1}{2}(r_+ + r_-) \right) \lambda^2 \right] d\lambda \, dx \,. \end{aligned}$$

The inner integral in this last expression may be evaluated by contour integration to obtain

$$\lim_{\hbar \to 0} H_2^\hbar = S\big(\rho(0), \mu(0)\big). \tag{5.29}$$

Upon collecting (5.27), (5.28), and (5.29), one finds that

$$\begin{aligned} S\big(\rho(t), \mu(t)\big) &\leq \liminf_{\hbar \to 0} S\big(\rho^\hbar(t), \mu^\hbar(t)\big) \\ &\leq \limsup_{\hbar \to 0} S\big(\rho^\hbar(t), \mu^\hbar(t)\big) \leq S\big(\rho(0), \mu(0)\big) \,, \end{aligned} \tag{5.30}$$

which is valid for all t.

The third step begins by noting that so long as $(\rho(x,t), \mu(x,t))$ is a solution of the hyperbolic system (1.14a), which will be the case up to the break-time t_b, the integral S is an invariant. Hence, one has

$$S\big(\rho(0), \mu(0)\big) = S\big(\rho(t), \mu(t)\big) \quad \text{for } 0 \leq t \leq t_b,$$

whereby (5.30) implies

$$\lim_{\hbar\to 0} S\big(\rho^{\hbar}(t), \mu^{\hbar}(t)\big) = S\big(\rho(t), \mu(t)\big)\,.$$

This, together with Lemma 5.2, establishes the theorem.

5.3 Postbreaking NLS Dynamics

In this section we describe the semiclassical (weak) limits of the densities for $t > t_b$. We begin with the assumption that the set I consists of a finite union of $n+1$ disjoint intervals:

$$I = (0, \widehat{\sigma}_1) \cup (\widehat{\sigma}_2, \widehat{\sigma}_3) \cup \cdots \cup (\widehat{\sigma}_{2n}, \pi)\,,$$

where $0 < \widehat{\sigma}_1 < \widehat{\sigma}_2 < \cdots < \widehat{\sigma}_{2n} < \pi$. Further, we denote by Λ the set of positive endpoints of I,

$$\Lambda = \{\widehat{\sigma}_1, \widehat{\sigma}_2, \ldots, \widehat{\sigma}_{2n}\}\,.$$

Next, in terms of Λ, we solve the Riemann-Hilbert problem (5.5) of Property 3 in Section 5.1. First, we change variables $\sigma \mapsto z \equiv \exp(i\sigma)$, $z_n = \exp(i\widehat{\sigma}_n)$, and with the natural abuse of notation, introduce the radical $\mathcal{R}(z)$:

$$\mathcal{R}^2(z) \equiv \prod_{m=1}^{2n} \left[\frac{z}{(1+z)^2} - \frac{z_m}{(1+z_m)^2}\right]. \tag{5.31}$$

Here $\mathcal{R}(\exp(i\sigma)) > 0$ for $\sigma \in (0, \widehat{\sigma}_1)$ and $i\mathcal{R}(\exp(i\sigma)) < 0$ for $\sigma \in (\widehat{\sigma}_1, \widehat{\sigma}_2)$. In addition, we introduce the functions $\mathcal{P}(z)$ and $\mathcal{Q}(z)$:

$$\mathcal{P}(z) = -\frac{1}{2}\mathcal{R}(0)\left\{\frac{1-z^2}{z} + \sum_{m=1}^{n} \alpha_m \frac{1-z}{1+z}\left[\frac{z}{(1+z)^2}\right]^{m-1}\right\}, \tag{5.32a}$$

$$\begin{aligned}\mathcal{Q}(z) = -\frac{1}{2}\mathcal{R}(0)\Bigg\{&\left(\frac{\mathcal{R}'(0)}{\mathcal{R}(0)} - 2\right)\frac{1-z^2}{z} + \frac{1-z^2}{z^2} + \frac{1-z^2}{z^2}(1+z)^2 \\ &+ \sum_{m=1}^{n} \beta_m \frac{1-z}{1+z}\left[\frac{z}{(1+z)^2}\right]^{m-1}\Bigg\},\end{aligned} \tag{5.32b}$$

where $\mathcal{R}'(0) = \partial_z \mathcal{R}(z)\big|_{z=0}$ and the coefficients $\{\alpha_m,\, m = 1, 2, \ldots, n\}$ and $\{\beta_m,\, m = 1, 2, \ldots n\}$ are the unique real constants defined in terms of $\Lambda = \{\widehat{\sigma}_1, \ldots, \widehat{\sigma}_{2n}\}$ by

$$\int_{\widehat{\sigma}_{2k-1}}^{\widehat{\sigma}_{2k}} \frac{\mathcal{P}(e^{i\sigma})}{\mathcal{R}(e^{i\sigma})}\, d\sigma = 0, \tag{5.33a}$$

$$\int_{\widehat{\sigma}_{2k-1}}^{\widehat{\sigma}_{2k}} \frac{\mathcal{Q}(e^{i\sigma})}{\mathcal{R}(e^{i\sigma})}\, d\sigma = 0, \qquad k = 1, 2, \ldots, n. \tag{5.33b}$$

In terms of these ingredients, we express our candidates for η_x^* and η_t^* that satisfy properties 2 and 3:

$$\eta_x^* = -\operatorname{Re}\left(\frac{\mathcal{P}(e^{i\sigma})}{\mathcal{R}(e^{i\sigma})}\right), \tag{5.34a}$$

$$\eta_t^* = \operatorname{Re}\left(\frac{\mathcal{Q}(e^{i\sigma})}{\mathcal{R}(e^{i\sigma})}\right). \tag{5.34b}$$

Next, a direct calculation shows that $\partial_t \eta_x^* = \partial_x \eta_t^*$ if and only if

$$\partial_t \hat{\lambda}_k + V_k(\Lambda)\partial_x \hat{\lambda}_k = 0\,, \quad k = 1, 2, \ldots, 2n,$$

where $\hat{\lambda}_k = \cos \widehat{\sigma}_k$ and

$$V_k(\Lambda) = V(\widehat{\sigma}_k; \Lambda) = \left.\frac{\mathcal{Q}(e^{i\sigma}; \Lambda)}{\mathcal{P}(e^{i\sigma}; \Lambda)}\right|_{\sigma = \widehat{\sigma}_k}. \tag{5.35}$$

Here we have explicitly indicated that the functions $\mathcal{P}$ and $\mathcal{Q}$ depend upon Λ. Equations (5.35) constitute a coupled system of $2n$ first-order partial differential equations. As in the KdV case, the speeds $V_k(\Lambda)$ are real and distinct; system (5.35) is a strictly hyperbolic, genuinely nonlinear system in Riemann invariant form [30]. The set Λ evolves according to (5.35), so long as the solutions of this system remain regular and n does not change. We have not yet stated how the set Λ and its cardinality $(= 2n)$ change when a singularity is encountered, nor have we addressed how $\Lambda(x,t)$ depends on the initial data. We shall deal with these issues by introducing a pair of functions $x^+(\sigma,t)$ and $x^-(\sigma,t)$ already defined for $t < t_b$ but here extended to $t > t_b$. We then construct the set $\Lambda(x,t)$ in terms of these two functions. We begin with a definition of a "crossing point."

DEFINITION 5.4 A pair of functions $x^+(\cdot), x^-(\cdot)$ cross the value x at σ if the union of the images of any neighborhood of σ under $x^+(\cdot), x^-(\cdot)$ is a neighborhood of x. Such a σ is called a *crossing point* of the pair at x. At time t, the set $\Lambda_c(x,t)$ is defined by

$$\Lambda_c(x,t) = \{\sigma : \sigma \text{ is a crossing point of } x^+(\cdot,t), x^-(\cdot,t) \text{ at } x\}\,. \tag{5.36}$$

Note that if $0 < \sigma < \pi$ and $x^+(\sigma) = x$ or $x^-(\sigma) = x$, then σ is a crossing point at x unless $x^+(\cdot)$ or $x^-(\cdot)$ has a local extremum at σ. Also note that for $t = 0$, $\Lambda_c(x,0)$ is given in terms of the initial data

$$\Lambda_c(x,0) = \{\cos^{-1}[\hat{\lambda}_1(x,0)], \cos^{-1}[\hat{\lambda}_2(x,0)]\} = \{\widehat{\sigma}_1(x,0), \widehat{\sigma}_2(x,0)\}\,. \tag{5.37}$$

Finally, note that the set $\Lambda_c(x,t)$ depends on the functions $x^+(\cdot,t)$ and $x^-(\cdot,t)$, a dependence that enables us to formulate the following initial value problem:

Consider the initial value problem for the functions $x^+(\sigma,t)$ and $x^-(\sigma,t)$:

$$\frac{d}{dt}\,x^+ = V\big(\sigma; \Lambda_c(x^+,t)\big), \quad \frac{d}{dt}\,x^- = V\big(\sigma; \Lambda_c(x^-,t)\big)\,, \tag{5.38}$$

with

$$x^+(\sigma,0) = x^+(\sigma) = \cos^{-1}(\hat\lambda_1(x,0)) = \widehat{\sigma}_1(x,0)\,, \tag{5.39}$$
$$x^-(\sigma,0) = x^-(\sigma) = \cos^{-1}(\hat\lambda_2(x,0)) = \widehat{\sigma}_2(x,0)\,.$$

We *assume* that this initial problem has a unique solution such that the following hold:

1. As functions of σ and t, x^+ and x^- are C^1 in $(0,\sigma_{\max}) \cup (\sigma_{\min},\pi) \times \mathbb{R}$ and continuous in $(0,\sigma_{\max}] \cup [\sigma_{\min},\pi) \times \mathbb{R}$.
2. The limits $\lim_{\sigma\to 0,\pi} x_\pm(\sigma,t)$ exist, possibly as $+\infty$ or $-\infty$. If finite, we call them the boundary value of x^+, x^- at $\sigma = 0,\pi$; denote them by $x^+(0,t)$, $x^-(0,t)$, and assume that they are C^1 with respect to t.
3. The number of critical points of $x^+(\sigma,t)$ and $x^-(\sigma,t)$ is finite for all t.
4. If, for some $\sigma, x^+(\sigma) = x^-(\sigma)$, then $x^+(\sigma,t) = x^-(\sigma,t)$ for all t.

With $x^+(\sigma,t)$ and $x^-(\sigma,t)$ defined through the initial value problem (5.38)–(5.39), we integrate $\eta^*_x(x,t)$, equation (5.34a), to obtain the maximizer.

THEOREM 5.5 *The solution of the variational problem is given by*

$$\eta^*(\sigma;x,t) = \begin{cases} 0 & \textit{for } x^+(\sigma,t) \le x, \\ \displaystyle\int_x^{x^+} \frac{\mathcal{P}(\sigma)}{\mathcal{R}(\sigma)}\,dy & \textit{for } x^-(\sigma,t) < x < x^+(\sigma,t), \\ \varphi(\sigma)\sin\sigma & \textit{for } x \le x^-(\sigma,t). \end{cases} \tag{5.40}$$

This theorem is proven by direct calculation.

In terms of the maximizer, the weak limit of the densities are given by

$$\mathcal{D}'(dx)\text{–}\lim_{\hbar\to 0}\rho_k^\hbar = (\sin\sigma\cos^k\sigma \mid \eta^*_x)\,. \tag{5.41}$$

Specifically, the semiclassical limits of the first density is given by

(5.42)
$$\begin{aligned}\mathcal{D}'(dx) - \lim_{\hbar\to 0}\rho_0^\hbar &= (\sin\sigma \mid \eta^*_x) \\ &= -\frac{2}{\pi}\int_I \sin\sigma\,\frac{\mathcal{P}(\sigma)}{\mathcal{R}(\sigma)}\,d\sigma \\ &= \Bigg[n^2 - n + \left(n - 1 + \frac{1}{2}\sum_1^{2n}\cos\widehat{\sigma}_k\right)\alpha_1 + \frac{1}{2}\alpha_2 \\ &\qquad + n\sum_1^{2n}\cos\widehat{\sigma}_k + \frac{1}{4}\left(\sum_1^{2n}\cos\widehat{\sigma}_k\right)^2 + \frac{1}{2}\sum_1^{2n}\cos^2\widehat{\sigma}_k\Bigg].\end{aligned}$$

There are similar (but more complicated) expressions for the higher densities, the next two of which may be found in [23].

6 Conclusion

In this paper we have given a complete global characterization of the semiclassical (as $\hbar \to 0$) limit of the one-dimensional, cubic, defocusing NLS equation. Our method adapts the Lax-Levermore procedure, which was developed for the KdV equation, to the NLS case. The present work goes further than the original KdV study in that for the first time it gives a simultaneous analysis of the infinite hierarchy of densities and fluxes. Moreover, it substantially reorganizes and clarifies the original proof [28] that the weak limit is characterized by a constrained quadratic maximization problem. We expect that recent results on the KdV zero-dispersion limit [10, 11, 31] will have defocusing NLS analogues. Those results characterize the maximizer as the limiting density of half-line Dirichlet spectra of the associated Schrödinger operator and strengthen the limits asserted for the conserved densities and fluxes.

When this work was completed [23, 24], it provided the second PDE example of the Lax-Levermore procedure; thus, defocusing NLS provided an indication of the generality of the Lax-Levermore method for the description of "semiclassical limits" of integrable nonlinear wave equations. Since then, the Lax-Levermore procedure has been applied to the Toda lattice [4, 25, 37, 42] and to the focusing mKdV equation [9], and the variational method has even been implemented numerically [36] for the KdV case.

Ultimately, one should be interested in more than weak limits. For example, one should understand the microstructure of the limiting solutions. Historically, modulation theory [12, 44] was built on postulating a particular microstructure. Venakides [41] realized, somewhat heuristically, that limiting microstructure could be obtained from a "higher order" approach to the Lax-Levermore formalism. Recently a powerful asymptotic method for analyzing integrable systems has been introduced by Deift and Zhou [2, 5, 7] and is currently being developed to extend the Lax-Levermore theory so as to establish the limiting microstructure for the zero-dispersion KdV case [6, 37].

These tools provide powerful asymptotic techniques for the rigorous global analysis of the limiting behavior in integrable nonlinear wave equations. Striking and detailed nonlinear phenomena, such as the formation and propagation of rapid oscillations, can now be understood completely. However, at present, rigorous techniques are almost exclusively limited to integrable equations. In the more general nonintegrable setting, numerical experiments, as well as the formal asymptotic methods, indicate similar behavior in some instances and new phenomena in others [3, 17, 18, 20, 21, 22, 32, 33, 40]. However, very little global rigorous analysis of semiclassical limits for nonintegrable dispersive wave equations currently exists, a situation that provides a challenge to PDE analysts. (Some recent analytical work in this direction includes that of [16, 17, 19, 33].)

Even for integrable cases, fundamental and difficult problems remain open and are related to striking behavior of nonlinear waves. The most important such open problem is the semiclassical behavior of the cubic *focusing* NLS equation [1, 24]. Here numerical experiments indicate the presence of spatial and temporal chaos but remain inconclusive at present. Modulation theory simply predicts instabilities, with the only indication of the existence of a limiting behavior restricted to extremely weak nonlinearities [1]. The integrable theories for *focusing* NLS are based upon a non-self-adjoint eigenvalue problem; however, non-self-adjointness of the Zakharov and Shabat operator is not the key obstacle as was established with the description of the semiclassical behavior of mKdV [9]. (Within the Lax-Levermore framework, the key obstacle is the analysis of the limit of the log-determinant when the matrix is *not* positive definite.) It is intriguing to note that hyperbolic structure, which is behind the many instabilities of focusing NLS [8, 34, 35] as well as behind the obstacles to its analysis, is itself a very stable and general phenomena. This structural stability leads us to believe that once the semiclassical limit of the integrable *focusing* NLS equation is complete, it will significantly enhance our understanding of the general semiclassical behavior of nonlinear dispersive waves.

This work about the behavior of solutions of the nonlinear Schrödinger equation in the semiclassical limit (as well as the earlier studies in references [1, 24]) was entirely motivated by a natural mathematical question; however, the problem is also of direct importance to basic physics and technology in nonlinear optics. As mentioned in [1], the semiclassical limit of defocusing NLS provides an idealized description of optical shocks and wave breaking in the nonlinear propagation of laser pulses in optical fibers [38, 39]. Very recently the semiclassical limit of the defocusing NLS equations has been applied [14, 26] to describe "nonreturn to zero" (NRZ) optical pulses in nonlinear fibers, a problem of central importance to the current technology of long distance (transoceanic) telephone communication. In fact, the mathematical techniques developed in this paper (and in [23, 24]) provide rigorous justification for the formal modulation approach of [26] to the semiclassical description of NRZ. It is very satisfying to see that the semiclassical limit of NLS, which is so natural mathematically, is also important technologically.

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