Numerical Methods II
Spring, 2017
Assignment 1: due February 14

1) Consider the inner product $\langle f, g\rangle = \int_0^\infty e^{-x} f(x)g(x)\,dx$. Generate the first four monic polynomials (i.e., with leading coefficient equal to one) that are orthogonal with respect to this inner product (the Laguerre polynomials).

2) The Euler-MacLaurin formula shows that the trapezoidal rule gives high order accuracy for periodic functions over an entire period.

a) Can you conclude that the trapezoidal rule is *exact* for C^∞ functions? Why or why not?

b) Compute $\int_{-\infty}^{\infty} e^{-x^2}\,dx$. For $x = \pm 4$, the integrand is less than $0.5 \cdot 10^{-6}$. Use the trapezoidal rule on $[-4, 4]$ with $h = 1$ and $h = 0.5$. (The correct value of the integral om the real line is $\sqrt{\pi} \approx 1.772454$.) What is the apparent convergence rate? How do you explain it?

c) Compute the sum $S = 1 + 2 + \ldots + n$ by using the Euler-MacLaurin formula.

3) In Richardson extrapolation, we are assuming that we have a method $A(h)$ for approximating A^*:

$$A^* = A(h) + c_n h^n + c_{n+1}h^{n+1} + \ldots$$

and that eliminating the $c_n h^n$ term will yield a better approximation. This clearly assumes that $c_n h^n$ is the dominant term in the error expansion. Let

$$R(h, k) = \frac{k^n A(h) - A(kh)}{k^n - 1}.$$

a) Show that $R(h, k) = A^* + O(h^{n+1})$. Show also that

$$(A_h - A_{kh})/(A_{kh} - A_{k^2h})$$

should be approximately k^{-n}.

b) Romberg integration is an extrapolation procedure and is susceptible to a kind of numerical instability. For a given positive integrer M and an interval $[a, b]$, it approximates the integral $\int_a^b f(x)\,dx$ to order $2M$ as follows:

for $m = 1, \dots, M$

$M = 2^{m-1}$

$h = (b-a)/M$

$T_{m,1} = \frac{h}{2}\left[f(a) + 2\sum_{m=1}^{M-1} f(a+mh) + f(b)\right]$

for $n = 2, \dots, m$

$T_{m,n} = T_{m,n-1} + \frac{T_{m,n-1} - T_{m-1,n-1}}{4^{n-1}-1}$

end

end

As a check we can calculate the ratios $R_{m,n} = (T_{m,n} - T_{m-1,n})/(T_{m+1,n} - T_{m,n})$. Usng part (a), show that these ratios should be approximately 4^n.

c) Write a Romberg integration code and check the validity of the estimate

$$(A_h - A_{kh})/(A_{kh} - A_{k^2h}) \approx k^{-n}.$$

d) Test your program on $\int_0^1 \sqrt{x}\,dx$ and on $\int_1^2 \sqrt{x}\,dx$. In your experiments, print out the number of mesh intervals, the number of extrapolation steps , the above ratios, the error (since you know the exact solution), etc. Explain the observed rates of convergence.

4) Write a *simple* program to solve

$$u_{xx} + \alpha u_x - \beta u = f$$

with periodic boundary conditions on $[0, 2\pi]$ using Fourier analysis and the FFT.

a) Consider the exact solution $\sin(\cos(x))$ with $\alpha = \beta = 1$, and compute the corresponding right-hand side f. Demonstrate that your code is spectrally accurate.

b) Are there values of α and/or β where your code fails?

c) Modify your program to solve

$$u_{xx} + \alpha u_x - \beta u = f$$

with homogeneous Dirichlet boundary conditions $u(0) = u(2\pi) = 0$ by using a sine series. This is a very small change to the code. Try the exact solution $\sin^2(x)$, again with $\alpha = \beta = 1$ and the corresponding right-hand side f. What is the rate of convergence of the method?

5) (Extra credit): This problem is devoted to the construction of an indefinite integral *transform*. Once you have an approximation of the form

$$f(x) = \sum_{k=0}^{n} \alpha_k T_k(x) \ ,$$

it is easy to construct a Chebyshev expansion for the indefinite integral. This follows from the fact that

$$\int T_k(x)\, dx = -\int \cos(k\theta) \sin\theta \, d\theta = \frac{1}{2}\left(\frac{T_{k+1}(x)}{k+1} - \frac{T_{k-1}(x)}{k-1}\right)$$

for $k \geq 2$.

Write a program which does the following:

a) Given a set of function values at the Chebyshev nodes, construct the coefficients α_k of the Chebyshev series via a cosine transform.

b) From the α_k, compute the coefficients β_k of the Chebyshev series

$$\int_{-1}^{x} f(t)\, dt = \sum_{k=0}^{n} \beta_k T_k(x) \ .$$

c) Evaluate the indefinite integral at the Chebyshev nodes by a cosine transform.

You now have an $N \log N$ algorithm for obtaining the indefinite integral with spectral accuracy.

d) Solve the differential equation $dx/dt = \cos^2 t, x(-1) = 1$ on the interval $[-1, 1]$. Compare your results with the analytic solution and discuss the convergence rate.