Conceptual dynamical models for turbulence
Supplementary material

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Table S1-S3 show climatological statistics of each mode from the negative large scale damping six dimensional conceptual models, $K = 5$, for various strengths of $\overline{F}$. In Table S1, the mean, variance, and standard deviation of $\overline{u}$ and effective damping of each small scale mode, $d_k + \gamma\langle\overline{u}\rangle$ are shown while Table S2 and S3 show variances and correlation times of the turbulent signal u, $\overline{u}$, and $u'_k, k = 1, 2, ..., 5$ respectively.

Table S1: Negative large scale damping with $K = 5$: mean, variance, and standard deviation of $\overline{u}$ and effective damping of small scale modes for various strengths of $\overline{F}$

$\overline{F}$	$\langle\overline{u}\rangle$	Var($\overline{u}$)	Std($\overline{u}$)	$d_1 + \gamma\langle\overline{u}\rangle$	$d_2 + \gamma\langle\overline{u}\rangle$	$d_3 + \gamma\langle\overline{u}\rangle$	$d_4 + \gamma\langle\overline{u}\rangle$	$d_5 + \gamma\langle\overline{u}\rangle$
0	1.4178	0.0000	0.0005	3.1467	3.2067	3.3067	3.4467	3.6267
-0.010	1.3654	0.0000	0.0005	3.0681	3.1281	3.2281	3.3681	3.5481
-0.054	0.8727	0.0000	0.0026	2.3290	2.3890	2.4890	2.6290	2.8090
-0.055	-0.6733	0.0397	0.1993	0.0025	0.0625	0.1625	0.3025	0.4825
-0.060	-0.6792	0.0416	0.2039	0.0013	0.0613	0.1613	0.3013	0.4813
-0.080	-0.6808	0.0511	0.2261	-0.0013	0.0587	0.1587	0.2987	0.4787
-0.100	-0.6814	0.0601	0.2452	-0.0021	0.0579	0.1579	0.2979	0.4779

Table S2: Negative large scale damping with $K = 5$: variances of the turbulent signal u, $\overline{u}$, and $u'_k, k = 1, 2, ..., 5$ for various strengths of $\overline{F}$

$\overline{F}$	Var(u)	Var($\overline{u}$)	Var(u'_1)	Var(u'_2)	Var(u'_3)	Var(u'_4)	Var(u'_5)
0	0.0005	0.0000	0.0002	0.0001	0.0001	0.0001	0.0000
-0.010	0.0005	0.0000	0.0002	0.0001	0.0001	0.0001	0.0000
-0.054	0.0006	0.0000	0.0003	0.0001	0.0001	0.0001	0.0001
-0.055	0.1069	0.0397	0.0446	0.0174	0.0049	0.0014	0.0005
-0.060	0.1128	0.0416	0.0468	0.0181	0.0051	0.0014	0.0005
-0.080	0.1353	0.0511	0.0557	0.0209	0.0058	0.0016	0.0005
-0.100	0.1571	0.0601	0.0660	0.0227	0.0063	0.0018	0.0006

Table S3: Negative large scale damping with $K = 5$: correlation times of the turbulent signal u, $\overline{u}$, and $u'_k, k = 1, 2, ..., 5$ for various strengths of $\overline{F}$

$\overline{F}$	$T_{cor}(u)$	$T_{cor}(\overline{u})$	$T_{cor}(u'_1)$	$T_{cor}(u'_2)$	$T_{cor}(u'_3)$	$T_{cor}(u'_4)$	$T_{cor}(u'_5)$
0	0.3030	5.0494	0.3174	0.3083	0.2937	0.2831	0.2734
-0.010	0.3102	5.6554	0.3318	0.3234	0.2917	0.2885	0.2799
-0.054	0.3895	36.07	0.4184	0.4073	0.3813	0.3717	0.3617
-0.055	24.95	34.18	29.41	15.53	6.373	4.122	2.833
-0.060	24.52	35.13	30.28	15.61	6.253	4.156	2.850
-0.080	20.32	36.13	33.18	17.88	4.817	4.037	2.987
-0.100	31.30	36.98	31.37	17.21	5.017	4.211	3.121

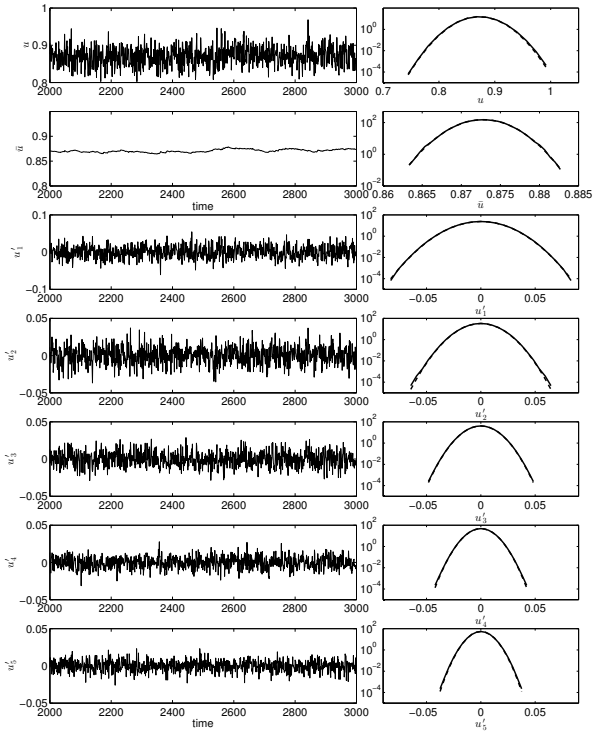


Fig. S1: Negative large scale damping with $K = 5$: time series (left column) and pdfs (right column) of the turbulent signal u, $\overline{u}$ and $u'_k, k = 1, 2, ..., 5$ with $\overline{F} = -0.054$. Note the logarithmic scale of pdfs in the y-axis. Dashed lines are Gaussian distributions with the same mean and variance.

Table S4-S6 show climatological statistics of each mode from the negative large scale damping six dimensional conceptual models, $K = 5$, for various strengths of $\overline{F}$. In Table S4, the mean, variance, and standard deviation of $\overline{u}$ and effective damping of each small scale mode, $d_k + \gamma\langle\overline{u}\rangle$ are shown while Table S5 and S6 show variances and correlation times of the turbulent signal u, $\overline{u}$, and $u'_k, k = 1, 2, ..., 5$ respectively.

Table S4: Positive large scale damping with $K = 5$: mean, variance, and standard deviation of $\overline{u}$ and effective damping of small scale modes for various strengths of $\overline{F}$

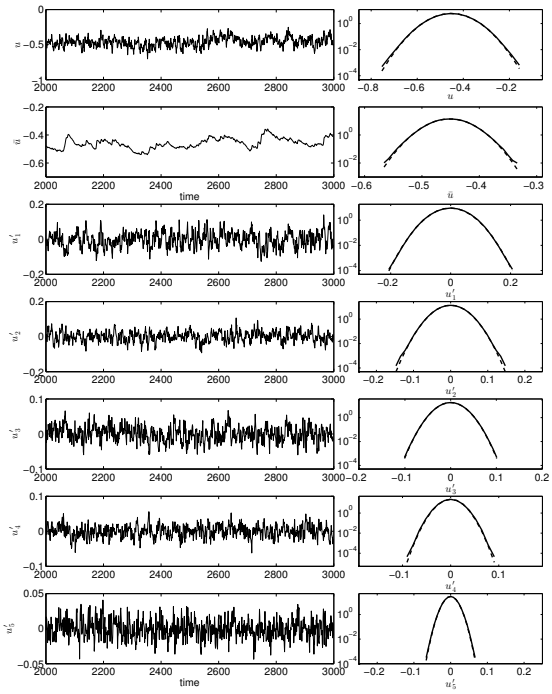
$\overline{F}$	$\langle\overline{u}\rangle$	Var($\overline{u}$)	Std($\overline{u}$)	$d_1 + \gamma\langle\overline{u}\rangle$	$d_2 + \gamma\langle\overline{u}\rangle$	$d_3 + \gamma\langle\overline{u}\rangle$	$d_4 + \gamma\langle\overline{u}\rangle$	$d_5 + \gamma\langle\overline{u}\rangle$
0	0.1700	0.0000	0.0070	1.2750	1.3350	1.4350	1.5750	1.7550
-0.010	-0.4519	0.0008	0.0289	0.3422	0.4022	0.5022	0.6422	0.8222
-0.030	-0.6504	0.0063	0.0793	0.0444	0.1044	0.2044	0.3444	0.5244
-0.054	-0.6779	0.0121	0.1102	0.0031	0.0631	0.1631	0.3031	0.4831
-0.055	-0.6784	0.0124	0.1113	0.0027	0.0624	0.1624	0.3024	0.4824
-0.060	-0.6808	0.0136	0.1167	-0.0012	0.0588	0.1588	0.2988	0.4788
-0.080	-0.6853	0.0186	0.1364	-0.0079	0.0521	0.1521	0.2921	0.4721
-0.100	-0.6870	0.0234	0.1530	-0.0105	0.0495	0.1495	0.2895	0.4695

Table S5: Positive large scale damping with $K = 5$: variances of the turbulent signal u, $\overline{u}$, and $u'_k, k = 1, 2, ..., 5$ for various strengths of $\overline{F}$

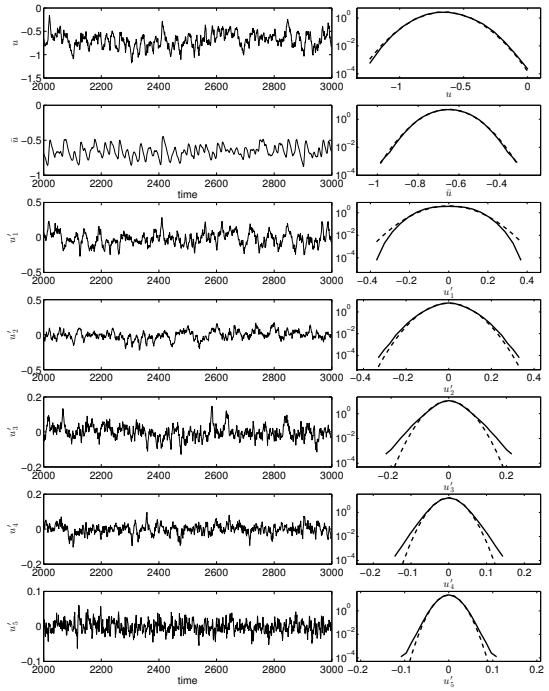
$\overline{F}$	Var(u)	Var($\overline{u}$)	Var(u'_1)	Var(u'_2)	Var(u'_3)	Var(u'_4)	Var(u'_5)
0	0.0012	0.0000	0.0005	0.0003	0.0002	0.0001	0.0001
-0.010	0.0045	0.0008	0.0018	0.0009	0.0005	0.0003	0.0002
-0.030	0.0218	0.0063	0.0092	0.0041	0.0015	0.0006	0.0003
-0.054	0.0436	0.0121	0.0190	0.0087	0.0025	0.0008	0.0004
-0.055	0.0445	0.0124	0.0195	0.0089	0.0026	0.0008	0.0004
-0.060	0.0488	0.0136	0.0216	0.0099	0.0028	0.0009	0.0004
-0.080	0.0667	0.0186	0.0311	0.0129	0.0033	0.0010	0.0004
-0.100	0.0849	0.0234	0.0421	0.0147	0.0037	0.0011	0.0004

Table S6: Positive large scale damping with $K = 5$: correlation times of the turbulent signal u, $\overline{u}$, and $u'_k, k = 1, 2, ..., 5$ for various strengths of $\overline{F}$

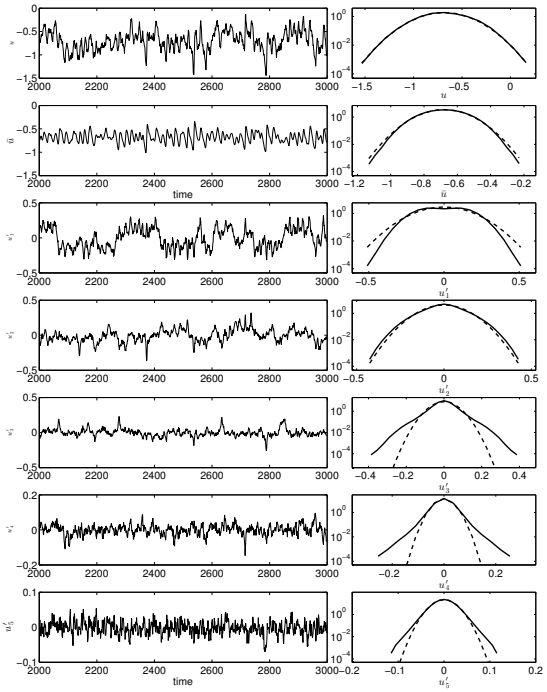
$\overline{F}$	$T_{cor}(u)$	$T_{cor}(\overline{u})$	$T_{cor}(u'_1)$	$T_{cor}(u'_2)$	$T_{cor}(u'_3)$	$T_{cor}(u'_4)$	$T_{cor}(u'_5)$
0	1.7502	79.34	0.7502	0.7213	0.6923	0.6280	0.5771
-0.010	3.1739	33.18	2.885	2.452	2.019	1.587	1.251
-0.030	7.502	16.98	9.013	8.800	6.059	3.404	2.019
-0.054	14.42	17.31	12.98	11.54	7.933	4.328	2.308
-0.055	14.00	18.00	12.83	11.25	7.817	4.353	2.400
-0.060	16.78	18.75	15.87	11.71	7.517	4.478	2.511
-0.080	23.01	22.15	28.35	13.31	6.780	4.517	2.671
-0.100	28.83	28.18	42.13	13.71	6.275	4.328	2.783



(a) $\overline{F} = -0.010$

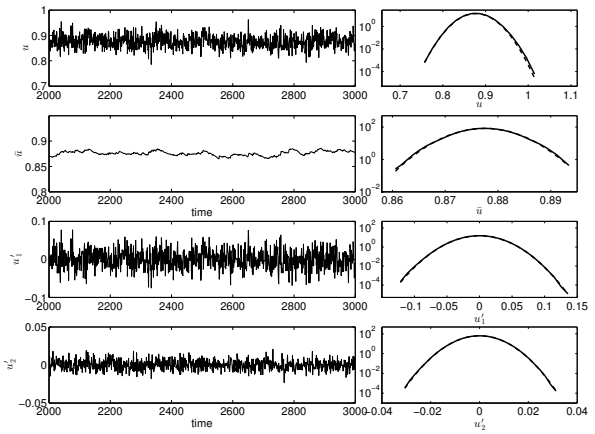
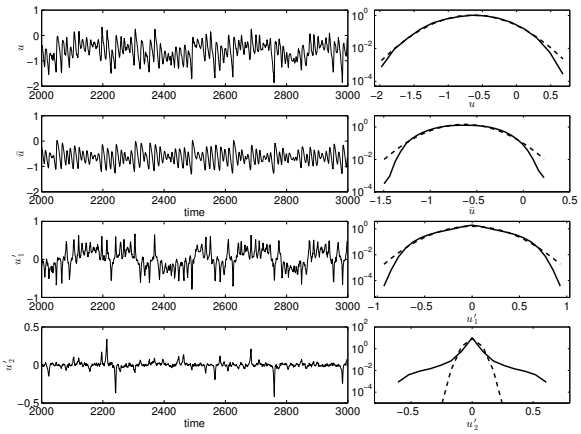


(b) $\overline{F} = -0.030$



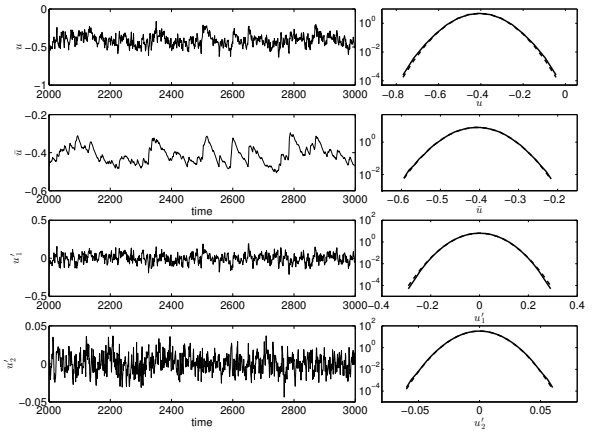
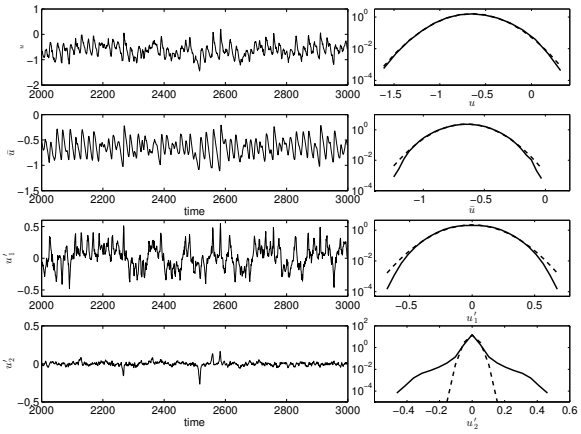
(c) $\overline{F} = -0.055$

Fig. S2: Positive large scale damping with $K = 5$: time series (left column) and pdfs (right column) of the turbulent signal u, $\overline{u}$ and $u'_k, k = 1, 2, ..., 5$ with $\overline{F} = -0.010, -0.030$, and -0.055. The tails of u'_3, u'_4, and u'_5 become fatter for larger $\overline{F}$ in magnitude. Note the logarithmic scale of pdfs in the y-axis. Dashed lines are Gaussian distributions with the same mean and variance.

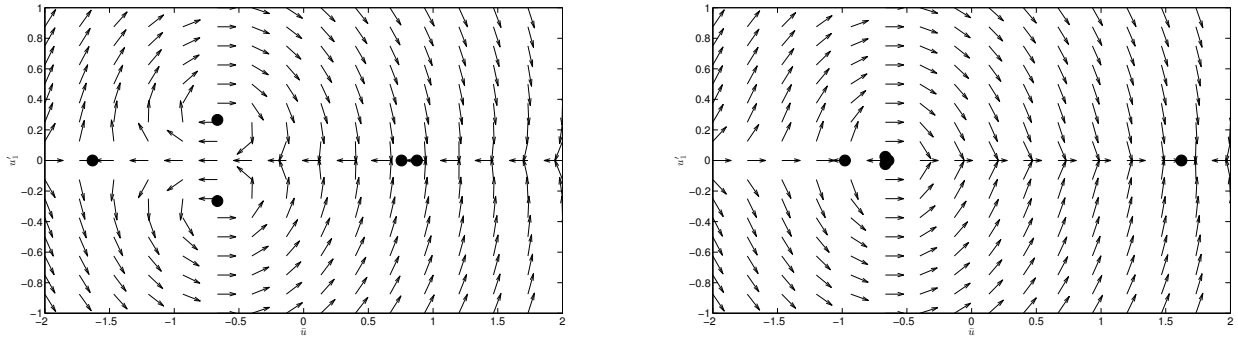
(a) $\overline{F} = -0.054$ (b) $\overline{F} = -0.055$

Fig. S3: Negative large scale damping with $K = 2$: time series (left column) and pdfs (right column) of the turbulent signal u, $\overline{u}$ and $u'_k, k = 1, 2$ with $\overline{F} = -0.054$ (a) and -0.055 (b). Note the logarithmic scale of pdfs in the y-axis. Dashed lines are Gaussian distributions with the same mean and variance.

(a) $\overline{F} = -0.010$ (b) $\overline{F} = -0.055$

Fig. S4: Positive large scale damping with $K = 2$: time series (left column) and pdfs (right column) of the turbulent signal u, $\overline{u}$ and $u'_k, k = 1, 2$ with $\overline{F} = -0.010$ (a) and -0.055 (b). Note the logarithmic scale of pdfs in the y-axis. Dashed lines are Gaussian distributions with the same mean and variance.



(a) $\overline{F} = -0.054$ (b) $\overline{F} = 0.051$

Fig. S5: Phase portrait of the two mode, $K = 1$, without noise **[7]** for $\overline{F} = -0.054$ (a) and $\overline{F} = 0.051$ (b). Critical points are marked with black circles. The rightmost critical points are stable