Assignment 6.

Given October 21, due October 28. Last revised, October 21.
Objective: Forward and Backward equations for Brownian motion.

The terms forward and backward equation refer to the equations the probability density of X_t and $E_{x,t}[V(X_T)]$ respectively. The integrals below are easily done if you use identities such as

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} x^{2n} e^{-x^2/2\sigma^2} dx = \sigma^{2n} \cdot (2n-1)(2n-3)\cdots 3 \; .$$

You should not have to do any actual integration for these problems.

1. Solve the forward equation with initial data

$$u_0(x) = \frac{x^2}{\sqrt{2\pi}} e^{-x^2/2} \; .$$

a. Assume the solution has the form

$$u(x,t) = \Big(A(t)x^2 + B(t)\Big)\, g(x,t) \;\; , \quad g(x,t) = G(0,x,t{+}1) = \frac{1}{\sqrt{2\pi(t+1)}} e^{-x^2/2(t+1)} \; . \tag{1}$$

Then find and solve the ordinary differential equations for $A(t)$ and $B(t)$ that make (1) a solution of the forward equation.

b. Compute the integrals

$$u(x,t) = \int_{-\infty}^{\infty} u_0(y) G(y,x,t) dy \; .$$

This should give the same answer as part a.

c. Sketch the probability density at time $t = 0$, for small time, and for large time. Rescale the large time plot so that it doesn't look flat.

d. Why does the structure seen in $u(x,t)$ for small time (the double hump) disappear for large t?

e. Show in a rough way that a similar phenomenon happens for any initial data of the form $u_0(x) = p(x)g(x,0)$, where $p(x)$ is an even nonnegative polynomial. When t is large, $u(x,t)$ looks like a simple Gaussian, no matter what p was.

2. Solve the backward equation with final data $V(x) = x^4$.

a. Write the solution in the form

$$f(x,t) = x^4 + a(t)x^2 + b(t) \; . \tag{2}$$

Then find and solve the differential equations that $a(t)$ and $b(t)$ must satisfy so that (2) is the solution of the backward equation.

b. Compute the integrals

$$f(x,t) = \int_{-\infty}^{\infty} G(x,y,T-t)V(y)dy \ .$$

This should be the same as your answer to part a.

c. Give a simple explanation for the form of the formula for $f(0,t) = b(t)$ in terms of moments of a Gaussian random variable.

3. Check that

$$\int_{-\infty}^{\infty} u(x,t)f(x,t)dx$$

is independent of t.