Statistical energy conservation principle for inhomogeneous turbulent dynamical systems

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Understanding the complexity of anisotropic turbulent processes over a wide range of spatiotemporal scales in engineering shear turbulence as well as climate atmosphere ocean science is a grand challenge of contemporary science with important societal impact. In such inhomogeneous turbulent dynamical systems there is a large dimensional phase space with a large dimension of unstable directions where a large-scale ensemble mean and the turbulent fluctuations exchange energy and strongly influence each other. These complex features strongly impact practical prediction and uncertainty quantification. A systematic energy conservation principle is developed here in a *Theorem* that precisely accounts for the statistical energy exchange between the mean flow and the related turbulent fluctuations. This statistical energy is a sum of the energy in the mean and the trace of the covariance of the fluctuating turbulence. This result applies to general inhomogeneous turbulent dynamical systems including the above applications. The *Theorem* involves an assessment of statistical symmetries for the nonlinear interactions and a self-contained treatment is presented below. *Corollary 1* and *Corollary 2* illustrate the power of the method with general closed differential equalities for the statistical energy in time either exactly or with upper and lower bounds, provided that the negative symmetric dissipation matrix is diagonal in a suitable basis. Implications of the energy principle for low-order closure modeling and automatic estimates for the single point variance are discussed below.

mean | fluctuations | interaction statistical symmetries

Understanding the complexity of anisotropic turbulent processes over a wide range of spatiotemporal scales in engineering shear turbulence as well as climate atmosphere ocean science is a grand challenge of contemporary science (1–4) with important societal impact. In such inhomogeneous turbulent dynamical systems, there is a large dimensional phase space with a large dimension of unstable directions where a large-scale ensemble mean and the turbulent fluctuations exchange energy and strongly influence each other. These complex features strongly impact practical prediction and uncertainty quantification. A systematic energy conservation principle is developed here in a *Theorem* that precisely accounts for the statistical energy exchange between the mean flow and the related turbulent fluctuations. This statistical energy is a sum of the energy in the mean and the trace of the covariance of the fluctuating turbulence. This result applies to general inhomogeneous turbulent dynamical systems including the above applications. The *Theorem* involves an assessment of statistical symmetries for the nonlinear interactions and a self-contained treatment is presented below. *Corollary 1* and *Corollary 2* illustrate the power of the method with general closed differential equalities for the statistical energy in time either exactly or with upper and lower bounds provided that the negative symmetric dissipation matrix is diagonal in a suitable basis. Implications of the energy principle for low-order closure modeling and automatic estimates for the single point variance are discussed below (5–7).

Most earlier work in statistical turbulence involve assumptions of homogeneity and isotropy without a mean flow (1, 3, 8) or approximate the nonlinear terms in the equations for the mean flow and fluctuating interactions by ad hoc linear stochastic models (1, 9). More complete formulations of statistical energetics for the Navier–Stokes equations can be found in ref. 10. The advantage of the energy principle in the *Theorem* is the exact treatment of the nonlinear statistical energy exchange between the mean and fluctuations, which are amenable to systematic low-order closure models for prediction and uncertainty quantification with high skill (5–7, 11).

The Mathematical Structure of Turbulent Dynamical Systems

Consider the statistical behavior of quadratic systems with conservative nonlinear dynamics and unstable directions. In particular, consider the general turbulent dynamical system:

$$\frac{d\mathbf{u}}{dt}=[L+D]\mathbf{u}+B(\mathbf{u},\mathbf{u})+\mathbf{F}(t)+\sigma_k(t)\dot{\mathbf{W}}_k(t;\omega), \qquad \mathbf{[1]}$$

acting on $\mathbf{u}\in\mathbb{R}^N$. In the above equation and for what follows, repeated indices will indicate summation. In some cases, the limits of summation will be given explicitly to emphasize the range of the index.

In the above equation, we have the following:

- L, being a skew-symmetric linear operator representing the β-effect of Earth's curvature, topography, etc., and satisfying,

$$L^*=-L. \qquad \mathbf{[2a]}$$

- D, being a negative definite symmetric operator,

$$D^*=D, \qquad \mathbf{[2b]}$$

representing dissipative processes such as surface drag, radiative damping, viscosity, etc.

Significance

Understanding the complexity of anisotropic turbulent processes over a wide range of spatiotemporal scales in engineering shear turbulence as well as climate atmosphere ocean science is a grand challenge of contemporary science with important societal impact. In such inhomogeneous turbulent dynamical systems, there is a large dimensional phase space with a large dimension of unstable directions where a large-scale ensemble mean and the turbulent fluctuations exchange energy and strongly influence each other. These complex features strongly impact practical prediction and uncertainty quantification. A systematic energy conservation principle is developed here in a *Theorem* that precisely accounts for the statistical energy exchange between the mean flow and the related turbulent fluctuations.

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- The quadratic operator $B(\mathbf{u},\mathbf{u})$ conserves the energy by itself so that it satisfies the following:

$$\mathbf{u}\cdot B(\mathbf{u},\mathbf{u})=0. \qquad \textbf{[2c]}$$

Here and below, the dot product in [**2c**] is the standard Euclidean product, and "*" is used for transpose. Finally, $\mathbf{F}(t)+\sigma_k(t)\dot{\mathbf{W}}_k(t;\omega)$ represents the effect of external forcing, such as solar forcing, which we will assume that it can be split into a mean component $\mathbf{F}(t)$ and a stochastic component with white-noise characteristics that can also mimic unresolved model error crudely. Turbulent dynamical systems are ubiquitous in geoscience, climate science, and engineering (1–4).

We use a finite-dimensional representation of the stochastic field consisting a N-dimensional, orthonormal basis, $\{\mathbf{e}_i\}_{i=1}^N$,

$$\mathbf{u}(t)=\bar{\mathbf{u}}(t)+\sum_{i=1}^{N}Z_i(t;\omega)\mathbf{e}_i,$$

where $\bar{\mathbf{u}}(t)\equiv\langle\mathbf{u}(t)\rangle$ represents the ensemble average of the response, i.e., the mean field, and $Z_i(t;\omega)$ are stochastic processes.

Exact Equation for Mean and Covariance. The exact mean field equation is given by the following:

$$\frac{d\bar{\mathbf{u}}}{dt}=[L+D]\bar{\mathbf{u}}+B(\bar{\mathbf{u}},\bar{\mathbf{u}})+R_{ij}B(\mathbf{e}_i,\mathbf{e}_j)+\mathbf{F}, \qquad \textbf{[3]}$$

where we have the covariance matrix given by $R_{ij}=\langle Z_iZ_j^*\rangle$ and $\langle\cdot\rangle$ denotes averaging over the ensemble members. As in typical turbulence, the mean equation is not closed and depends on the fluctuations through the covariance matrix.

Moreover, the random component of the solution, $\mathbf{u}'=Z_i(t;\omega)\mathbf{e}_i$ satisfies the following:

$$\begin{aligned}\frac{d\mathbf{u}'}{dt}=&[L+D]\mathbf{u}'+B(\bar{\mathbf{u}},\mathbf{u}')+B(\mathbf{u}',\bar{\mathbf{u}})+B(\mathbf{u}',\mathbf{u}')\\&-R_{jk}B(\mathbf{e}_j,\mathbf{e}_k)+\sigma_k(t)\dot{\mathbf{W}}_k(t;\omega).\end{aligned} \qquad \textbf{[4]}$$

By projecting the above equation to each basis element $\mathbf{e}_i$, we obtain the following:

$$\begin{aligned}\frac{dZ_i}{dt}=&Z_j\left([L+D]\mathbf{e}_j+B(\bar{\mathbf{u}},\mathbf{e}_j)+B(\mathbf{e}_j,\bar{\mathbf{u}})\right)\cdot\mathbf{e}_i\\&+\left(B(\mathbf{u}',\mathbf{u}')-R_{jk}B(\mathbf{e}_j,\mathbf{e}_k)\right)\cdot\mathbf{e}_i+\sigma_k\dot{\mathbf{W}}_k\cdot\mathbf{e}_i.\end{aligned} \qquad \textbf{[5]}$$

From the last equation, we directly obtain the exact evolution of the covariance matrix $R=\langle\mathbf{Z}\mathbf{Z}^*\rangle$:

$$\frac{dR}{dt}=L_vR+RL_v^*+Q_F+Q_\sigma, \qquad \textbf{[6]}$$

where we have the following terms in [**6**]:

i) the linear dynamics operator expressing energy transfers between the mean field and the stochastic modes (effect due to B), as well as energy dissipation (effect due to D), and non-normal dynamics (effect due to $L, D, \bar{\mathbf{u}}$):

$$\{L_v\}_{ij}=\left([L+D]\mathbf{e}_j+B(\bar{\mathbf{u}},\mathbf{e}_j)+B(\mathbf{e}_j,\bar{\mathbf{u}})\right)\cdot\mathbf{e}_i; \qquad \textbf{[7]}$$

ii) the positive definite operator expressing energy transfer due to external stochastic forcing:

$$\{Q_\sigma\}_{ij}=(\mathbf{e}_i\cdot\sigma_k)(\sigma_k\cdot\mathbf{e}_j); \qquad \textbf{[8]}$$

iii) as well as the energy flux between different modes due to non-Gaussian statistics (or nonlinear terms) given exactly through third-order moments:

$$Q_F=\overline{Z_mZ_nZ_j}B(\mathbf{e}_m,\mathbf{e}_n)\cdot\mathbf{e}_i+\overline{Z_mZ_nZ_i}B(\mathbf{e}_m,\mathbf{e}_n)\cdot\mathbf{e}_j. \qquad \textbf{[9]}$$

With energy conservation, the nonlinear terms satisfy the statistical symmetry requirement:

$$\mathrm{tr}Q_F\equiv0, \qquad \textbf{[10]}$$

because with $\mathbf{u}'=Z_i\mathbf{e}_i$, $\mathrm{tr}Q_F=\overline{\mathbf{u}\cdot B(\mathbf{u},\mathbf{u})}=0$ by energy conservation.

Detailed Triad Energy Conservation Symmetries. For many applications in geosciences and engineering, the orthonormal basis $\mathbf{e}_i$ consists of Fourier modes or spherical harmonics and the nonlinear term, $B(\mathbf{u},\mathbf{u})$, satisfies a detailed conservation of energy principle. To understand this, we consider the dynamics for the nonlinear terms alone,

$$\mathbf{u}_t=B(\mathbf{u},\mathbf{u}),\ \mathbf{u}\in\mathbb{R}^N, \qquad \textbf{[11]}$$

and the truncation $\mathbf{u}_\Lambda=\mathscr{P}_\Lambda\mathbf{u}$ where $\mathscr{P}_\Lambda$ is any L^2 projection, and the associated Galerkin truncation dynamics:

$$(\mathbf{u}_\Lambda)_t=\mathscr{P}_\Lambda B(\mathbf{u}_\Lambda,\mathbf{u}_\Lambda), \qquad \textbf{[12]}$$

with energy $E_\Lambda=(1/2)\mathbf{u}_\Lambda\cdot\mathbf{u}_\Lambda$. It follows from [**2c**] that the projected energy E_Λ is also conserved, because with [**2c**]:

$$\frac{dE_\Lambda}{dt}=\mathbf{u}_\Lambda\cdot\mathscr{P}_\Lambda B(\mathbf{u}_\Lambda,\mathbf{u}_\Lambda)=\mathbf{u}_\Lambda\cdot B(\mathbf{u}_\Lambda,\mathbf{u}_\Lambda)\equiv0. \qquad \textbf{[13]}$$

Now consider the 3D Galerkin projection of [**12**] to the subspace spanned by the triad, $\mathbf{e}_i,\mathbf{e}_j,\mathbf{e}_k$. We have the following.

Proposition 1. *Consider the 3D Galerkin projected dynamics spanned by the triad $\mathbf{e}_i,\mathbf{e}_j,\mathbf{e}_k$ for $1\le i,j,k\le N$. Assume the following*:

A) *The self interactions vanish*,

$$B(\mathbf{e}_i,\mathbf{e}_i)\equiv0,\quad 1\le i\le N; \qquad \textbf{[14]}$$

B) *The dyad interaction coefficients vanish through the symmetry*,

$$\mathbf{e}_i\cdot[B(\mathbf{e}_l,\mathbf{e}_i)+B(\mathbf{e}_i,\mathbf{e}_l)]=0,\quad \textit{for any } i,l. \qquad \textbf{[15]}$$

Then the 3D Galerkin truncation in [**12**] *becomes the "triad interaction equations" for* $\mathbf{u}=(u_i,u_j,u_k)=(\mathbf{u}_\Lambda\cdot\mathbf{e}_i,\mathbf{u}_\Lambda\cdot\mathbf{e}_j,\mathbf{u}_\Lambda\cdot\mathbf{e}_k)$:

$$\frac{du_i}{dt}=A_{ijk}u_ju_k, \qquad \textbf{[16a]}$$

$$\frac{du_j}{dt}=A_{jki}u_ku_i, \qquad \textbf{[16b]}$$

$$\frac{du_k}{dt}=A_{kij}u_iu_j, \qquad \textbf{[16c]}$$

with coefficient satisfying the following:

$$A_{ijk}+A_{jik}+A_{kji}=0, \qquad \textbf{[17]}$$

which is the detailed triad energy conservation symmetry, *because*

$$\begin{aligned}A_{ijk}+A_{jki}+A_{kij}\equiv&\,\mathbf{e}_i\cdot\left[B(\mathbf{e}_j,\mathbf{e}_k)+B(\mathbf{e}_k,\mathbf{e}_j)\right]\\&+\mathbf{e}_j\cdot[B(\mathbf{e}_k,\mathbf{e}_i)+B(\mathbf{e}_i,\mathbf{e}_k)]\\&+\mathbf{e}_k\cdot\left[B(\mathbf{e}_i,\mathbf{e}_j)+B(\mathbf{e}_j,\mathbf{e}_i)\right]=0.\end{aligned} \qquad \textbf{[18]}$$

The forms in [**16a–16c**] for [**12**] is a direct expansion of $\mathbf{e}_m \cdot \mathscr{P}_\Lambda B(\mathbf{u},\mathbf{u})$ for $m=i,j,k$, using the properties in [**14**] and [**15**]. The property in [**17**] is a direct consequence of the energy conservation in [**13**] for an equation with the form in [**16a–16c**] because

$$0 \equiv \frac{dE_\Lambda}{dt} = \left[A_{ijk}+A_{jki}+A_{kij}\right]u_iu_ju_k.$$

The detailed symmetry in [**18**] follows from [**17**] for the explicit expansion coefficients.

To show the importance of the requirements in [**14**] and [**15**], consider the dyad interaction equation:

$$\frac{\partial u_1}{\partial t} = \gamma_1 u_1 u_2 + \gamma_2 u_2^2, \qquad [19a]$$

$$\frac{\partial u_2}{\partial t} = -\gamma_1 u_1^2 - \gamma_2 u_1 u_2. \qquad [19b]$$

These 2D equations conserve energy with [**14**] and [**15**] both nonzero and obviously do not have the form in [**16a–16c**]. For nonlinear advection term in two dimensions, so that $B(\mathbf{u}_\Lambda,\mathbf{u}_\Lambda) = -\nabla^\perp\psi_\Lambda \cdot \nabla\omega_\Lambda$ suitably scaled where the stream function satisfies $\Delta\psi_\Lambda = \omega_\Lambda$, the conditions in [**14**], [**15**], and the symmetry properties in [**18**] are satisfied provided the basis $\mathbf{e}_i$ is Fourier modes or spherical harmonics (1, 3, 4, 8). Terms with dyad interactions naturally arise with inhomogeneities when the orthonormal basis comes from data such as in empirical orthogonal function (EOF) analysis. The properties of turbulent dynamical systems with both dyad and triad interactions have been studied earlier (12, 13). The conceptual model for turbulence introduced in ref. 14 is an interesting system with only nonlinear dyad interactions.

A General Statistical Energy Conservation Principle

Consider the mean energy first, $\bar{E} = (1/2)|\bar{\mathbf{u}}|^2 = (1/2)\bar{\mathbf{u}}\cdot\bar{\mathbf{u}}$. From [**3**], we have immediately the following.

Proposition 2. *The change of mean energy* $\bar{E} = (1/2)(\bar{\mathbf{u}}\cdot\bar{\mathbf{u}})$ *satisfies the following*:

$$\frac{d}{dt}\left(\frac{1}{2}|\bar{\mathbf{u}}|^2\right) = \bar{\mathbf{u}}\cdot D\bar{\mathbf{u}} + \bar{\mathbf{u}}\cdot\mathbf{F} + \frac{1}{2}R_{ij}\bar{\mathbf{u}}\cdot\left[B(\mathbf{e}_i,\mathbf{e}_j)+B(\mathbf{e}_j,\mathbf{e}_i)\right]. \qquad [20]$$

The last term represents the effect of the fluctuation on the mean, $\bar{\mathbf{u}}$.

Now consider the fluctuating energy $E' = (1/2)\mathrm{tr}(R_{ij})$. From $\mathrm{tr}Q_F = 0$ in [**10**], we have using [**6**] that

$$\frac{dE'}{dt} = \frac{1}{2}\frac{d\mathrm{tr}R}{dt} = \mathrm{tr}\left(\frac{L_\nu R + RL_\nu^*}{2}\right) + \frac{1}{2}\mathrm{tr}Q_\sigma. \qquad [21]$$

With [**8**],

$$\mathrm{tr}Q_\sigma = \sum_{i,k}(\mathbf{e}_i\cdot\sigma_k)(\sigma_k\cdot\mathbf{e}_i). \qquad [22]$$

Use the formula in [**7**] to rewrite [**21**] as follows:

$$\mathrm{tr}\left(\frac{L_\nu R + RL_\nu^*}{2}\right) = \mathrm{tr}\left(\frac{[\tilde{L}+\tilde{D}]R + R[\tilde{L}^*+\tilde{D}^*]}{2}\right) + \frac{1}{2}\mathrm{tr}\left[B(\bar{\mathbf{u}},\hat{R}) + B(\hat{R},\bar{\mathbf{u}}) + B(\bar{\mathbf{u}},\hat{R}^T) + B(\hat{R}^T,\bar{\mathbf{u}})\right]. \qquad [23]$$

Here $\tilde{L}$ and $\tilde{D}$ are the linear operator representations under basis $\mathbf{e}_i$, that is,

$$\tilde{L}_{ij} = \mathbf{e}_i\cdot L\mathbf{e}_j, \tilde{D}_{ij} = \mathbf{e}_i\cdot D\mathbf{e}_j, \qquad [24]$$

with the properties in [**2a**] and [**2b**] preserved; $\hat{R}$ is the tensor representation of the covariance matrix with $\mathbf{u}' = Z_i\mathbf{e}_i$,

$$\hat{R} = \langle\mathbf{u}'\otimes\mathbf{u}'\rangle = \sum_{i,j}R_{ij}\mathbf{e}_i\otimes\mathbf{e}_j; \qquad [25]$$

the matrix $B(\bar{\mathbf{u}},\hat{R})$ is defined as the componentwise interaction with each column of $\hat{R}$. Because $\tilde{L}$ is skew symmetric $\mathrm{tr}(\tilde{L}R + R\tilde{L}^*) = \mathrm{tr}(\tilde{L}R - R\tilde{L}) = 0$, so in [**23**] and below, the skew-symmetric terms make no direct contribution to the change in the trace of statistical energy fluctuations, but directly alter the mean. Next, expand $\bar{\mathbf{u}}$ by $\bar{\mathbf{u}} = \bar{u}_M\mathbf{e}_M$. With [**25**], we get the identity,

$$\begin{aligned} &B(\bar{\mathbf{u}},\hat{R}) + B(\hat{R},\bar{\mathbf{u}}) + \text{‘}\textit{transpose part}\text{'} \\ &= R_{ij}\bar{u}_M\left[B(\mathbf{e}_M,\mathbf{e}_i)\otimes\mathbf{e}_j + \mathbf{e}_i\otimes B(\mathbf{e}_j,\mathbf{e}_M) + \text{‘}\textit{transpose part}\text{'}\right]. \end{aligned} \qquad [26]$$

So using that $\mathrm{tr}(\mathbf{a}\otimes\mathbf{b}) = \mathbf{a}\cdot\mathbf{b}$ and [**25**], we have the following:

$$\begin{aligned} &\frac{1}{2}\mathrm{tr}\left[B(\bar{\mathbf{u}},\hat{R}) + B(\hat{R},\bar{\mathbf{u}}) + \text{‘}\textit{transpose part}\text{'}\right] \\ &= \frac{1}{2}R_{ij}\bar{u}_M\left[\mathbf{e}_i\cdot B(\mathbf{e}_M,\mathbf{e}_j) + B(\mathbf{e}_i,\mathbf{e}_M)\cdot\mathbf{e}_j + \mathbf{e}_j\cdot B(\mathbf{e}_M,\mathbf{e}_i) + B(\mathbf{e}_j,\mathbf{e}_M)\cdot\mathbf{e}_i\right]. \end{aligned} \qquad [27]$$

Now use the detailed triad conservation structure in [**18**]. The expression in bracket in [**27**] is given by the following formula:

$$\begin{aligned} &\mathbf{e}_i\cdot\left[B(\mathbf{e}_j,\mathbf{e}_M) + B(\mathbf{e}_M,\mathbf{e}_j)\right] + \mathbf{e}_j\cdot\left[B(\mathbf{e}_M,\mathbf{e}_i) + B(\mathbf{e}_i,\mathbf{e}_M)\right] \\ &= -\mathbf{e}_M\cdot\left[B(\mathbf{e}_i,\mathbf{e}_j) + B(\mathbf{e}_j,\mathbf{e}_i)\right]. \end{aligned}$$

Now, use $\bar{\mathbf{u}} = \bar{u}_M\mathbf{e}_M$ and get that the sum in [**17**] for this contribution to the trace is exactly the following:

$$-\frac{1}{2}R_{ij}\bar{\mathbf{u}}\cdot\left[B(\mathbf{e}_i,\mathbf{e}_j) + B(\mathbf{e}_j,\mathbf{e}_i)\right]. \qquad [28]$$

With [**21**], [**23**], and [**28**], we deduce the following.

Proposition 3. *Under the structure assumption in* [**14**] *and* [**15**] *on the basis* $\mathbf{e}_i$, *the fluctuating energy*, $E' = (1/2)\mathrm{tr}R$, *for any turbulent dynamical system satisfies the following*:

$$\begin{aligned} \frac{dE'}{dt} &= \frac{1}{2}\mathrm{tr}\left(\tilde{D}R + R\tilde{D}^*\right) + \frac{1}{2}\mathrm{tr}Q_\sigma \\ &\quad - \frac{1}{2}R_{ij}\bar{\mathbf{u}}\cdot\left[B(\mathbf{e}_i,\mathbf{e}_j) + B(\mathbf{e}_j,\mathbf{e}_i)\right], \end{aligned} \qquad [29]$$

where R satisfies the exact covariance equation in [**6**].

Note that the nonlinear energy transfer to the fluctuations by the mean in *Proposition 3* exactly balances the nonlinear energy transfer from the fluctuations to the mean.

Thus, adding the results in *Proposition 2* and *Proposition 3*, we have the following.

Theorem. (*Statistical Energy Conservation Principle*) *Under the structural assumption* [**14**], [**15**] *on the basis* $\mathbf{e}_i$, *for any turbulent dynamical systems in* [**1**], *the total statistical energy*, $E = \bar{E} + E' = (1/2)\bar{\mathbf{u}}\cdot\bar{\mathbf{u}} + (1/2)\mathrm{tr}R$, *satisfies the following:*

$$\frac{dE}{dt} = \bar{\mathbf{u}}\cdot D\bar{\mathbf{u}} + \bar{\mathbf{u}}\cdot\mathbf{F} + \mathrm{tr}(\tilde{D}R) + \frac{1}{2}\mathrm{tr}Q_\sigma, \qquad [30]$$

where R satisfies the exact covariance equation in [**6**].

Other interesting identities for energy exchange between the mean and the fluctuations for the Navier–Stokes equation have been derived (15, 16).

Illustrative General Examples and Applications

We have the interesting immediate corollary of the *Theorem*.

Corollary 1. *Under the assumption of the* Theorem, *assume* $D = -dI$, *with* $d > 0$, *then the turbulent dynamical system satisfies the closed statistical energy equation for* $E = (1/2)\bar{\mathbf{u}} \cdot \bar{\mathbf{u}} + (1/2)\mathrm{tr}R$,

$$\frac{dE}{dt} = -2dE + \bar{\mathbf{u}} \cdot \mathbf{F} + \frac{1}{2}\mathrm{tr}Q_\sigma. \qquad \textbf{[31]}$$

In particular, if the external forcing vanishes so that $F \equiv 0, Q_\sigma \equiv 0$, *for random initial conditions, the statistical energy decays exponentially in time and satisfies* $E(t) = \exp(-2dt)E_0$.

Assume the symmetric dissipation matrix, D, satisfies the upper and lower bounds,

$$-d_+|\mathbf{u}|^2 \geq \mathbf{u} \cdot D\mathbf{u} \geq -d_-|\mathbf{u}|^2, \qquad \textbf{[32]}$$

with $d_-, d_+ > 0$. Typical general dissipation matrices $\tilde{D}$ are diagonal in basis with Fourier modes or spherical harmonics (4). Now for any diagonal matrix $\tilde{D}$ and any positive symmetric matrix $R \geq 0$, we have the a priori bounds,

$$-d_+\mathrm{tr}R \geq \mathrm{tr}\left(\frac{\tilde{D}R + R\tilde{D}^*}{2}\right) \geq -d_-\mathrm{tr}R. \qquad \textbf{[33]}$$

Thus, with the *Theorem* and *Corollary 1*, we immediately have the following.

Corollary 2. *Assume* $\tilde{D}$ *is diagonal and satisfies the upper and lower bounds in* **[32]**, *then the statistical energy in* **[30]** *in the* Theorem, $E(t)$, *satisfies the upper and lower bounds* $E_+(t) \geq E(t) \geq E_-(t)$, *where* $E_\pm(t)$ *satisfy the differential equality in Corollary 1 with* $d \equiv d_\pm$. *In particular, the statistical energy is a statistical Lyapunov function for the turbulent dynamical system in* **[1]**. *Also, if the external forcings* F, Q_σ *vanish, the statistical energy decays exponentially with these upper and lower bounds.*

In standard fashion if some bound is known on the statistical mean in **[31]**, then this also provides control of the total variance and in particular $\mathrm{tr}R \equiv \sum_k \langle |Z_k|^2 \rangle$. Consider the Gaussian approximation to the one point statistics; recall that $\mathbf{u} = \bar{\mathbf{u}} + Z_i\mathbf{e}_i$ so at the location x, the mean and variance are given by the following:

$$\bar{\mathbf{u}}(x) = \bar{u}_M \mathbf{e}_M(x), \mathrm{var}(\mathbf{u}(x)) = \langle Z_j Z_k^* \rangle \mathbf{e}_j(x) \otimes \mathbf{e}_k(x).$$

We have control over the variance of the average over the domain, denoted by $\mathbb{E}_x$ because $\mathbb{E}_x \mathbf{e}_j(x) \otimes \mathbf{e}_k(x) = \delta_{jk} I$; thus, the average of the single point variance is bounded by $\mathrm{tr}R$, which is controlled by E. See ref. 5 for an explicit demonstration.

In many geophysical applications with strong inhomogeneities through topography, the symmetric matrix D in **[1]** from the background linear operator is neither diagonal nor negative definite. In these situations, although the equation for R is not closed according to **[6]** in general, the *Theorem* can be used as a framework for obtaining bounds to imperfect closure models that overcome the curse of dimensionality for uncertainty quantification (UQ). Such closure models for turbulent dynamical systems can be combined with an information-theoretic framework to calibrate the imperfect models in a training phase (17, 18) for such important problems as the response to the change in external forcing. This strategy for imperfect models using such an energy principle has recently been tested on the Lorenz 96 (L-96) system (19) for a family of low-order closure models (5) where *Corollary 1* above is derived in a different explicit fashion for these models for homogeneous statistics. Other examples of using low-order closure for UQ include the modified quasilinear Gaussian closure (6, 7) and applications include the L-96 model and linearly unstable two-layer baroclinic turbulence, which is another complex turbulent dynamical system satisfying all of the assumptions of the *Theorem*. Other complex turbulent dynamical systems where the *Theorem* should prove useful for UQ include the following turbulent dynamical systems, which satisfy the assumptions of the *Theorem*:

A) The turbulent Navier–Stokes equations in two or three dimensions with shear, in periodic or channel geometry;
B) Two-layer or even multilayer stratified flows with topography and shear in periodic, channel geometry or on the sphere (1, 4);
C) The rotating stratified Boussinesq equations with both gravity waves and vortices (1).

The details of these diverse physical settings present intriguing applications for the near future. Here the use of energy and enstrophy metrics or these combinations that satisfy the statistical symmetries for the nonlinear terms has been avoided for simplicity in exposition but could be crucial in these more sophisticated applications (1, 4, 8).

We conclude our discussion with a concrete geophysical demonstration. Consider spatially periodic β-plane turbulence in two dimensions (4) given by the dynamics:

$$\begin{aligned} \frac{d\omega_\Lambda}{dt} &= -\mathscr{P}_\Lambda(\nabla^\perp\psi_\Lambda \cdot \nabla\omega_\Lambda) - \beta(\psi_\Lambda)_x + D(\Delta)\omega_\Lambda + F_\Lambda(x,t) \\ &\quad + \dot{W}_k \sigma_{k,\Lambda}(t), \\ \Delta\psi_\Lambda &= \omega_\Lambda. \end{aligned} \qquad \textbf{[34]}$$

The equation in **[34]** is the Galerkin projection of the standard β-plane equation to a reduced finite dimensional subspace of 2D Fourier modes, which form the orthonormal basis $\mathbf{e}_i$ with $\mathscr{P}_\Lambda u = u_\Lambda$, the L^2 projection on these Fourier modes. The term $\beta(\psi_\Lambda)_x$ is a skew-symmetric operator. $D(\Delta)$ is diagonal in the Fourier basis and satisfies the assumption in **[32]** of *Corollary 2* for general combination of Ekman damping and hyperviscosity (4). The nonlinear terms define B with dynamics alone:

$$\frac{\partial\omega_\Lambda}{\partial t} = -\mathscr{P}_\Lambda(\nabla^\perp\psi_\Lambda \cdot \nabla\omega_\Lambda), \Delta\psi_\Lambda = \omega_\Lambda,$$

which satisfy both the conservation of energy, E_Λ, and enstrophy, $\mathscr{E}_\Lambda$ (4), with

$$E_\Lambda = -\frac{1}{2}\int \psi_\Lambda\omega_\Lambda d\mathbf{x}, \quad \text{and} \quad \mathscr{E}_\Lambda = \frac{1}{2}\int \omega_\Lambda^2 d\mathbf{x}. \qquad \textbf{[35]}$$

It is well known that, in a Fourier basis, the symmetry conditions of *Proposition 1* are satisfied for either the L^2 inner product for $\mathscr{E}_\Lambda$ or with a rescaled version of E_Λ (1, 8):

$$\begin{aligned} \langle \mathbf{u}_1, \mathbf{u}_2 \rangle_{\mathscr{E}} &\equiv \int \omega_1\omega_2 d\mathbf{x} = \int \Delta\psi_1 \Delta\psi_2 d\mathbf{x}, \\ \langle \mathbf{u}_1, \mathbf{u}_2 \rangle_E &\equiv -\int \psi_1\omega_2 d\mathbf{x} = \int \nabla\psi_1 \cdot \nabla\psi_2 d\mathbf{x}, \end{aligned}$$

with $\mathbf{u} = \nabla^\perp\psi$. There is a separate statistical energy identity for both the energy and enstrophy, which should be useful in practice. Thus, all of the assumptions of *Theorem* and *Corollary 2* are satisfied for the turbulent dynamical system in **[34]**, so these general results on statistical energy exchange between mean and fluctuations apply. It is well-known that there are interesting statistical bifurcations between jets and vortices as parameters vary and it is a contemporary challenge to explain these with statistical theory (4, 20). The systematic statistical energy and enstrophy principles developed here could be useful to gain insight on these issues.

Concluding Discussion

A systematic statistical energy conservation principle has been developed here in a *Theorem* that assesses the statistical energy exchange between an ensemble averaged mean flow and the related turbulent fluctuations. This statistical energy is a sum of the energy in the mean and the trace of the covariance of the fluctuations. This result applies to general inhomogeneous turbulent dynamical systems, which are ubiquitous in engineering and climate atmosphere ocean science. It involves an assessment of the detailed energy conservation principles of the deterministic nonlinear dynamics, which induce detailed statistical symmetries for the nonlinear terms yielding the exact statistical energy conservation principle. *Corollary 1* and *Corollary 2* illustrate the power of the method with general closed differential equalities for the statistical energy either exactly or bounding the statistical energy in time under the natural hypothesis of a bounded negative definite diagonal dissipation matrix. In particular, the statistical energy is a stochastic Lyapunov functional. A concrete example of β-plane turbulence is briefly discussed here as a simple illustration. In general, the statistical energy principle is not closed and bounds on the full fluctuating covariance are required. Nevertheless, this formulation of the energy principle is useful for systematic closure models on low-order subspaces (5–7), a problem with great practical interest. The systematic energy principle illustrates the fact that accurate prediction of the mean with a low-order imperfect model automatically guarantees accurate prediction of the mean and variance of the single-point statistics provided the imperfect model respects suitable statistical symmetries. This has been demonstrated recently (5) for the 40-mode L-96 model, which serves as a template for future applications of the energy principle to realistic complex inhomogeneous turbulent dynamical systems in engineering and climate atmosphere ocean science.

The statistical energy principle including the statistical energy of the mean and the trace of the covariance of the fluctuations has been developed here for finite dimensional inhomogeneous turbulent dynamical systems. An important mathematical problem is to extend this to the infinite dimensional setting for the Navier–Stokes equation and use the statistical Lyapunov functional in *Corollary 2* to help prove geometric ergodicity. At the present time, there is the celebrated proof of geometry ergodicity of the stochastic Navier–Stokes equations (21) under hypotheses of minimal stochastic forcing but that make the mean flow vanish. The only rigorous result with a mean flow and rigorous small random fluctuations interacting involves the random bombardment of the Navier–Stokes equations by coherent vortices (22).

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