Low-Dimensional Reduced-Order Models for Statistical Response and Uncertainty Quantification in Turbulent Systems

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joint work with Di Qi

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Challenges for turbulent dynamical systems

Uncertainty quantification (UQ) deals with the probabilistic characterization of all the possible evolutions of a dynamical system given an *initial set of possible states* as well as *the random forcing or parameters*.

- Turbulent dynamical systems are characterized by *a large dimensional phase space* and *high degrees of internal instability*.
- Instabilities through *energy-conserving nonlinear interactions* result in a statistical steady state that is usually *non-Gaussian*.
- Accurate quantification for the statistical variability to *general external perturbations* is important in climate change sciences.

Major Task of this work:

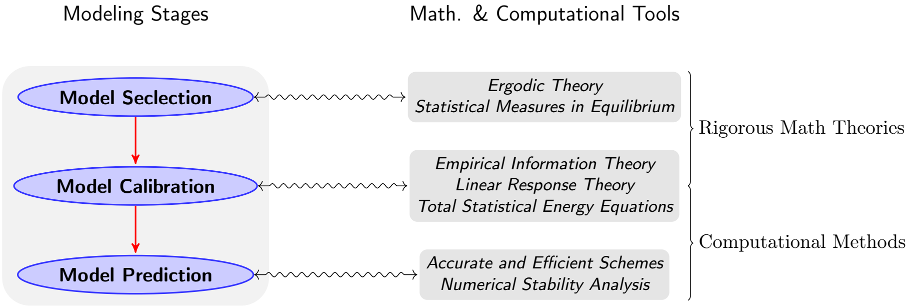
Investigate a concise systematic framework for measuring and optimizing consistency and sensitivity of imperfect dynamical models.

General framework for statistical modeling

The system setup will be a finite-dimensional system of, $\mathbf{u} \in \mathbb{R}^N$, with linear dynamics and an energy preserving quadratic part

$$\frac{d\boldsymbol{u}}{dt} = \mathscr{L}[\boldsymbol{u}] = (L+D)\boldsymbol{u} + B(\boldsymbol{u},\boldsymbol{u}) + \boldsymbol{F}(t) + \sigma_k(t)\dot{W}_k(t;\omega), \tag{1}$$

$$L^* = -L; \quad D \leq 0; \quad \mathbf{u}\cdot B(\boldsymbol{u},\boldsymbol{u}) \equiv 0.$$



Related Works and Papers

- Recent new developments
  - Majda, *Introduction to turbulent dynamical systems in complex systems*, Springer, 2016.
  - Majda and Qi, *New strategies for reduced-order models for predicting the statistical responses and uncertainty quantification in complex turbulent dynamical systems*, SIAM Review, 2017.
- Statistical theories
  - Majda, *Statistical energy conservation principle for inhomogeneous turbulent dynamical systems*. PNAS, 2015.
  - Majda and Gershgorin, *Link between statistical equilibrium fidelity and forecasting skill for complex systems with model error*. PNAS, 2011.
  - Majda and Wang, *Linear response theory for statistical ensembles in complex systems with time-periodic forcing*. CMS, 2010.
- Improving imperfect model skill
  - Majda and Qi, *Improving prediction skill of imperfect turbulent models through statistical response and information theory*, Journal of Nonlinear Science, 2015.
  - Qi and Majda, *Low-dimensional reduced-order models for statistical response and uncertainty quantification: two-layer baroclinic turbulence,* JAS, 2016.
  - Qi and Majda, *Low-dimensional reduced-order models for statistical response and uncertainty quantification: barotropic turbulence with topography,* Physica D, 2016.

Outline

1. A two-layer quasi-geostrophic model for baroclinic turbulence
   - Two-layer baroclinic turbulence in ocean and atmosphere regimes
2. Reduced-order statistical models for general turbulent systems
   - Formulation of the exact statistical moment dynamics
   - A reduced-order statistical model with consistency and sensitivity
   - Model calibration for optimal performance
3. Low-dimensional reduced-order models for the two-layer system
   - Statistical equations for the two-layer model
   - Tuning imperfect model parameters in the training phase
   - Imperfect model prediction in various dynamical regimes

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The two-layer flow with forcing and dissipation

The **two-layer quasi-geostrophic model** with baroclinic instability is one simple but fully nonlinear fluid model capable in capturing the essential physics *in ocean and atmosphere science*.

Two-layer model

$$\frac{\partial q_\psi}{\partial t} + J(\psi, q_\psi) + J(\tau, q_\tau) + \beta \frac{\partial \psi}{\partial x} + U \frac{\partial \Delta \tau}{\partial x} = -\frac{\kappa}{2}\Delta(\psi - \tau) - \nu \Delta^s q_\psi + \mathscr{F}_\psi,$$

$$\frac{\partial q_\tau}{\partial t} + J(\psi, q_\tau) + J(\tau, q_\psi) + \beta \frac{\partial \tau}{\partial x} + U \frac{\partial}{\partial x}\left(\Delta \psi + k_d^2 \psi\right) = -\frac{\kappa}{2}\Delta(\tau - \psi) - \nu \Delta^s q_\tau + \mathscr{F}_\tau.$$

Barotropic and baroclinic modes:

$$q_\psi = \Delta \psi,\ \psi = \frac{1}{2}(\psi_1 + \psi_2),$$

$$q_\tau = \Delta \tau - k_d^2 \tau,\ \tau = \frac{1}{2}(\psi_1 - \psi_2).$$

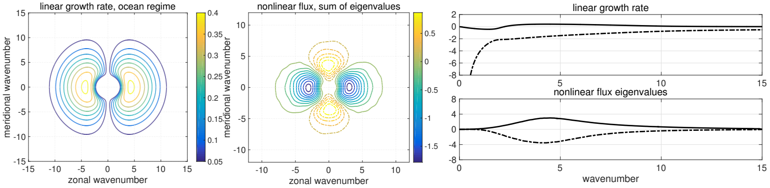
Normalized energy-consistent modes:

$$p_{\psi,\mathbf{k}} = \frac{q_{\psi,\mathbf{k}}}{|\mathbf{k}|} = -|\mathbf{k}|\,\psi_{\mathbf{k}},$$

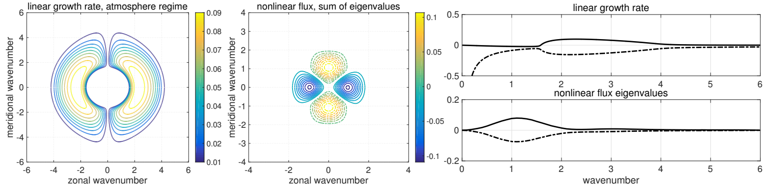
$$p_{\tau,\mathbf{k}} = \frac{q_{\tau,\mathbf{k}}}{\sqrt{|\mathbf{k}|^2 + k_d^2}} = -\sqrt{|\mathbf{k}|^2 + k_d^2}\,\tau_{\mathbf{k}}.$$

Flow in high-latitude homogeneous regimes

regime	N	β	k_d	U	κ	$(k_{\min}, k_{\max})$	$\sigma_{\max}$	$(k_x, k_y)_{\max}$
ocean, high lat.	256	10	10	1	9	$(2.25, 14.61)$	0.411	(4, 0)
atmosphere, high lat.	256	1	4	0.2	0.2	$(1.58, 6.78)$	0.099	(2, 0)

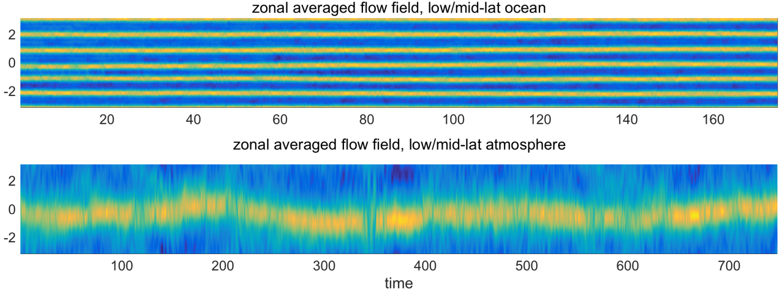


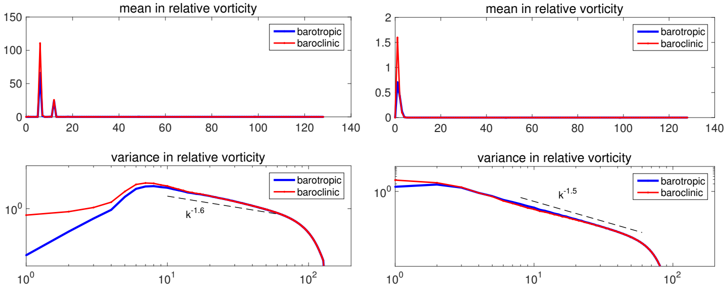
(a) high-latitude ocean regime



(b) high-latitude atmosphere regime

Flow in low/mid-latitude regimes with zonal jets





Outline

General setup of turbulent systems with quadratic nonlinearities

The system setup will be a finite-dimensional system with linear dynamics and an energy preserving quadratic part with $\boldsymbol{u} \in \mathbb{R}^N$

$$\frac{d\boldsymbol{u}}{dt} = \mathscr{L}\left[\boldsymbol{u}(t;\omega);\omega\right] = (L+D)\boldsymbol{u} + B(\boldsymbol{u},\boldsymbol{u}) + \boldsymbol{F}(t) + \sigma_k(t)\dot{W}_k(t;\omega), \tag{2}$$

$$\boldsymbol{u}(t_0;\omega) = \boldsymbol{u}_0(\omega). \tag{3}$$

- L being a **skew-symmetric linear operator** $L^* = -L$, representing the β-effect of Earth's curvature, topography etc.
- D being a **negative definite symmetric operator** $D^* = D$, representing dissipative processes such as surface drag, radiative damping, viscosity etc.
- $B(\boldsymbol{u},\boldsymbol{u})$ being a **quadratic operator** which conserves the energy by itself so that it satisfies $B(\boldsymbol{u},\boldsymbol{u})\cdot\boldsymbol{u} = 0$.
- $\boldsymbol{F}(t) + \sigma_k(t)\dot{W}_k(t;\omega)$ being the **effects of external forcing**, i.e. solar forcing, seasonal cycle, which can be split into a mean component $\boldsymbol{F}(t)$ and a stochastic component with white noise characteristics.

Exact statistical moment equations

Statistical mean and covariance dynamics, $\boldsymbol{u} = \bar{\boldsymbol{u}} + Z_i \boldsymbol{v}_i$, $R_{ij} = \left\langle Z_i Z_j^* \right\rangle$,

$$\frac{d\bar{\boldsymbol{u}}}{dt} = (L+D)\,\bar{\boldsymbol{u}} + B\left(\bar{\boldsymbol{u}},\bar{\boldsymbol{u}}\right) + R_{ij}B\left(\boldsymbol{v}_i,\boldsymbol{v}_j\right) + \boldsymbol{F}(t),$$

$$\frac{dR}{dt} = L_v R + R L_v^* + Q_F + Q_\sigma.$$

- the linear dynamics operator L_v expressing energy transfers between the *mean field and the stochastic modes* (B), as well as *energy dissipation* (D), and *non-normal dynamics* (L)

$$\left\{L_v\right\}_{ij} = \left[(L+D)\,\boldsymbol{v}_j + B\left(\bar{\boldsymbol{u}},\boldsymbol{v}_j\right) + B\left(\boldsymbol{v}_j,\bar{\boldsymbol{u}}\right)\right]\cdot \boldsymbol{v}_i.$$

- the positive definite operator Q_σ expressing energy transfer due to external stochastic forcing

$$\left\{Q_\sigma\right\}_{ij} = \boldsymbol{v}_i^* \sigma_k^* \sigma_k \boldsymbol{v}_j.$$

- the third-order moments expressing the energy flux between different modes due to non-linear terms

$$Q_F = \left\langle Z_m Z_n Z_j \right\rangle B\left(\boldsymbol{v}_m,\boldsymbol{v}_n\right)\cdot \boldsymbol{v}_i + \left\langle Z_m Z_n Z_i \right\rangle B\left(\boldsymbol{v}_m,\boldsymbol{v}_n\right)\cdot \boldsymbol{v}_j.$$

  note that energy is still conserved in this nonlinear interaction part

$$\begin{aligned}\mathrm{Tr}\left[Q_F\right] &= 2\left\langle Z_m Z_n Z_i \right\rangle B\left(\boldsymbol{v}_m,\boldsymbol{v}_n\right)\cdot \boldsymbol{v}_i \\ &= 2\left\langle B\left(Z_m\boldsymbol{v}_m, Z_n\boldsymbol{v}_n\right)\cdot Z_i\boldsymbol{v}_i\right\rangle = 2\left\langle B\left(\boldsymbol{u}',\boldsymbol{u}'\right)\cdot \boldsymbol{u}'\right\rangle = 0.\end{aligned}$$

Reduced-Order Statistical Energy Closure

The true statistical model

$$\frac{d\bar{\boldsymbol{u}}}{dt} = (L+D)\,\bar{\boldsymbol{u}} + B(\bar{\boldsymbol{u}},\bar{\boldsymbol{u}}) + R_{ij}B\left(\boldsymbol{v}_i,\boldsymbol{v}_j\right) + \boldsymbol{F}(t),\ \bar{\mathbf{u}} \in \mathbb{R}^N,$$

$$\frac{dR}{dt} = L_v(\bar{\mathbf{u}})\,R + RL_v^*(\bar{\mathbf{u}}) + Q_F + Q_\sigma, \quad R \in \mathbb{R}^{N\times N}.$$

$$\boxed{Q_{F,ij} = \left\langle Z_m Z_n Z_j\right\rangle B(\boldsymbol{v}_m,\boldsymbol{v}_n)\cdot\boldsymbol{v}_i + \left\langle Z_m Z_n Z_i\right\rangle B(\boldsymbol{v}_m,\boldsymbol{v}_n)\cdot\boldsymbol{v}_j}$$

Reduced-Order Statistical Energy Closure

The reduced-order approximation $\bar{\mathbf{u}}_M \in \mathbb{R}^M$, $M \ll N$

$$\begin{aligned}
\frac{d\bar{\mathbf{u}}_M}{dt} &= (L+D)\,\bar{\mathbf{u}}_M + B\left(\bar{\mathbf{u}}_M, \bar{\mathbf{u}}_M\right) + R_{M,ij}B\left(\mathbf{v}_i, \mathbf{v}_j\right) + \mathbf{F}, \\
\frac{dR_M}{dt} &= L_v R_M + R_M L_v^* + Q_F^M + Q_\sigma,
\end{aligned}$$

A preferred approach for the nonlinear flux Q_F^M combining both the *detailed model energy mechanism* and *control over model sensitivity* is proposed

$$\boxed{Q_F^M = Q_F^{M,-} + Q_F^{M,+} = f_1(E)\left[-\left(N_{M,\mathrm{eq}} + d_M I_N\right) R_M\right] + f_2(E)\left[Q_{F,\mathrm{eq}}^{+} + \Sigma_M\right].}$$

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- *Higher-order corrections from equilibrium statistics*:

$$Q_{F,\mathrm{eq}} = Q_{F,\mathrm{eq}}^{-} + Q_{F,\mathrm{eq}}^{+} = -L_v(\bar{\mathbf{u}}_{\mathrm{eq}}) R_{\mathrm{eq}} - R_{\mathrm{eq}} L_v^{*}(\bar{\mathbf{u}}_{\mathrm{eq}}) - Q_\sigma, \quad N_{M,\mathrm{eq}} = \frac{1}{2} Q_{F,\mathrm{eq}}^{-} R_{\mathrm{eq}}^{-1}.$$

- *Additional damping and noise to model nonlinear flux*:

$$Q_M^{\mathrm{add}} = -d_M R_M + \Sigma_M.$$

- *Statistical energy-consistent scaling to improve model sensitivity*:

$$f_1(E) = \left(\frac{E}{E_{\mathrm{eq}}}\right)^{1/2}, \quad f_2(E) = \left(\frac{E}{E_{\mathrm{eq}}}\right)^{3/2}.$$

Climate fidelity for equilibrium

Equilibrium fidelity refers to the convergence to the same final unperturbed statistical equilibrium R_{eq} in the reduced-order models R_M in each resolved component.

Specifically, it requires that the model nonlinear flux correction term Q_M converges to the truth, $Q_M \to Q_{F,\mathrm{eq}}$, when no external perturbation is added

$$\frac{dR_{M,\mathrm{eq}}}{dt} = 0 = L_v\left(\bar{\mathbf{u}}_{\mathrm{eq}}\right) R_{M,\mathrm{eq}} + R_{M,\mathrm{eq}} L_v^*\left(\bar{\mathbf{u}}_{\mathrm{eq}}\right) + Q_{F,\mathrm{eq}}^M + Q_\sigma \ \to \ R_{M,\mathrm{eq}} = R_{\mathrm{eq}}.$$

- the first component $\left(N_{M,\mathrm{eq}}, Q_{F,\mathrm{eq}}^+\right)$ comes from the true equilibrium statistics.
- climate consistency requires the second component correction makes no contribution in the unperturbed case

$$\Sigma_M = \frac{1}{2} d_M R_{\mathrm{eq}}, \quad f_1\left(E_{\mathrm{eq}}\right) = 1,\ f_2\left(E_{\mathrm{eq}}\right) = 1.$$

Statistical energy conservation principle[1]

Theorem

*(**Statistical Energy Conservation Principle**) Under the structural symmetries on the basis $\mathbf{v}_i$, for any turbulent dynamical systems in the form (1) the total statistical energy, $E = \bar{E} + E' = \frac{1}{2}\bar{\mathbf{u}}\cdot\bar{\mathbf{u}} + \frac{1}{2}\mathrm{tr}R$, satisfies*

$$\frac{dE}{dt} = \bar{\mathbf{u}}\cdot D\bar{\mathbf{u}} + \bar{\mathbf{u}}\cdot\mathbf{F} + \mathrm{tr}\left(DR\right) + \frac{1}{2}\mathrm{tr}Q_\sigma,$$

where R satisfies the exact covariance equation.

Corollary

Under the assumption of the Theorem, assume $D = -dI$, with $d > 0$, then the turbulent dynamical system satisfies the closed statistical energy equation for $E = \frac{1}{2}\bar{\mathbf{u}}\cdot\bar{\mathbf{u}} + \frac{1}{2}\mathrm{tr}R$,

$$\frac{dE}{dt} = -2dE + \bar{\mathbf{u}}\cdot\mathbf{F} + \frac{1}{2}\mathrm{tr}Q_\sigma.$$

[1] Majda, *Statistical energy conservation principle for inhomogeneous turbulent dynamical systems*, PNAS, 2015.

Model calibration blending *statistical response* and *information theory*

Accurate modeling about the model sensitivity to various external perturbations requires the imperfect reduced-order models to correctly reflect the true system's "memory" about its previous states.

- *the linear response operator* can characterize the model sensitivity involving the nonlinear effects in the system regardless of the specific forms of the external perturbations.
- *empirical information theory* can be used as the distance between these two operators to calculate the unbiased and invariant measure for model distributions.

Linear response operator $\mathscr{R}_A(t)$

Linear response theory: The linear response of a system, $\mathbf{u}_t = \mathbf{f}(\mathbf{u})$, $\mathbf{f}^\delta = \mathbf{f} + \delta\mathbf{f}'(t)$, can be predicted by observing appropriate statistics of the system in equilibrium π_{eq}

$$\mathbb{E}^\delta A(\mathbf{u}) = E_{\mathrm{eq}}(A) + \delta E'_A + O\left(\delta^2\right),$$

$$\delta E'_A = \int_0^t \mathscr{R}_A(t-s)\delta f'(s)\,ds.$$

without the need of applying any perturbations.

kicked response: For δ small enough, the linear response operator $\mathscr{R}_A(t)$ can be calculated by solving the unperturbed system with a perturbed initial distribution

$$\pi\mid_{t=0} = \pi_{\mathrm{eq}}(\mathbf{u} - \delta\mathbf{u}) = \pi_{\mathrm{eq}} - \delta\mathbf{u}\cdot\nabla\pi_{\mathrm{eq}} + O\left(\delta^2\right).$$

$$\delta\mathscr{R}_A(t) \equiv \delta\mathbf{u}\cdot\mathscr{R}_A = \int A(\mathbf{u})\,\delta\pi' + O\left(\delta^2\right).$$

Link between equilibrium fidelity and forecasting skill

Given the optimal model for the unperturbed climate π_{eq}, how can we assess the error in the climate change prediction

$$\mathscr{P}\left(\pi,\pi^M\right)=\int\pi\ln\frac{\pi}{\pi^M},$$

based on the unperturbed climate?

Under assumptions with diagonal covariance matrices $R=\mathrm{diag}\left(R_k\right)$ and equilibrium model fidelity $\mathscr{P}\left(\pi_G,\pi_G^M\right)=0$

$$\begin{aligned}\mathscr{P}\left(\pi_\delta,\pi_\delta^M\right) &= \mathscr{S}\left(\pi_{G,\delta}\right)-\mathscr{S}\left(\pi_\delta\right)\\ &+\frac{1}{2}\sum_k\left(\delta\bar{u}_k-\delta\bar{u}_{M,k}\right)R_k^{-1}\left(\delta\bar{u}_k-\delta\bar{u}_{M,k}\right)\\ &+\frac{1}{4}\sum_k R_k^{-2}\left(\delta R_k-\delta R_{M,k}\right)^2+O\left(\delta^3\right).\end{aligned}$$

R_k is the equilibrium variance in k-th component, and $\delta\bar{u}_k$ and δR_k are the linear response operators for the mean and variance in k-th component. (Majda & Gershogorin, PNAS, 2011)

Optimization in the training phase

To summarize, consider a class of imperfect models, $\mathscr{M}$. The *optimal model* $M^* \in \mathscr{M}$ that ensures best information consistent responses is characterized with the smallest additional information in the linear response operator $\mathscr{R}_A$, such that

$$\left\| \mathscr{P}\left(p_\delta, p_\delta^{M^*}\right) \right\|_{L^1([0,T])} = \min_{M \in \mathscr{M}} \left\| \mathscr{P}\left(p_\delta, p_\delta^{M}\right) \right\|_{L^1([0,T])},$$

- p_δ^M can be achieved through a kicked response procedure in the training phase compared with the actual observed data p_δ in nature;
- the information distance between perturbed responses $\mathscr{P}\left(p_\delta, p_\delta^M\right)$ can be calculated through the expansion formula;
- the information distance $\mathscr{P}\left(p_\delta(t), p_\delta^M(t)\right)$ is measured at each time instant, so the entire error is averaged under the L^1-norm inside a time window $[0,T]$.

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Exact statistical moment equations for the two-layer model

The rescaled set of equations of (1) can be summarized in the abstract form

$$\frac{d\mathbf{p_k}}{dt} = B_{\mathbf{k}}(\mathbf{p_k},\mathbf{p_k}) + (\mathscr{L}_{\mathbf{k}} - \mathscr{D}_{\mathbf{k}})\mathbf{p_k} + \mathscr{F}_{\mathbf{k}}, \quad \mathbf{p_k} = (p_{\psi,\mathbf{k}}, p_{\tau,\mathbf{k}})^T, \quad \sum_{\mathbf{k}} \mathbf{p_k} \cdot B_{\mathbf{k}}(\mathbf{p_k},\mathbf{p_k}) \equiv 0,$$

where the normalized state variable $\mathbf{p_k} = (p_{\psi,\mathbf{k}}, p_{\tau,\mathbf{k}})^T$ is in barotropic and baroclinic mode, the linear operator is decomposed into non-symmetric part $\mathscr{L}_{\mathbf{k}}$ involving β-effect and shear flow U and dissipation part $\mathscr{D}_{\mathbf{k}}$, together with the forcing $\mathscr{F}_{\mathbf{k}}$ combining deterministic component and stochastic component.

$$\mathscr{L}_{\mathbf{k}} = \begin{bmatrix} \frac{ik_x\beta}{|\mathbf{k}|^2} & -\frac{ik_xU}{\sqrt{1+(k_d/|\mathbf{k}|)^2}} \\ -ik_xU\frac{1-(k_d/|\mathbf{k}|)^2}{\sqrt{1+(k_d/|\mathbf{k}|)^2}} & \frac{ik_x\beta}{|\mathbf{k}|^2+k_d^2} \end{bmatrix}, \quad \mathscr{D}_{\mathbf{k}} = \frac{\kappa}{2}\begin{bmatrix} -1 & \frac{1}{\sqrt{1+(k_d/|\mathbf{k}|)^2}} \\ \frac{1}{\sqrt{1+(k_d/|\mathbf{k}|)^2}} & -\frac{1}{1+(k_d/|\mathbf{k}|)^2} \end{bmatrix},$$

$$B_{\mathbf{k}}(\mathbf{p_k},\mathbf{p_k}) = \begin{bmatrix} B_{\psi,\mathbf{k}} \\ B_{\tau,\mathbf{k}} \end{bmatrix} = \begin{bmatrix} \sum_{\mathbf{m}+\mathbf{n}=\mathbf{k}} \frac{\mathbf{m}^\perp\cdot\mathbf{n}}{|\mathbf{k}|}\left(\frac{|\mathbf{n}|}{|\mathbf{m}|}p_{\psi,\mathbf{m}}p_{\psi,\mathbf{n}} + \sqrt{\frac{|\mathbf{n}|^2+k_d^2}{|\mathbf{m}|^2+k_d^2}}p_{\tau,\mathbf{m}}p_{\tau,\mathbf{n}}\right) \\ \sum_{\mathbf{m}+\mathbf{n}=\mathbf{k}} \frac{\mathbf{m}^\perp\cdot\mathbf{n}}{\sqrt{|\mathbf{k}|^2+k_d^2}}\left(\frac{\sqrt{|\mathbf{n}|^2+k_d^2}}{|\mathbf{m}|}p_{\psi,\mathbf{m}}p_{\tau,\mathbf{n}} + \frac{|\mathbf{n}|}{\sqrt{|\mathbf{m}|^2+k_d^2}}p_{\tau,\mathbf{m}}p_{\psi,\mathbf{n}}\right) \end{bmatrix}.$$

Exact statistical moment equations

Statistical energy in each spectral mode

$$R_{\mathbf{k}} = \overline{\mathbf{p_k}^* \mathbf{p_k}} = \begin{bmatrix} \overline{\left|p_{\psi,\mathbf{k}}\right|^2} & \overline{p^*_{\psi,\mathbf{k}} p_{\tau,\mathbf{k}}} \\ \overline{p_{\psi,\mathbf{k}} p^*_{\tau,\mathbf{k}}} & \overline{\left|p_{\tau,\mathbf{k}}\right|^2} \end{bmatrix}, \quad \overline{p^*_{1,\mathbf{k}} p_{2,\mathbf{k}}} = \bar{p}^*_{1,\mathbf{k}} \bar{p}_{2,\mathbf{k}} + \overline{p'^*_{1,\mathbf{k}} p'_{2,\mathbf{k}}}.$$

$R_{\mathbf{k}}$ combines the variability in both *mean* and *variance*. The true statistical dynamical equations form a 2×2 system about $R_{\mathbf{k}} \in \mathbb{C}^{2\times 2}$

$$\boxed{\frac{dR_{\mathbf{k}}}{dt} = \left(\mathscr{L}_{\mathbf{k}} - \mathscr{D}_{\mathbf{k}}\right) R_{\mathbf{k}} + Q_{F,\mathbf{k}} + Q_{\sigma,\mathbf{k}} + c.c., \quad |\mathbf{k}| \leq N,}$$

$$Q_{F,\mathbf{k}} = \overline{\mathbf{p_k}^* B_{\mathbf{k}}\left(\mathbf{p_k}, \mathbf{p_k}\right)} = \begin{bmatrix} \overline{\mathbf{p}^*_{\psi,\mathbf{k}} B_{\psi,\mathbf{k}}} & \overline{\mathbf{p}^*_{\psi,\mathbf{k}} B_{\tau,\mathbf{k}}} \\ \overline{\mathbf{p}^*_{\tau,\mathbf{k}} B_{\psi,\mathbf{k}}} & \overline{\mathbf{p}^*_{\tau,\mathbf{k}} B_{\tau,\mathbf{k}}} \end{bmatrix}, \quad \sum_{\mathbf{k}} \mathrm{tr} Q_{F,\mathrm{k}} \equiv 0.$$

Statistical energy conservation principle

The total statistical energy dynamical equation concerns the evolution of the *total variability in mean and variance* in response to external perturbations

$$E=\frac{1}{2}\sum_{1\leq|\mathbf{k}|\leq N}|\mathbf{k}|^2\overline{|\psi_{\mathbf{k}}|^2}+\left(|\mathbf{k}|^2+k_d^2\right)\overline{|\tau_{\mathbf{k}}|^2}=\frac{1}{2}\sum_{1\leq|\mathbf{k}|\leq N}\overline{\left|p_{\psi,\mathbf{k}}\right|^2}+\overline{\left|p_{\tau,\mathbf{k}}\right|^2}.$$

The exact dynamics for the statistical energy can be derived as

$$\frac{dE}{dt}+H_f=-\kappa E+\frac{\kappa}{2}F-\nu H+Q_\sigma.$$

H_f is the meridional heat flux due to baroclinic instability, F is the additional damping effects due to the non-symmetry in Ekman drag only applied on the bottom layer

$$H_f=k_d^2U\int\overline{\psi_x\tau}=k_d^2U\sum ik_x\overline{\psi_{\mathbf{k}}^*\tau_{\mathbf{k}}},\quad F=\sum k_d^2\overline{|\tau_{\mathbf{k}}|^2}+2\,|\mathbf{k}|^2\,\mathfrak{Re}\,\overline{\psi_{\mathbf{k}}^*\tau_{\mathbf{k}}}.$$

Set-up for the numerical problem

- The true statistics is calculated by a pseudo-spectra code with 128 spectral modes zonally and meridionally, corresponding to $256 \times 256 \times 2$ grid points in total.
- In the reduced-order methods, only the large-scale modes $|\mathbf{k}| \leq 10$ are resolved, which is about 0.15% of the full model resolution.

External forcing in stochastic and deterministic component:

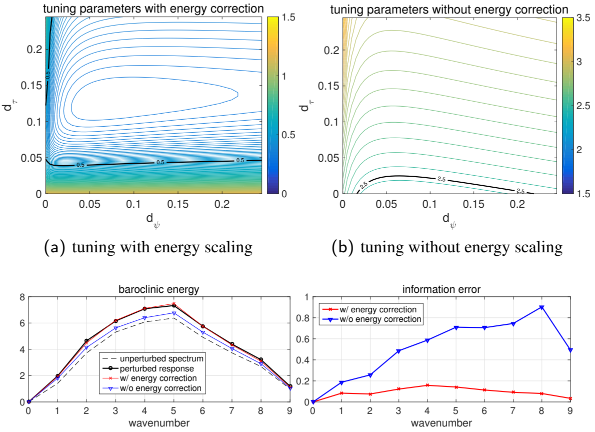
- The amplitude of the *stochastic forcing* $\sigma_k \dot{W}_k$ is introduced according to the equilibrium energy so that

$$\sigma^2_{\psi,\mathbf{k}} = \delta\sigma_0^2 \overline{\left|q_{\psi,\mathbf{k}}\right|^2}_{\mathrm{eq}}, \quad \sigma^2_{\tau,\mathbf{k}} = \delta\sigma_0^2 \overline{\left|q_{\tau,\mathbf{k}}\right|^2}_{\mathrm{eq}}.$$

- The *deterministic forcing* is introduced through a perturbation in the background shear $U_\delta = U + \delta U$

$$\delta f_{\psi,\mathbf{k}} = \delta U i k_x \left(-\left|\mathbf{k}\right|^2\right) \tau_{\mathbf{k}}, \quad \delta f_{\tau,\mathbf{k}} = \delta U i k_x \left(-\left|\mathbf{k}\right|^2 + k_d^2\right) \psi_{\mathbf{k}}.$$

Tuning parameters in the training phase



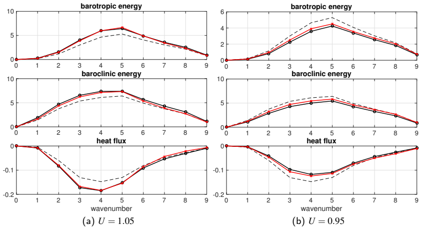
(a) tuning with energy scaling (b) tuning without energy scaling

(c) prediction and info. errors

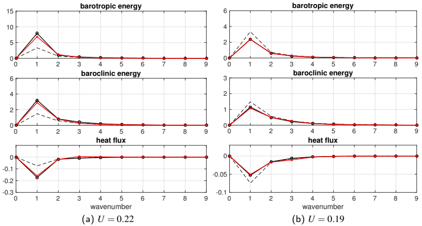
Figure: Tuning imperfect model parameters in the training phase. The information errors with varying model parameters, $d_M = (d_\psi, d_\tau)$, are plotted for stochastic barotropic perturbation case.

High-latitude: Mean shear flow perturbation

- The model is perturbed by changing the background zonal flow strength U;
- The entire spectral is perturbed due to the mean flow advection in each spectral mode.



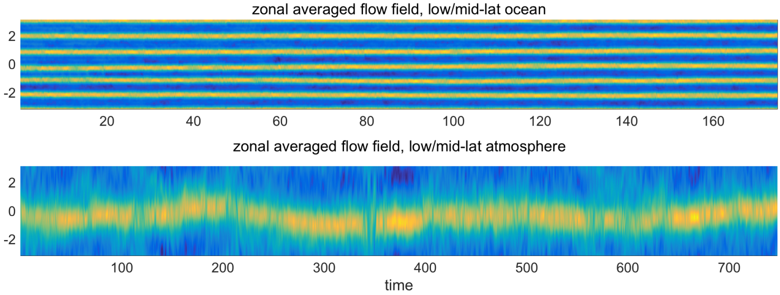
(a) ocean regime $\delta U = \pm 0.05$, $U_0 = 1$



(b) atmos. regime $\delta U = \pm 0.02$, $U_0 = 0.2$

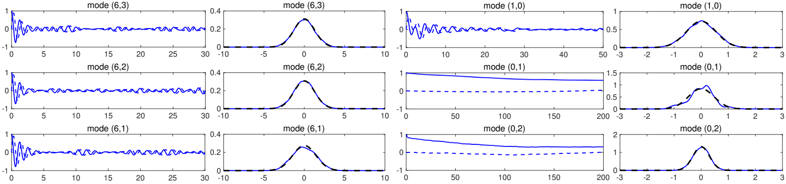
Flow in low-latitude regimes with zonal jets

regime	N	β	k_d	U	κ	$(k_{\min}, k_{\max})$	$\sigma_{\max}$	$(k_x, k_y)_{\max}$
ocean, high lat.	256	100	10	1	1	$(7.14, 15.63)$	0.104	(2, 8)
atmosphere, high lat.	256	2.5	4	0.2	0.05	$(2.51, 7.06)$	0.053	(3, 0)

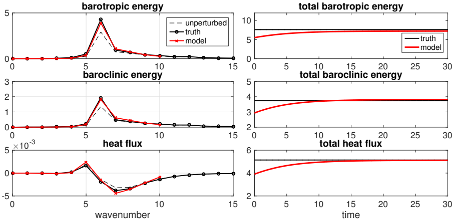


Low-latitude: Stochastic perturbation with $\delta\sigma_0^2 = 0.2$

autocorrelation functions and probability density functions

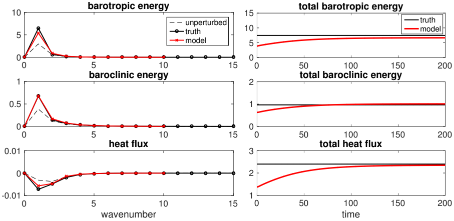


(a) ocean (b) atmosphere



(a) spectra for the energy and heat flux (b) time-series for total responses

(c) ocean regime



(a) spectra for the energy and heat flux (b) time-series for total responses

(d) atmosphere regime

Current and future work

- Reduced-order stochastic modeling strategies to capture passive scalar intermittency
  - Majda & Tong, *Intermittency in turbulent diffusion models with a mean gradient*, Nonlinearity, 2015
  - Majda & Gershgorin, *Elementary models for turbulent diffusion with complex physical features: eddy diffusivity, spectrum and intermittency*, Phil. Trans. R. Soc. A, 2013
  - Qi & Majda, *Predicting fat-tailed intermittent probability distributions in passive scalar turbulence with imperfect models through empirical information theory*, Comm. Math. Sci, 2015.
  - Qi & Majda, *Predicting Extreme Events for Passive Scalar Turbulence in Two-Layer Baroclinic Flows through Reduced-Order Stochastic Models*, Comm. Math. Sci, 2017.
- Design of a mitigation control strategy by using novel low-order statistical models;
  - Majda & Qi, *Effective control of complex turbulent dynamical systems through statistical functionals,* PNAS, 2017
  - Majda & Qi, *Using Statistical Functionals for Effective Control of Inhomogeneous Complex Turbulent Dynamical Systems*, submitted to Physica D.
- Rigorous statistical UQ for turbulent geophysical flows
  - Majda & Qi, Rigorous statistical uncertainty quantification for one-layer turbulent geophysical flows, in preparation.

Related Works and Papers

- Recent new developments
  - Majda, *Introduction to turbulent dynamical systems in complex systems*, Springer, 2016.
  - Majda and Qi, *New strategies for reduced-order models for predicting the statistical responses and uncertainty quantification in complex turbulent dynamical systems*, SIAM Review, 2017.
- Statistical theories
  - Majda, *Statistical energy conservation principle for inhomogeneous turbulent dynamical systems*. PNAS, 2015.
  - Majda and Gershgorin, *Link between statistical equilibrium fidelity and forecasting skill for complex systems with model error*. PNAS, 2011.
  - Majda and Wang, *Linear response theory for statistical ensembles in complex systems with time-periodic forcing*. CMS, 2010.
- Improving imperfect model skill
  - Majda and Qi, *Improving prediction skill of imperfect turbulent models through statistical response and information theory*, Journal of Nonlinear Science, 2015.
  - Qi and Majda, *Low-dimensional reduced-order models for statistical response and uncertainty quantification: two-layer baroclinic turbulence,* JAS, 2016.
  - Qi and Majda, *Low-dimensional reduced-order models for statistical response and uncertainty quantification: barotropic turbulence with topography,* Physica D, 2016.