Blended Particle Filters for Large Dimensional Chaotic Dynamical Systems – Supplementary material

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1 Proof of Proposition 1

Proposition 1. *Assume the prior distribution from the forecast is the blended particle filter conditional Gaussian distribution*

$$p_-(\boldsymbol{u}) = \sum_{j=1}^{Q} p_{j,-}\delta(\boldsymbol{u}_1 - \boldsymbol{u}_{1,j})\mathcal{N}\left(\bar{\boldsymbol{u}}_{2,j}^-, R_{2,j}^-\right), \quad [1.1]$$

and assume the observations have the structure

$$\boldsymbol{v} = G(\boldsymbol{u}) + \boldsymbol{\sigma}_0 = G_0(\boldsymbol{u}_1) + G_1(\boldsymbol{u}_1)\boldsymbol{u}_2 + \boldsymbol{\sigma}_0, \quad [1.2]$$

then the posterior distribution in the analysis step taking into account the observations in [1.2] is also a blended particle filter conditional Gaussian distribution, i.e. there are explicit formulas for the updated weights, $p_{j,+}$, $1 \le j \le Q$, and conditional mean, $\bar{\boldsymbol{u}}_{2,j}^+$, and covariance, $R_{2,j}^+$, so that

$$p_+(\boldsymbol{u}) = \sum_{j=1}^{Q} p_{j,+}\delta(\boldsymbol{u}_1 - \boldsymbol{u}_{1,j})\mathcal{N}\left(\bar{\boldsymbol{u}}_{2,j}^+, R_{2,j}^+\right). \quad [1.3]$$

In fact, the distributions $\mathcal{N}\left(\bar{\boldsymbol{u}}_{2,j}^+, R_{2,j}^+\right)$ are updated by suitable Kalman filter formulas with the mean update for $\bar{\boldsymbol{u}}_{2,j}$ depending nonlinearly on $\boldsymbol{u}_{1,j}$ in general.

Proof. Let $\boldsymbol{u} = (\boldsymbol{u}_1, \boldsymbol{u}_2)$, $\boldsymbol{u}_j \in \mathbb{R}^{N_j}$, $N_1 + N_2 = N$, and $\boldsymbol{u}_1 = \{\boldsymbol{u}_{1,j}\}_{j=1}^{Q}$ are sampled in Q discrete states such that [1.1] is satisfied. Using discrete representation of $\boldsymbol{u}_1$, the posterior distribution can be written as

$$p_+(\boldsymbol{u} \mid \boldsymbol{v}) = \sum_{j=1}^{Q} p_+(\boldsymbol{u}, \boldsymbol{u}_{1,j} \mid \boldsymbol{v}) = \sum_{j=1}^{Q} p_+(\boldsymbol{u} \mid \boldsymbol{v}, \boldsymbol{u}_{1,j})\, p_{j,+}, \quad [1.4]$$

where $p_{j,+} = p_+(\boldsymbol{u}_{1,j} \mid \boldsymbol{v})$ is the posterior probability of the j-th state.

1. First, consider the general likelihood function $p(\boldsymbol{v} \mid \boldsymbol{u})$ (not possibly Gaussian). A simply application of Bayes Theorem gives

$$\begin{aligned} p_+(\boldsymbol{u} \mid \boldsymbol{v}, \boldsymbol{u}_{1,j}) &= \frac{p(\boldsymbol{v} \mid \boldsymbol{u}, \boldsymbol{u}_{1,j})\, p_-(\boldsymbol{u}, \boldsymbol{u}_{1,j})}{p(\boldsymbol{v}, \boldsymbol{u}_{1,j})} \\ &= \frac{p(\boldsymbol{v} \mid \boldsymbol{u}, \boldsymbol{u}_{1,j})\, p_-(\boldsymbol{u} \mid \boldsymbol{u}_{1,j})}{\int p(\boldsymbol{v} \mid \boldsymbol{u}_{1,j}, \boldsymbol{u}_2)\, p(\boldsymbol{u}_2 \mid \boldsymbol{u}_{1,j})\, d\boldsymbol{u}_2} \\ &= \frac{p(\boldsymbol{v} \mid \boldsymbol{u}_{1,j}, \boldsymbol{u}_2)\, \Pi_{j,-}}{\tilde{I}_j}, \end{aligned} \quad [1.5]$$

with $\tilde{I}_j = \int p(\boldsymbol{v} \mid \boldsymbol{u}_{1,j}, \boldsymbol{u}_2) \mathcal{N}\left(\bar{\boldsymbol{u}}_{2,j}^-, R_{2,j}^-\right) d\boldsymbol{u}_2,\ 1 \le j \le Q$, and $\Pi_{j,-} = p_-(\boldsymbol{u} \mid \boldsymbol{u}_{1,j}) = \delta(\boldsymbol{u}_1 - \boldsymbol{u}_{1,j}) \mathcal{N}\left(\bar{\boldsymbol{u}}_{2,j}^-, R_{2,j}^-\right)$ by the conditional Gaussian assumption.

2. Then, to determine the posterior weights $p_{j,+}$, the formula of conditional probability for posterior gives

$$\begin{aligned} p(\boldsymbol{u}, \boldsymbol{u}_{1,j}, \boldsymbol{v}) &= p(\boldsymbol{u} \mid \boldsymbol{v}, \boldsymbol{u}_{1,j}) p_+(\boldsymbol{u}_{1,j} \mid \boldsymbol{v}) p(\boldsymbol{v}) \\ &= p(\boldsymbol{u} \mid \boldsymbol{v}, \boldsymbol{u}_{1,j}) p_{j,+} p(\boldsymbol{v}), \end{aligned} \quad [1.6]$$

and for prior

$$\begin{aligned} p(\boldsymbol{u}, \boldsymbol{u}_{1,j}, \boldsymbol{v}) &= p(\boldsymbol{v} \mid \boldsymbol{u}, \boldsymbol{u}_{1,j}) p_-(\boldsymbol{u}, \boldsymbol{u}_{1,j}) \\ &= p(\boldsymbol{v} \mid \boldsymbol{u}, \boldsymbol{u}_{1,j}) p_{j,-} \Pi_{j,-}. \end{aligned} \quad [1.7]$$

Compare [1.6] with [1.7], we get the identity

$$p(\boldsymbol{v} \mid \boldsymbol{u}, \boldsymbol{u}_{1,j}) p_{j,-} \Pi_{j,-} = p(\boldsymbol{u} \mid \boldsymbol{v}, \boldsymbol{u}_{1,j}) p_{j,+} p(\boldsymbol{v}). \quad [1.8]$$

Use the fact $\sum_{j=1}^{Q} \int p(\boldsymbol{u} \mid \boldsymbol{v}, \boldsymbol{u}_{1,j}) p_{j,+} d\boldsymbol{u}_2 = 1$ and integrate both sides of [1.8]

$$p(\boldsymbol{v}) = \sum_{j=1}^{Q} \int p(\boldsymbol{v} \mid \boldsymbol{u}_{1,j}, \boldsymbol{u}_2) p_{j,-} \Pi_{j,-} d\boldsymbol{u}_2 = \sum_{j=1}^{Q} p_{j,-} \tilde{I}_j. \quad [1.9]$$

Therefore, substitute [1.5] and [1.9] into [1.8],

$$\begin{aligned} p_{j,+} &= \frac{p_{j,-} \int p(\boldsymbol{v} \mid \boldsymbol{u}_{1,j}, \boldsymbol{u}_2) p(\boldsymbol{u}_2 \mid \boldsymbol{u}_{1,j}) d\boldsymbol{u}_2}{p(\boldsymbol{v})} \\ &= \frac{p_{j,-} \tilde{I}_j}{\sum_{k=1}^{Q} p_{k,-} \tilde{I}_k}. \end{aligned} \quad [1.10]$$

3. Finally with observations in the form of [1.2] and Gaussian likelihood function $p(\boldsymbol{v} \mid \boldsymbol{u}_{1,j}, \boldsymbol{u}_2) = p_G(\boldsymbol{v} - G_0(\boldsymbol{u}_{1,j}) - G_1(\boldsymbol{u}_{1,j}) \boldsymbol{u}_2)$, $p_G \sim \mathcal{N}(\boldsymbol{0}, R_0)$, the numerator of [1.5] becomes the standard process of Kalman filter. That is,

$$\begin{aligned} p_+(\boldsymbol{u} \mid \boldsymbol{v}, \boldsymbol{u}_{1,j}) &= \frac{\delta(\boldsymbol{u}_1 - \boldsymbol{u}_{1,j}) \mathcal{N}\left(\bar{\boldsymbol{u}}_{2,j}^-, R_{2,j}^-\right) p_G(\boldsymbol{v} - G_0(\boldsymbol{u}_{1,j}) - G_1(\boldsymbol{u}_{1,j}) \boldsymbol{u}_2)}{\tilde{I}_j} \\ &= \delta(\boldsymbol{u}_1 - \boldsymbol{u}_{1,j}) \mathcal{N}\left(\bar{\boldsymbol{u}}_{2,j}^+, R_{2,j}^+\right), \end{aligned} \quad [1.11]$$

where $\left(\bar{\boldsymbol{u}}_{2,j}^+, R_{2,j}^+\right)$, depending nonlinearly on $\boldsymbol{u}_{1,j}$ in general, are the posterior mean and covariance through suitable Kalman filtering process. Also the weight updating factor $\tilde{I}_j$ can be calculated explicitly by integrating the multiplication of two Gaussian distributions

$$\begin{aligned} \tilde{I}_j &= \int \exp\left(-\frac{1}{2}(\boldsymbol{v} - G(\boldsymbol{u}))^{\mathrm{T}} R_0^{-1} (\boldsymbol{v} - G(\boldsymbol{u}))\right) \exp\left(-\frac{1}{2}\left(\boldsymbol{u}_2 - \bar{\boldsymbol{u}}_{2,j}^-\right)^{\mathrm{T}} R_{2,j}^- \left(\boldsymbol{u}_2 - \bar{\boldsymbol{u}}_{2,j}^-\right)\right) d\boldsymbol{u}_2 \\ &= \det\left(R_{2,j}^+\right)^{-\frac{1}{2}} \exp\left[\frac{1}{2}\left(\bar{u}_{2,j}^{+\mathrm{T}} R_{2,j}^{+-1} \bar{u}_{2,j}^+ - \bar{u}_{2,j}^{-\mathrm{T}} R_{2,j}^{--1} \bar{u}_{2,j}^- - \left(v - GEu_{1,j}^-\right)^{\mathrm{T}} R_0^{-1} \left(v - GEu_{1,j}^-\right)\right)\right]. \end{aligned} \quad [1.12]$$

Therefore, the expression for the posterior distribution [1.4] is derived explicitly in the form [1.10], [1.11], and [1.12]. □

2 Details of Blended Filter Algorithms

Here we describe the details about the blended filter algorithms. After the forecast step, we get the predicted values for the mean states $\bar{\boldsymbol{u}}$ and covariance matrix R, together with the particle presentation under DO basis $E(t) = \{\boldsymbol{e}_1(t), ..., \boldsymbol{e}_s(t)\}$,

$$\boldsymbol{u}(t) = \bar{\boldsymbol{u}}(t) + \sum_{i=1}^{s} Y_i(t; \omega) \boldsymbol{e}_i(t),$$

where $\{\boldsymbol{e}_i(t)\}_{i=1}^{s}$ is the subdimensional dynamical basis, and $\{Y_i\}$ is the corresponding stochastic coefficients achieved through Monte-Carlo simulations. The particle statistics presented by Y_i must be consistent with the covariance matrix R, and Y_i , Y_j are normalized to be independent with each other, that is

$$\left\langle Y_i Y_j^* \right\rangle = \boldsymbol{e}_i \cdot R\boldsymbol{e}_j \delta_{ij}, \qquad 1 \le i, j \le s.$$

For the analysis step, filtering is implemented in two separate subspaces. The projection operator is calculated by completing the dynamical basis $P = \left[E, E^{\perp}\right]$. The orthogonal complement $E^{\perp}$ are chosen freely here. Initially each particle is uniformly weighted as $(Y_i, p_i) = \left(Y_i, \frac{1}{Q}\right)$. Following is the analysis step algorithm for the blended methods.

Algorithm. *Blended filtering (analysis step)*

- *Project the mean and covariance matrix to the two subspaces*

$$\begin{pmatrix} \bar{\boldsymbol{u}}_1 \\ \bar{\boldsymbol{u}}_2 \end{pmatrix} = P^{\mathrm{T}} \bar{\boldsymbol{u}}, \qquad R_t = P^{\mathrm{T}} R P = \begin{pmatrix} R_1 & R_{12} \\ R_{12}^{\mathrm{T}} & R_2 \end{pmatrix}. \tag{2.1}$$

- *Solve the following linear system to find the conditional mean* $\bar{\boldsymbol{u}}_2(\boldsymbol{u}_{1,j}) = \bar{\boldsymbol{u}}_{2,j}^{-}$ *(denote* $\boldsymbol{u}_1' = \boldsymbol{u}_1 - \langle \boldsymbol{u}_1 \rangle$, $\bar{\boldsymbol{u}}_2'(\boldsymbol{u}_1) = \bar{\boldsymbol{u}}_2(\boldsymbol{u}_1) - \langle \boldsymbol{u}_2 \rangle$ *as the fluctuations about the mean)*

$$\begin{bmatrix} p_1 u_{1,1}'^{1} & \cdots & p_Q u_{1,1}'^{Q} \\ \vdots & \ddots & \vdots \\ p_1 u_{1,N_1}'^{1} & \cdots & p_Q u_{1,N_1}'^{Q} \\ p_1 & \cdots & p_Q \end{bmatrix}_{(N_1+1)\times Q} \begin{bmatrix} u_{2,1}'^{1} & \cdots & u_{2,N_2}'^{1} \\ u_{2,1}'^{2} & \cdots & u_{2,N_2}'^{2} \\ \vdots & \ddots & \vdots \\ u_{2,1}'^{Q} & \cdots & u_{2,N_2}'^{Q} \end{bmatrix}_{Q\times N_2} = \begin{bmatrix} R_{12} \\ 0 \end{bmatrix}. \tag{2.2}$$

Note that this is an underdetermined system for a sufficiently large number of particles, $Q \gg N_1 + 1$, *and '–' notation is suppressed here.*

- *Calculate the conditional covariance* R_2^- *in* $\boldsymbol{u}_2$ *subspace by*

$$\begin{aligned} R_2^- &= R_2 + \langle \boldsymbol{u}_2 \rangle \otimes \langle \boldsymbol{u}_2 \rangle - \int \bar{\boldsymbol{u}}_2(\boldsymbol{u}_1) \otimes \bar{\boldsymbol{u}}_2(\boldsymbol{u}_1)\, p(\boldsymbol{u}_1)\, d\boldsymbol{u}_1 \\ &= R_2 - \int \bar{\boldsymbol{u}}_2'(\boldsymbol{u}_1) \otimes \bar{\boldsymbol{u}}_2'(\boldsymbol{u}_1)\, p(\boldsymbol{u}_1)\, d\boldsymbol{u}_1 \\ &= R_2 - \sum_j \bar{\boldsymbol{u}}_{2,j}' \otimes \bar{\boldsymbol{u}}_{2,j}' p_{j,-}. \end{aligned} \tag{2.3}$$

- *Use Kalman filter updates in the* $\boldsymbol{u}_2$ *subspace*

$$\bar{\boldsymbol{u}}_{2,j}^{+} = \bar{\boldsymbol{u}}_{2,j}^{-} + K\left(\boldsymbol{v} - GE\boldsymbol{u}_{1,j}^{-} - GE^{\perp}\bar{\boldsymbol{u}}_{2,j}^{-}\right), \tag{2.4a}$$

$$\tilde{R}_2^{+} = \left(I - KGE^{\perp}\right) R_2^{-}, \tag{2.4b}$$

$$K = R_2^{-}\left(GE^{\perp}\right)^{\mathrm{T}} \left(GE^{\perp} R_2^{-} \left(GE^{\perp}\right)^{\mathrm{T}} + R_0\right)^{-1}, \tag{2.4c}$$

with the (linear) observation operator $G(\boldsymbol{u}_1, \boldsymbol{u}_2) = GE\boldsymbol{u}_1 + GE^{\perp}\boldsymbol{u}_2$, *where* R_0 *is the covariance matrix for the observation noise.*

- *Update the particle weights in the* $\boldsymbol{u}_1$ *subspace by* $p_{j,+} \propto p_{j,-} I_j$, *with*

$$I_j = \exp\left[\frac{1}{2}\left(\bar{\boldsymbol{u}}_{2,j}^{+\mathrm{T}} \left(\tilde{R}_2^{+}\right)^{-1} \bar{\boldsymbol{u}}_{2,j}^{+} - \bar{\boldsymbol{u}}_{2,j}^{-\mathrm{T}} \left(R_2^{-}\right)^{-1} \bar{\boldsymbol{u}}_{2,j}^{-} - \left(\boldsymbol{v} - GE\boldsymbol{u}_{1,j}\right)^{\mathrm{T}} R_0^{-1} \left(\boldsymbol{v} - GE\boldsymbol{u}_{1,j}\right)\right)\right]. \tag{2.5}$$

- *Normalize the weights* $p_{j,+} = \frac{p_{j,-} I_j}{\sum_j p_{j,-} I_j}$ *and use residual resampling.*

- *Get the posterior mean and covariance matrix from the posterior particle presentation*

$$\bar{\boldsymbol{u}}_1^+ = \sum_j \boldsymbol{u}_{1,j} p_{j,+},\ \bar{\boldsymbol{u}}_2^+ = \sum_j \bar{\boldsymbol{u}}_{2,j}^+ p_{j,+}, \tag{2.6a}$$

and

$$R_{1,ij}^+ = \sum_k Y_{i,k} Y_{j,k}^* p_{k,+},\ 1 \le i,j \le s, \tag{2.6b}$$

$$R_{12,ij}^+ = \sum_k Y_{i,k} \bar{u}_{j,k}^{\prime +*} p_{k,+},\ 1 \le i \le s,\ s+1 \le j \le N, \tag{2.6c}$$

$$R_2^+ = \tilde{R}_2^+ + \sum_j \bar{\boldsymbol{u}}_{2,j}^{\prime +} \otimes \bar{\boldsymbol{u}}_{2,j}^{\prime +} p_{j,+}. \tag{2.6d}$$

- *Rotate the stochastic coefficients and basis to principal directions in the s-dimensional dynamical subspace.*

Remark. 1. As derived in [1.12] of Proposition 1, the weight updating factor $\tilde{I}_j$ has an additional component $\det\left(R_{2,j}^+\right)^{-\frac{1}{2}}$, which is expensive to compute given the high dimensionality of the orthogonal subspace $\boldsymbol{u}_2$. For the blended algorithms in the main text, R_2^- is independent of the choice of particles j, so R_2^+ is also independent of j by [2.4b]. Therefore the expensive determinant term can be neglected in [2.5] as a normalization factor for the updated weights $p_{j,+}$.

2. By the blended forecast model, the dynamical basis $\boldsymbol{e}_j(t)$ keeps tracking the principal directions of the system (that is, the direction with the largest eigenvalues of the covariance matrix $R(t)$). However, after the analysis step, the principal directions are changed according to the observation data. Therefore, it is necessary to rotate the basis according to the posterior covariance after every analysis step as described in the last step above. Although numerical tests show reasonable results without the rotation, corrections at every step can make sure that the scheme is robust and more accurate.

2.1 The conditional covariance $R_2(\boldsymbol{u}_1)$ given particles in $\boldsymbol{u}_1$ subspace

The forecast model gives the particle representation $\boldsymbol{u}_{1,j}$ in the low dimensional subspace. And the conditional Gaussian mixture $(\bar{\boldsymbol{u}}_{2,j}, R_{2,j})$ in the orthogonal subspace is achieved by solving the least squares solution of the linear system [2.2] with minimum weighted L^2-norm, $\sum_{j=1}^{Q} \left|\bar{\boldsymbol{u}}_{2,j}'\right|^2 p_j$. Such idea comes from the maximum entropy principle. Another advantage of this max-entropy solution is that it concludes that the conditional covariance $R_2(\boldsymbol{u}_1) = R_2^-$ is independent of the particles in $\boldsymbol{u}_1$ subspace.

One important point to note is that the prior covariance R_2 from [2.1] is different from the conditional covariance $R_2(\boldsymbol{u}_1) = R_2^-$ given the particle values in the $\boldsymbol{u}_1$ subspace. This can be seen clearly from their definitions

$$R_2 = \iint (\boldsymbol{u}_2 - \langle \boldsymbol{u}_2 \rangle) \otimes (\boldsymbol{u}_2 - \langle \boldsymbol{u}_2 \rangle)\, p(\boldsymbol{u}_2 \mid \boldsymbol{u}_1)\, p(\boldsymbol{u}_1)\, d\boldsymbol{u}_2 d\boldsymbol{u}_1, \tag{2.7}$$

$$R_2(\boldsymbol{u}_1) = \int (\boldsymbol{u}_2 - \bar{\boldsymbol{u}}_2(\boldsymbol{u}_1)) \otimes (\boldsymbol{u}_2 - \bar{\boldsymbol{u}}_2(\boldsymbol{u}_1))\, p(\boldsymbol{u}_2 \mid \boldsymbol{u}_1)\, d\boldsymbol{u}_2. \tag{2.8}$$

with $\langle \boldsymbol{u}_2 \rangle = \int \bar{\boldsymbol{u}}_2(\boldsymbol{u}_1)\, p(\boldsymbol{u}_1)\, d\boldsymbol{u}_1$. By a simple calculation combining [2.7] and [2.8] we have

$$\begin{aligned} \int R_2(\boldsymbol{u}_1)\, p(\boldsymbol{u}_1)\, d\boldsymbol{u}_1 &= R_2 + \langle \boldsymbol{u}_2 \rangle \otimes \langle \boldsymbol{u}_2 \rangle - \int \bar{\boldsymbol{u}}_2(\boldsymbol{u}_1) \otimes \bar{\boldsymbol{u}}_2(\boldsymbol{u}_1)\, p(\boldsymbol{u}_1)\, d\boldsymbol{u}_1 \\ &= R_2 - \int \bar{\boldsymbol{u}}_2'(\boldsymbol{u}_1) \otimes \bar{\boldsymbol{u}}_2'(\boldsymbol{u}_1)\, p(\boldsymbol{u}_1)\, d\boldsymbol{u}_1, \end{aligned}$$

with $\bar{\boldsymbol{u}}_2'(\boldsymbol{u}_1) = \bar{\boldsymbol{u}}_2(\boldsymbol{u}_1) - \langle \boldsymbol{u}_2 \rangle$. Noting that $R_2(\boldsymbol{u}_1) = R_2^-$ is independent of the variable $\boldsymbol{u}_1$, we get the approximation for the conditional covariance

$$\begin{aligned} R_2^- &= R_2 - \int \bar{\boldsymbol{u}}_2'(\boldsymbol{u}_1) \otimes \bar{\boldsymbol{u}}_2'(\boldsymbol{u}_1)\, p(\boldsymbol{u}_1)\, d\boldsymbol{u}_1 \\ &= R_2 - \sum_j \bar{\boldsymbol{u}}_{2,j}' \otimes \bar{\boldsymbol{u}}_{2,j}' p_j. \end{aligned} \tag{2.9}$$

This conditional covariance formula is theoretically consistent, while it will introduce the problem of realizability. Then as discussed in the main text, we may need introduce a realizability check and correction method for R_2^- after each analysis step, or simply inflate the matrix as $R_2^- = R_2$ (labeled as 'inflated R_2' in the figures below). Both strategies work well for MQG-DO method while QG-DO method requires the larger inflation approach in practical implementations as shown in the results below.

2.2 Resampling strategies

Here we add some details about the resampling process. In the resampling step, particles with small weights are abandoned, while the other particles are duplicated according to their weights to keep the ensemble size unchanged. One useful and popular resample approach is the residual resampling method. The idea is, replicating the j-th particle to have $\lfloor p_j Q \rfloor$ copies according to its weight in the first step; and for the rest $Q - \sum_j \lfloor p_j Q \rfloor$ members particle values are assigned randomly with probability proportional to their residual weights $p_j Q - \lfloor p_j Q \rfloor$.

An additional issue arises in the process of duplicating particles. After resampling, we will get several particles with the same value. For stochastic systems, this will not be a problem since the random external forcing term will add uncertainty to each particle and produce different prediction results even though the initial values are the same. However for deterministic systems with internal instability (for example, the L-96 system), such resampling would be of no real value, since no external randomness is introduced to the particles to generate different outputs in the next step. One possible strategy to avoid this problem is to add small perturbations to the duplicated particles. Due to the internal instability inside the system, small perturbations in the initial time can end up with large deviations in the forecasts before next analysis step so that the duplicated particles can get separated. The amplitude of the perturbation is added according to the uncertainties of the variables. This can be approximated by the filtered results before the resampling step. After updating the weights, we get the set of particles together with their new weights $\left\{\boldsymbol{u}_j', p_j\right\}$, then a perturbation is added to each particle as a white noise with variance $\sigma^2 = \sum_j \boldsymbol{u}_j'^2 p_j$. Note that this perturbation can be viewed as an inflation to the particles. We will calculate the posterior covariance matrix according to the resampled particles, therefore the covariance will also get inflated accordingly. From numerical simulations it is shown that this strategy is effective to avoid filter divergence.

3 Blended Filter Performance on L-96 model

Here we present the additional results for the blended filter methods compared with the MQG filter as well as EAKF with inflation and localization. The L-96 model is used as the test model

$$\frac{du_i}{dt} = u_{i-1}\left(u_{i+1} - u_{i-2}\right) - u_i + F, \quad i = 0, \cdots, J-1 \qquad [3.1]$$

with $J = 40$ and F the deterministic forcing parameter. We will test the filter performances in both weakly chaotic regime ($F = 5$) and strongly chaotic regime ($F = 8$) as described in the main text. The statistics for these two cases are shown in Figure 3.1. Strong non-Gaussian statistics can be seen from the weakly chaotic regime $F = 5$ while near Gaussian distribution is shown in strongly chaotic regime $F = 8$. Figure 3.2 shows the RMS errors and pattern correlations for various filtering methods in regime $F = 5$ with moderate observational error and observation frequency $r_0 = 2, \Delta t = 1, p = 4$. And Figure 3.3 gives the joint distributions of the first two principal Fourier modes $\hat{u}_7$ and $\hat{u}_8$ in this regime by scatter plots to further visualize the non-Gaussian statistics captured through different schemes. Finally Figure 3.4 gives additional examples for the performance of the filters in regimes with sparse infrequent high quality observations with slightly larger observation noise $r_0 = 0.25$. In addition, we compare the performances of the QG-DO method with crude covariance inflation $R_2^- = R_2$. Incorrect energy transfers in the QG forecast model for the long forecast time using the original covariance approximation as in [2.9] in this case end up with serious filter divergence.

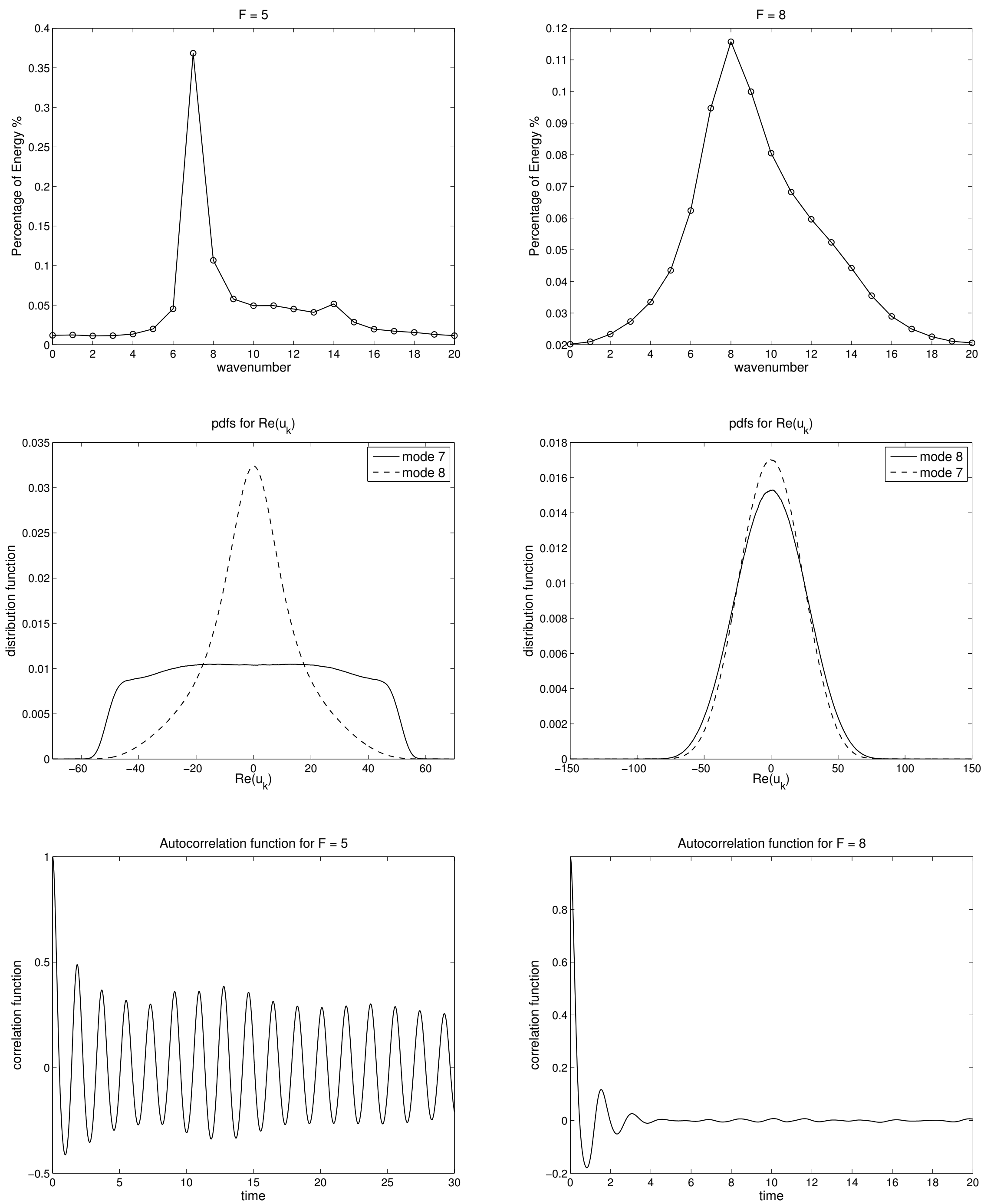
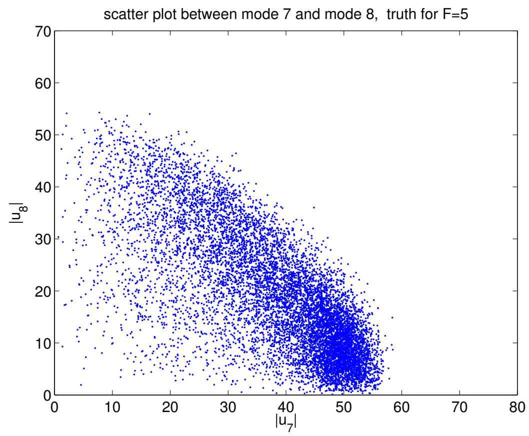
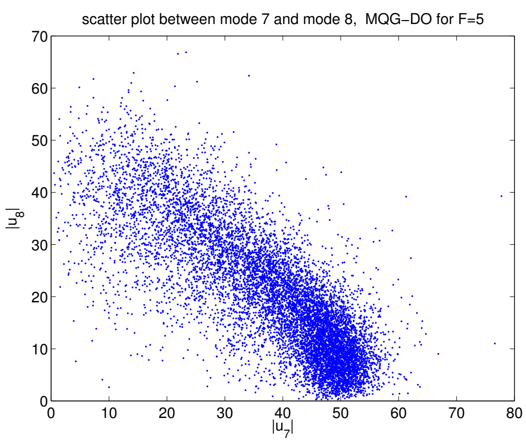


Fig. 3.1: Statistics for the L-96 model: energy spectra in Fourier space (first row); the pdfs for the first two principal Fourier modes (second row); and the autocorrelation functions of the state variable (third row). Regimes for weakly chaotic ($F = 5$, left) and strongly chaotic ($F = 8$, right) dynamics are shown.

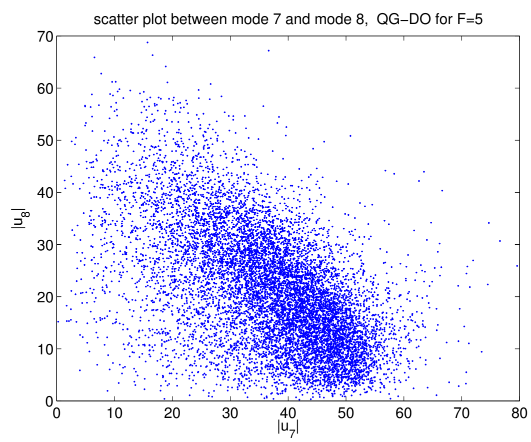
Fig. 3.2: RMS errors and pattern correlations in the weakly chaotic regime ($F = 5$) with parameters $r_0 = 2, \Delta t = 1, p = 4$. Results by different filtering methods are compared.



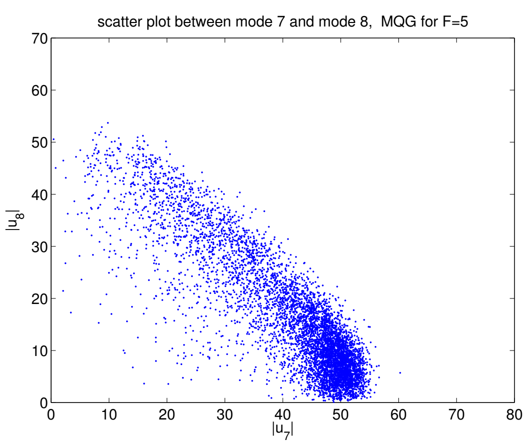
(a) truth



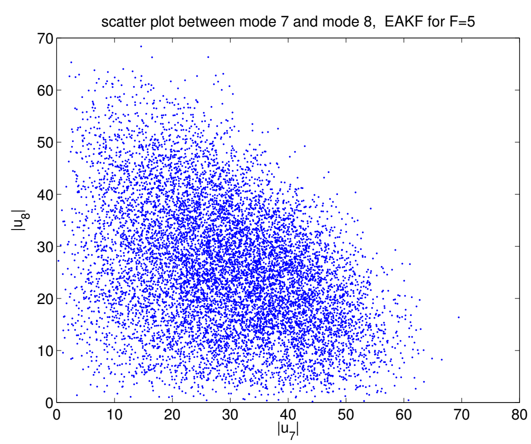
(b) MQG-DO



(c) QG-DO



(d) MQG



(e) EAKF

Fig. 3.3: Joint distributions of the first two principal Fourier modes $\hat{u}_7$ and $\hat{u}_8$ shown by scatter plots with different filter schemes in regime $F = 5$.

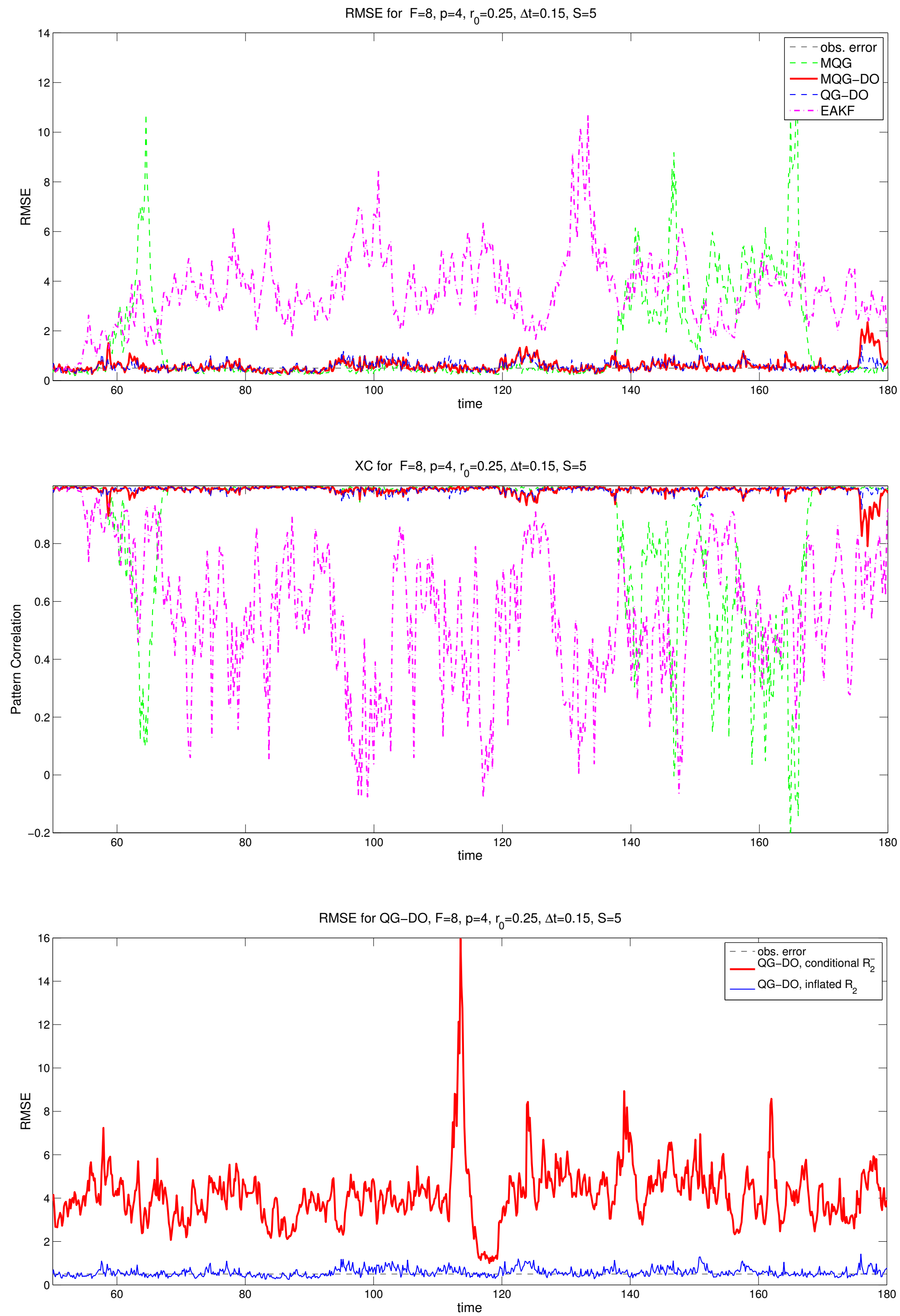


Fig. 3.4: RMS errors and pattern correlations in the strongly chaotic regime ($F = 8$) with parameters $r_0 = 0.25, \Delta t = 0.15, p = 4$. Results by different filtering methods are compared, and in addition the last row gives the comparison of the RMSEs for QG-DO method with inflated (blue) and original (red) conditional covariance R_2^-.