ADAMS' INEQUALITY WITH THE EXACT GROWTH CONDITION IN $\mathbb{R}^4$

NADER MASMOUDI AND FEDERICA SANI

Contents

Abstract: Adams' inequality is an extension of the Trudinger-Moser inequality to the case when the Sobolev space considered has more than one derivative. The goal of this paper is to give the optimal growth rate of the exponential type function in Adams' inequality when the problem is considered in the whole space $\mathbb{R}^4$.

1. Introduction

Trudinger-Moser inequality is a subsitute to the well known Sobolev embedding theorem when the critical exponent is infinity since $W_0^{1,n}(\Omega) \nsubseteq L^\infty(\Omega)$ when $\Omega \subset \mathbb{R}^n, n \geq 2$, is a bounded domain. Adams' inequality is an extension of the Trudinger-Moser inequality to the case when the Sobolev space considered has more than one derivative. The goal of this paper is to give the optimal growth rate of the exponential type function when the problem is considered in the whole space $\mathbb{R}^4$.

1.1. Trudinger-Moser inequality.

Let $\Omega \subset \mathbb{R}^n, n \geq 2$, be a bounded domain. The Sobolev embedding theorem asserts that, for $p < n$,

$$W_0^{1,p}(\Omega) \subset L^q(\Omega) \quad 1 \leq q \leq \frac{np}{n-p}.$$

If we look at the limiting Sobolev case, namely $p = n$, then

$$W_0^{1,n}(\Omega) \subset L^q(\Omega) \quad \forall q \geq 1,$$

but it is well known that

$$W_0^{1,n}(\Omega) \nsubseteq L^\infty(\Omega).$$

In other words, in the limiting case the Sobolev embedding theorem does not give a notion of critical growth. In 1970, J. Moser in [28] refined a result proved independently by V. I. Yudovich [39], S. I. Pohozaev [32] and N. S. Trudinger [38] and obtained a notion of critical growth for the Sobolev spaces $W_0^{1,p}(\Omega)$ in the limiting case $p = n$.

Theorem 1.1 ([28], Theorem 1)**.** *There exists a constant $C_n > 0$ such that*

$$\sup_{u \in W_0^{1,n}(\Omega),\, \|\nabla u\|_n \leq 1} \int_\Omega e^{\alpha |u|^{\frac{n}{n-1}}}\, dx \leq C_n |\Omega| \qquad \forall \alpha \leq \alpha_n \tag{1.1}$$

where $\|\nabla u\|_n^n = \int_\Omega |\nabla u|^n\, dx$ denotes the Dirichlet norm of u, $\alpha_n := n\omega_{n-1}^{1/(n-1)}$ and ω_{n-1} is the surface measure of the unit sphere $S^{n-1} \subset \mathbb{R}^n$. Furthermore (1.1) *is sharp, i.e. if $\alpha > \alpha_n$ then the supremum in* (1.1) *is infinite.*

In the literature (1.1) is known under the name *Trudinger-Moser inequality.* This inequality has been extended in various directions. Related results and variants can be found in several papers, see e.g. Y. Li [22, 23] for generalizations to functions on compact Riemannian manifolds and A. Cianchi [13] for a sharp inequality concerning functions without boundary conditions. For Trudinger-Moser type inequalities in other function spaces, in particular Orlicz and Lorentz spaces, see e.g. A. Cianchi [11, 12], A. Alvino, V. Ferone and G. Trombetti [5], B. Ruf and C. Tarsi [35], H. Bahouri, M. Majdoub and N. Masmoudi [6, 7], Adimurthi and K. Tintarev [4] and O. Druet [3]. Concentration and compactness properties of sequences bounded in $W^{1,n}$ were also studied in [10, 26, 29, 6]. We also mention that in a recent paper [15], D. G. De Figueiredo, J. M. Do Ó and B. Ruf gave an interesting overview of results concerning Trudinger-Moser type inequalities with applications to related equations and systems [16]. In addition to applications in elliptic systems, this inequality was also applied to Klein-Gordon equation with exponential nonlinearities [19, 20]. In this paper, firstly we will focus our attention on last developments in the study of Trudiger-Moser type inequalities which are domain independent, thus valid in the whole space $\mathbb{R}^n$ and, then we will consider possible generalizations at the case of Sobolev spaces involving second order derivatives.

An interesting extension of (1.1) is to construct Trudinger-Moser type inequalities for domains with infinite measure. In fact, we can notice that the bound in (1.1) becomes infinite, even in the case $\alpha \leq \alpha_n$, for domains $\Omega \subseteq \mathbb{R}^n$ with $|\Omega| = +\infty$ and consequently the original form of the Trudinger-Moser inequality is not available in these cases. One of the first results in this direction is due to S. Adachi and K. Tanaka [2]:

Theorem 1.2 ([2], Theorem 0.1 and Theorem 0.2)**.** *Let $\psi(t) := e^t - \sum_{j=0}^{n-2} \frac{t^j}{j!}$. For $\alpha \in (0, \alpha_n)$ there exists a constant $C(\alpha) > 0$ such that*

$$\int_{\mathbb{R}^n} \psi(\alpha |u|^{\frac{n}{n-1}})\, dx \leq C(\alpha) \|u\|_n^n \quad \forall u \in W^{1,n}(\mathbb{R}^n) \text{ with } \|\nabla u\|_n \leq 1\,, \tag{1.2}$$

and this inequality is false for $\alpha \geq \alpha_n$.

The limit exponent α_n is excluded in (1.2), which is quite different from Moser's result (see Theorem 1.1). However, in the case $n = 2$ (i.e. for $W_0^{1,2}(\Omega)$ with $\Omega \subseteq \mathbb{R}^2$), B. Ruf [33] showed that if the Dirichlet norm is replaced by the standard Sobolev norm, namely

$$\|u\|_{W^{1,n}}^n := \|\nabla u\|_n^n + \|u\|_n^n,$$

then the exponent $\alpha = 4\pi$ becomes admissible. Strengthening the norm in this way, the result of Moser (Theorem 1.1) can be fully extended to unbounded domains and the supremum in (1.1) is uniformly bounded independently of the domain $\Omega \subseteq \mathbb{R}^2$:

Theorem 1.3 ([33], Theorem 1.1)**.** *There exists a constant $C > 0$ such that for any domain $\Omega \subseteq \mathbb{R}^2$*

$$\sup_{u \in W_0^{1,2}(\Omega),\, \|u\|_{W^{1,2}} \leq 1} \int_\Omega (e^{4\pi u^2} - 1)\, dx \leq C \tag{1.3}$$

and this inequality is sharp.

In [24], Y. Li and B. Ruf extended Theorem 1.3 to arbitrary dimensions $n \geq 3$, i.e. to $W_0^{1,n}(\Omega)$ with $\Omega \subseteq \mathbb{R}^n$ not necessarily bounded and $n \geq 3$.

In short, the failure of the original Trudinger-Moser inequality (1.1) on $\mathbb{R}^n$ can be recovered either by weakening the exponent $\alpha = n\omega_{n-1}^{1/(n-1)}$ or by strengthening the Dirichlet norm $\|\nabla u\|_n$. Then a natural question arises:

What if we keep both the conditions $\alpha = n\omega_{n-1}^{1/(n-1)}$ and $\|\nabla u\|_n \leq 1$?

In the 2-dimensional case, an answer to this question is due to S. Ibrahim, N. Masmoudi and K. Nakanishi [21] and the idea is to weaken the exponential nonlinearity as follows

Theorem 1.4 ([21], Proposition 1.4)**.** *There exists a constant $C > 0$ such that*

$$\int_{\mathbb{R}^2} \frac{e^{4\pi u^2} - 1}{(1 + |u|)^2}\, dx \leq C\|u\|_2^2 \quad \forall u \in W^{1,2}(\mathbb{R}^2) \ \text{ with } \|\nabla u\|_2 \leq 1\,. \tag{1.4}$$

Moreover, this fails if the power 2 in the denominator is replaced with any $p < 2$.

Obviously this last inequality implies (1.2) and it is interesting to notice that it also implies inequality (1.3), which is not so obvious. This improved inequality has some application to the Klein-Gordon equation with exponential nonlinearity [20].

Remark 1. When one deals with Trudinger-Moser type inequalities, as (1.1), (1.2), (1.3) and (1.4), the target space is a Sobolev space involving first order derivatives, i.e. $W^{1,n}$. In these spaces it is easy to reduce the problem to the radial case considering the rearrangements of functions. In fact, given a function $u \in W_0^{1,n}(\Omega)$ with $\Omega \subseteq \mathbb{R}^n$, the spherically symmetric decreasing rearrangement $u^\sharp$ of u (see Section 3.1 for a precise definition) is such that $u^\sharp \in W_0^{1,n}(\Omega^\sharp)$, $\|u^\sharp\|_n = \|u\|_n$ and

$$\int_\Omega g(|u|)\, dx = \int_{\Omega^\sharp} g(u^\sharp)\, dx,$$

for any Borel measurable function $g : \mathbb{R} \to \mathbb{R}$ such that $g \geq 0$. Moreover, accordingly to the Pólya-Szëgo inequality,

$$\|\nabla u^\sharp\|_n \leq \|\nabla u\|_n. \tag{1.5}$$

The reduction of the problem to the radial case by mean of the rearrangements of functions is a key tool in the proof of all the previous results.

1.2. **Adams' inequality.** In 1988 D. R. Adams [1] obtained another interesting extension of (1.1) for Sobolev spaces involving higher order derivatives. Let $\Omega \subset \mathbb{R}^n$ with $n \geq 2$, if we consider spaces of the form $W_0^{2,2}(\Omega)$ then the limiting case in the Sobolev embedding theorem corresponds to the dimension $n = 4$ and in this particular case Adams' result can be stated as follows.

Theorem 1.5 ([1], Theorem 1)**.** *Let $\Omega \subset \mathbb{R}^4$ be a bounded domain. Then there exists a constant $C > 0$ such that*

$$\sup_{u \in W_0^{2,2}(\Omega),\, \|\Delta u\|_2 \leq 1} \int_\Omega e^{32\pi^2 u^2}\, dx \leq C|\Omega| \tag{1.6}$$

and this inequality is sharp.

We point out that some authors (see [25], [37] and [34]) have remarked that the proof of Adams' can, in fact, be adapted in order to obtain a stronger inequality involving a larger class of functions, i.e. the Sobolev space $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$. More precisely,

Theorem 1.6 ([25], [37] and [34])**.** *Let $\Omega \subset \mathbb{R}^4$ be a bounded domain. Then there exists a constant $C > 0$ such that*

$$\sup_{u \in W^{2,2} \cap W_0^{1,2}(\Omega),\, \|\Delta u\|_2 \le 1} \int_\Omega e^{32\pi^2 u^2}\, dx \le C|\Omega|$$

and this inequality is sharp.

In particular, we bring the reader's attention to the paper [37] where C. Tarsi suggests a proof of Theorem 1.6 which differs esserntially from Adams' one and relies on sharp embeddings into Zygmund spaces.

We recall that L. Fontana [17] proved that the complete analogue of Adams' theorem, Theorem 1.5, is valid for every compact smooth Riemannian manifold M. See also L. Fontana and C. Morpurgo [18] for a generalization of Adams' result to functions defined on arbitrary measure spaces.

As in the case of first order derivatives, one notes that the bound in (1.6) becomes infinite for domains Ω with $|\Omega| = +\infty$. In [34] the authors show that, replacing the norm $\|\Delta u\|_2$ by the full Sobolev norm

$$\|u\|^2_{W^{2,2}} := \|\Delta u\|_2^2 + \|\nabla u\|_2^2 + \|u\|_2^2,$$

the supremum in (1.6) is bounded by a constant independent of the domain $\Omega \subseteq \mathbb{R}^4$.

Theorem 1.7 ([34], Theorem 1.4)**.** *There exists a constant $C > 0$ such that for any domain $\Omega \subseteq \mathbb{R}^4$*

$$\sup_{u \in W_0^{2,2}(\Omega),\, \|u\|_{W^{2,2}} \le 1} \int_\Omega (e^{32\pi^2 u^2} - 1)\, dx \le C \tag{1.7}$$

and this inequality is sharp.

Remark 2. Dealing with higher order derivatives, the problem cannot be reduced directly to radial case. In fact, given a function $u \in W^{2,2}(\Omega)$, we do not know whether or not $u^\sharp$ still belongs to $W^{2,2}(\Omega^\sharp)$ and, even in the case $u^\sharp \in W^{2,2}(\Omega^\sharp)$, no inequality of the form (1.5) is known to hold for higher order derivatives. Adams' approach to this difficulty is to express u as Riesz potential of its Laplacian and then apply a result of O'Neil on non-increasing rearrangements for convolution integrals (see also Section 8). To overcome the same difficulty, in [34] the authors apply a suitable comparison principle, which leads to compare a function $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ not necessarily radial with a radial function $v \in W^{2,2}(\Omega^\sharp) \cap W_0^{1,2}(\Omega^\sharp)$, preserving a full Sobolev norm equivalent to the standard one and increasing the integral, i.e.

$$\int_\Omega (e^{32\pi^2 u^2} - 1)\, dx \le \int_{\Omega^\sharp} (e^{32\pi^2 v^2} - 1)\, dx.$$

2. Main Results

As in the case of first order derivatives, we ask the question:

How can we modify the exponential nonlinearity in order to obtain an analogue of Theorem 1.4 for the Sobolev space $W^{2,2}(\mathbb{R}^4)$?

The main result of this paper gives an answer to this question:

Theorem 2.1. *(***Adams' inequality with the exact growth.***) There exists a constant $C > 0$ such that*

$$\int_{\mathbb{R}^4} \frac{e^{32\pi^2 u^2} - 1}{(1+|u|)^2}\, dx \le C\|u\|_2^2 \quad \forall u \in W^{2,\,2}(\mathbb{R}^4) \;\; \textit{with} \; \|\Delta u\|_2 \le 1\,. \tag{2.1}$$

Moreover, this fails if the power 2 in the denominator is replaced with any $p < 2$.

We recall that T. Ogawa and T. Ozawa [30] proved the existence of positive constants α and C such that

$$\int_{\mathbb{R}^4} (e^{\alpha u^2} - 1)\, dx \le C\|u\|_2^2 \quad \forall u \in W^{2,\,2}(\mathbb{R}^4) \text{ with } \|\Delta u\|_2 \le 1\,. \tag{2.2}$$

The proof of this result follows the original idea of N. S. Trudinger; making use of the power series expansion of the exponential function, the problem reduces to majorizing each term of the expansion in terms of the Sobolev norms in order that the resulting power series should converge. But with these arguments the authors of [30] did not find the best possible exponent for this type of inequalities. However, as a simple consequence of Theorem 2.1, we have the following Adachi-Tanaka type inequality in the space $W^{2,\,2}(\mathbb{R}^4)$:

Theorem 2.2. *For any $\alpha \in (0,\, 32\pi^2)$ there exists a constant $C(\alpha) > 0$ such that*

$$\int_{\mathbb{R}^4} (e^{\alpha u^2} - 1)\, dx \le C(\alpha)\|u\|_2^2 \quad \forall u \in W^{2,\,2}(\mathbb{R}^4) \;\; \textit{with} \; \|\Delta u\|_2 \le 1\,, \tag{2.3}$$

and this inequality is false for $\alpha \ge 32\pi^2$.

Moreover, the arguments introduced in [21], to show that the Trudinger-Moser inequality with the exact growth condition (Theorem 1.4) implies the Trudinger-Moser inequality in $W^{1,\,2}(\mathbb{R}^2)$ (Theorem 1.3), will enable us to deduce from Theorem 2.1 a version of the Adams' inequality in $W^{2,\,2}(\mathbb{R}^4)$ involving the norm

$$\|\cdot\|_2^2 + \|\cdot\|_2^2$$

instead of the full Sobolev norm $\|\cdot\|_{W^{2,\,2}}$, for a precise statement see Theorem 6.1. Obviously, this newer version of Adams' inequality in $\mathbb{R}^4$ implies Theorem 1.7.

This paper is organized as follows. In order to prove Theorem 2.1, we, first, recall some preliminaries on rearrangements of functions (Section 3.1). Second, we introduce a useful inequality involving rearrangements, Proposition 3.5, that will enable us to avoid the difficulty of the reduction of the problem to the radial case (see Remark 2). Then, following [21], a key ingredient is the study in Section 3.3 of the growth, in the exterior of balls, of rearrangements of functions belonging to $W^{2,\,2}(\mathbb{R}^4)$. In Section 4, we prove inequality (2.1) and Section 5 is devoted to the proof of the sharpness of (2.1). Finally, in Section 6 we show that the Adams' inequality with the exact growth condition expressed by Theorem 2.1 implies inequality (1.7) and, as already mentioned, we obtain an improved version of Theorem 1.7.

We complete this paper with two Appendices. In the first one, we suggest an alternative proof of Adams' inequality in its original form (Theorem 1.5) which, at the same time, gives a simplified version of Adams' proof and involves the Sobolev space $W^{2,\,2}(\Omega) \cap W_0^{1,\,2}(\Omega)$, where $\Omega \subset \mathbb{R}^4$ is a bounded domain. In the second Appendix, we propose a variant of the proof of the Trudinger-Moser inequality with the exact growth condition, namely of Theorem 1.4, which simplifies the critical part of the estimate.

3. Preliminaries

3.1. **Rearrangements.** Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, be a measurable set, we denote by $\Omega^\sharp$ the open ball $B_R \subset \mathbb{R}^n$ centered at $0 \in \mathbb{R}^n$ of radius $R > 0$ such that

$$|\Omega| = |B_R|$$

where $|\cdot|$ denotes the n-dimensional Lebesgue measure. Let $u : \Omega \to \mathbb{R}$ be a real-valued measurable function in Ω. Then the *distribution function* of u is the function $\mu_u : [0, +\infty) \to [0, +\infty]$ defined as

$$\mu_u(t) := \big|\{x \in \Omega \mid |u(x)| > t\}\big| \qquad \text{for } t \geq 0,$$

and the *decreasing rearrangement* of u is the right-continuous non-increasing function $u^* : [0, +\infty) \to [0, +\infty]$ which is equimeasurable with u, namely

$$u^*(s) := \sup\{t \geq 0 \ \mid \mu_u(t) > s\} \qquad \text{for } s \geq 0.$$

We can notice that the support of u^* satisfies supp $u^* \subseteq [0, |\Omega|]$. Since u^* is non-increasing, the *maximal function* u^{**} of the rearrangement u^*, defined by

$$u^{**}(s) := \frac{1}{s}\int_0^s u^*(t)\, dt \qquad \text{for } s \geq 0,$$

is also non-increasing and $u^* \leq u^{**}$. Moreover

Proposition 3.1. *If $u \in L^p(\mathbb{R}^n)$ with $1 < p < +\infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$ then*

$$\Big(\int_0^{+\infty} [u^{**}(s)]^p\, ds\Big)^{\frac{1}{p}} \leq p'\Big(\int_0^{\infty} [u^*(s)]^p\, dt\Big)^{\frac{1}{p}}.$$

In particular, if supp $u \subseteq \Omega$ with Ω domain in $\mathbb{R}^n$, then

$$\Big(\int_0^{|\Omega|} [u^{**}(s)]^p\, ds\Big)^{\frac{1}{p}} \leq p'\Big(\int_0^{|\Omega|} [u^*(s)]^p\, dt\Big)^{\frac{1}{p}}.$$

This is an immediate consequence of the following well known result of G. H. Hardy (see e.g. [8], Chapter 3, Lemma 3.9)

Lemma 3.2 (Hardy's inequality). *If $1 < p < +\infty$, $\frac{1}{p} + \frac{1}{p'} = 1$ and f is a non-negative function, then*

$$\Big(\int_0^{+\infty} \Big[\frac{s^{\frac{1}{p}}}{s}\int_0^s f(t)\, dt\Big]^p \frac{ds}{s}\Big)^{\frac{1}{p}} \leq p'\Big(\int_0^{+\infty} \big[s^{\frac{1}{p}} f(s)\big]^p \frac{ds}{s}\Big)^{\frac{1}{p}}.$$

Finally, we will denote by $u^\sharp : \Omega^\sharp \to [0, +\infty]$ the *spherically symmetric decreasing rearrangement* of u:

$$u^\sharp(x) := u^*(\sigma_n |x|^n) \qquad \text{for } x \in \Omega^\sharp,$$

where σ_n is the volume of the unit ball in $\mathbb{R}^n$.

3.2. **A useful inequality involving rearrangements.** Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded open set. As a consequence of the coarea formula and the isoperimetric inequality, we have the following well-known result

Lemma 3.3 ([36], Inequality (32)). *If $u \in W_0^{1,2}(\Omega)$ then*

$$1 \leq \frac{-\mu_u'(t)}{[n\sigma_n^{\frac{1}{n}} \mu_u^{\frac{n-1}{n}}]^2}\Big(-\frac{d}{dt}\int_{\{|u|>t\}} |\nabla u|^2\, dx\Big)$$

for a.e. $t > 0$.

Sketch of the proof. For fixed t, $h > 0$, applying Hölder's inequality, we get

$$\frac{1}{h}\int_{\{t<|u|\le t+h\}} |\nabla u|\,dx \le \Big(\frac{1}{h}\int_{\{t<|u|\le t+h\}} |\nabla u|^2\,dx\Big)^{\frac{1}{2}}\Big(\frac{\mu_u(t)-\mu_u(t+h)}{h}\Big)^{\frac{1}{2}}$$

and, letting $h \downarrow 0^+$, we obtain

$$-\frac{d}{dt}\int_{\{t<|u|\}} |\nabla u|\,dx \le \Big(-\frac{d}{dt}\int_{\{t<|u|\}} |\nabla u|^2\,dx\Big)^{\frac{1}{2}}\big(-\mu_u'(t)\big)^{\frac{1}{2}}. \tag{3.1}$$

On the other hand, from the Coarea Formula and the isoperimetric inequality

$$-\frac{d}{dt}\int_{\{t<|u|\}} |\nabla u|\,dx \ge n\sigma_n^{\frac{1}{n}}\mu_u^{\frac{1}{n'}}(t)$$

and this inequality together with (3.1) gives the desired estimate. □

Let $f \in L^2(\Omega)$, we consider the following Dirichlet problem:

$$\begin{cases} -\Delta u = f \in L^2(\Omega) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

It is well-known that

Lemma 3.4 ([36], Inequality (30))**.** *Let $u \in W_0^{1,2}(\Omega)$ be the unique weak solution to (3.2), then*

$$-\frac{d}{dt}\int_{\{|u|>t\}} |\nabla u|^2\,dx \le \int_0^{\mu_u(t)} f^*(s)\,ds$$

for a.e. $t > 0$.

Sketch of the proof. For fixed t, $h > 0$, we define

$$\phi(x) := \begin{cases} 0 & \text{if } |u| \le t, \\ (|u|-t)\,\text{sign }(u) & \text{if } t \le |u| \le t+h, \\ h\,\text{sign }(u) & \text{if } t+h < |u|. \end{cases}$$

Then $\phi \in W_0^{1,2}(\Omega)$ and

$$\int_\Omega \nabla u \cdot \nabla\phi\,dx = \int_\Omega f\phi\,dx,$$

since u is a weak solution to (3.2). Now, easy computations show that

$$\int_{\{t<|u|\le t+h\}} |\nabla u|^2\,dx = \int_{\{t<|u|\le t+h\}} f\cdot(|u|-t)\,\text{sign }(u)\,dx + \int_{\{t+h<|u|\}} fh\,\text{sign }(u)\,dx. \tag{3.3}$$

Dividing through by h in (3.3) and letting $h \downarrow 0^+$, we obtain

$$-\frac{d}{dt}\int_{\{t<|u|\}} |\nabla u|^2\,dx \le -\frac{d}{dt}\int_{\{t<|u|\}} |f|\cdot(|u|-t)\,dx = \int_{\{t<|u|\}} |f|\,dx \le \int_0^{\mu_u(t)} f^*\,ds.$$

□

If $u \in W_0^{1,2}(\Omega)$ is the unique weak solution to (3.2) then, combining Lemma 3.3 and Lemma 3.4, we get

$$1 \le \frac{-\mu_u'(t)}{[n\sigma_n^{\frac{1}{n}}\mu_u^{\frac{n-1}{n}}]^2}\,F(\mu_u(t))$$

for a.e. $t>0$, where $F(\xi):=\xi f^{**}(\xi)=\int_0^\xi f^*(s)\,ds$. Let $0<s'<s$, integrating both sides of this last inequality from s' to s and making the change of variable $\xi=\mu_u(t)$, we obtain that

$$s-s'\le\frac{1}{[n\sigma_n^{\frac{1}{n}}]^2}\int_{\mu_u(s)}^{\mu_u(s')}\frac{F(\xi)}{\xi^{2\frac{n-1}{n}}}\,d\xi=\frac{1}{[n\sigma_n^{\frac{1}{n}}]^2}\int_{\mu_u(s)}^{\mu_u(s')}\frac{f^{**}(\xi)}{\xi^{1-\frac{2}{n}}}\,d\xi. \tag{3.4}$$

From inequality (3.4) we deduce the following result.

Proposition 3.5. *Let $u\in W_0^{1,2}(\Omega)$ be the unique weak solution to* (3.2) *then*

$$u^*(t_1)-u^*(t_2)\le\frac{1}{[n\sigma_n^{\frac{1}{n}}]^2}\int_{t_1}^{t_2}\frac{f^{**}(\xi)}{\xi^{1-\frac{2}{n}}}\,d\xi\qquad\text{for a.e. } 0<t_1\le t_2\le|\Omega|.$$

Equivalently, if $\Omega^\sharp=B_R$ then

$$u^\sharp(R_1)-u^\sharp(R_2)\le\frac{1}{[n\sigma_n^{\frac{1}{n}}]^2}\int_{|B_{R_1}|}^{|B_{R_2}|}\frac{f^{**}(\xi)}{\xi^{1-\frac{2}{n}}}\,d\xi\qquad\text{for a.e. } 0<R_1\le R_2\le R.$$

Proof. Let $0<t_1<t_2\le|\Omega|$, without loss of generality we may assume that $u^*(t_1)>u^*(t_2)$. Then there exists $\overline{\eta}>0$ such that

$$u^*(t_1)>u^*(t_2)+\eta\quad\forall\eta\in[0,\overline{\eta}).$$

Arbitrarily fixed $\eta\in[0,\overline{\eta})$, we set

$$s:=u^*(t_1)-\frac{\eta}{2}\qquad\text{and}\qquad s':=u^*(t_2)+\frac{\eta}{2},$$

so that $0<s'<s$. Now applying inequality (3.4), we get

$$u^*(t_1)-u^*(t_2)-\eta\le\frac{1}{[n\sigma_n^{\frac{1}{n}}]^2}\int_{\mu_u(s)}^{\mu_u(s')}\frac{f^{**}(\xi)}{\xi^{1-\frac{2}{n}}}\,d\xi\le\frac{1}{[n\sigma_n^{\frac{1}{n}}]^2}\int_{t_1}^{t_2}\frac{f^{**}(\xi)}{\xi^{1-\frac{2}{n}}}\,d\xi, \tag{3.5}$$

in fact, recalling the definition of decreasing rearrangement of a function, since $s<u^*(t_1)$ then $\mu_u(s)>t_1$ and, similarly, since $s'>u^*(t_2)$ then $\mu_u(s')<t_2$.

From the arbitrary choice of $\eta\in[0,\overline{\eta})$, inequality (3.5) holds for any $\eta\in[0,\overline{\eta})$ and passing to the limit as $\eta\downarrow 0^+$, we obtain the desired estimate. □

Remark 3. It is easy to prove that an analogue of Proposition 3.5 holds also for functions in $W^{2,2}(\mathbb{R}^n)$. In fact, if $u\in W^{2,2}(\mathbb{R}^n)$ then there exists a sequence $\{u_k\}_{k\ge1}\subset\mathcal{C}_0^\infty(\mathbb{R}^n)$ such that $u_k\to u$ in $W^{2,2}(\mathbb{R}^n)$. Thus, in particular,

$$f_k\to f\ \text{ in } L^2(\mathbb{R}^n)\qquad\text{where } f_k:=-\Delta u_k,\ f:=-\Delta u$$

and $u_k\to u$ in $L^2(\mathbb{R}^n)$. Since the decreasing rearrangement operator is nonexpansive in L^2 (see e.g. G. Chiti [14]), we have

$$f_k^*\to f^*\qquad\text{and}\qquad u_k^*\to u^*\qquad\text{in } L^2(\mathbb{R}^n). \tag{3.6}$$

In particular, up to the passage to a subsequence, we have also that $u_k^*\to u^*$ a.e. in $\mathbb{R}^n$ and, from Proposition 3.5, it follows that for fixed $0<t_1\le t_2$

$$u^*(t_1)-u^*(t_2)=\lim_{k\to+\infty}u_k^*(t_1)-u_k^*(t_2)\le\lim_{k\to+\infty}\frac{1}{[n\sigma_n^{\frac{1}{n}}]^2}\int_{t_1}^{t_2}\frac{f_k^{**}(\xi)}{\xi^{1-\frac{2}{n}}}\,d\xi.$$

On the other hand,

$$\lim_{k\to+\infty}\frac{1}{[n\sigma_n^{\frac{1}{n}}]^2}\int_{t_1}^{t_2}\frac{f_k^{**}(\xi)}{\xi^{1-\frac{2}{n}}}\,d\xi=\frac{1}{[n\sigma_n^{\frac{1}{n}}]^2}\int_{t_1}^{t_2}\frac{f^{**}(\xi)}{\xi^{1-\frac{2}{n}}}\,d\xi,$$

as a direct consequence of Hölder's inequality and (3.6).

3.3. **Optimal descending growth condition.** The following exponential version of the radial Sobolev inequality will be a key tool in the proof of Theorem 2.1.

Proposition 3.6. *Let $u \in W^{2,2}(\mathbb{R}^4)$ and let $R > 0$. If $u^\sharp(R) > 1$ and $f := -\Delta u$ in $\mathbb{R}^4$ satisfies*

$$\int_{|B_R|}^{+\infty} [f^{**}(s)]^2 \, ds \leq 4K,$$

for some $K > 0$, then

$$\frac{e^{\frac{32\pi^2}{K}[u^\sharp(R)]^2}}{[u^\sharp(R)]^2} R^4 \leq C \frac{\|u^\sharp\|^2_{L^2(\mathbb{R}^4 \setminus B_R)}}{K^2}$$

where $C > 0$ is a universal constant independent of u, R and K.

Fix h, R, $K > 0$. Let

$$\mu(h, R, K) := \inf\Big\{ \|u^\sharp\|_{L^2(\mathbb{R}^4 \setminus B_R)} \;\Big|\; u \in W^{2,2}(\mathbb{R}^4),\ u^\sharp(R) = h \text{ and } \int_{|B_R|}^{+\infty} [f^{**}(s)]^2 \, ds \leq 4K \text{ where } f := -\Delta u \text{ in } \mathbb{R}^4 \Big\}$$

and $\mu(h) := \mu(h, 1, 1)$. Using scaling properties, it is easy to see that

$$\mu(h, R, K) = R^2 \mu(h, 1, K) = R^2\sqrt{K}\mu\Big(\frac{h}{\sqrt{K}}, 1, 1\Big) = R^2\sqrt{K}\mu\Big(\frac{h}{\sqrt{K}}\Big).$$

We will prove the following result.

Proposition 3.7. *If $h > 1$ then*

$$\mu(h) \gtrsim \frac{e^{16\pi^2 h^2}}{h}.$$

Obviously, Proposition 3.6 is a direct consequence of Proposition 3.7. In order to prove Proposition 3.7, it suffices to consider the discrete version. Given any sequence $a := \{a_k\}_{k \geq 0}$, following [21], we introduce the notations

$$\|a\|_p^p := \sum_{k=0}^{+\infty} |a_k|^p, \quad \|a\|^2_{(e)} := \sum_{k=0}^{+\infty} a_k^2 e^k$$

and, for $h > 1$, we let

$$\mu_d(h) := \inf\{\|a\|_{(e)} \mid \|a\|_1 = h,\ \|a\|_2 \leq 1\}.$$

Lemma 3.8. *For any $h > 1$, we have $\mu(h) \gtrsim \mu_d(\sqrt{32\pi^2}h)$.*

Proof. Fix $h > 1$ and let $u \in W^{2,2}(\mathbb{R}^4)$ be such that $u^\sharp(1) = h$ and

$$\int_{|B_1|}^{+\infty} [f^{**}(s)] \, ds \leq 4,$$

where $f := -\Delta u$ in $\mathbb{R}^4$. Then we can define a sequence $a := \{a_k\}_{k \geq 0}$ as follows:

$$a_k := h_k - h_{k+1}, \quad \text{where } h_k := \sqrt{32\pi^2} u^\sharp(e^{\frac{k}{4}}).$$

By construction $\|a\|_1 = \sqrt{32\pi^2}u^\sharp(1) = \sqrt{32\pi^2}h$. In order to estimate $\|a\|_2$, we can notice that applying Proposition 3.5 (see also Remark 3) and Hölder's inequality, we get

$$\begin{aligned} 0 \le h_k - h_{k+1} &= \sqrt{32\pi^2}[u^\sharp(e^{\frac{k}{4}}) - u^\sharp(e^{\frac{k+1}{4}})] \le \sqrt{32\pi^2}\frac{\sqrt{2}}{16\pi}\int_{\left|B_{e^{k/4}}\right|}^{\left|B_{e^{(k+1)/4}}\right|} \frac{[f^{**}(s)]}{\sqrt{s}}\,ds \le \\ &\le \frac{1}{2}\Big(\int_{\left|B_{e^{k/4}}\right|}^{\left|B_{e^{(k+1)/4}}\right|}[f^{**}(s)]^2\,ds\Big)^{\frac{1}{2}}. \end{aligned}$$

Therefore

$$\|a\|_2^2 = \sum_{k=0}^{+\infty}(h_k - h_{k+1})^2 \le \frac{1}{4}\sum_{k=0}^{+\infty}\int_{\left|B_{e^{k/4}}\right|}^{\left|B_{e^{(k+1)/4}}\right|}[f^{**}(s)]^2\,ds = \frac{1}{4}\int_{|B_1|}^{+\infty}[f^{**}(s)]^2\,ds \le 1.$$

Finally,

$$\begin{aligned} \|u^\sharp\|^2_{L^2(\mathbb{R}^4\setminus B_1)} &= 2\pi^2\sum_{k=0}^{+\infty}\int_{e^{\frac{k}{4}}}^{e^{\frac{k+1}{4}}}[u^\sharp(r)]^2r^3\,dr \ge 2\pi^2\sum_{k=0}^{+\infty}[u^\sharp(e^{\frac{k+1}{4}})]^2\frac{e^{k+1}-e^k}{4} \gtrsim \\ &\gtrsim \sum_{k=0}^{+\infty}[u^\sharp(e^{\frac{k+1}{4}})]^2e^{k+1} = \frac{1}{32\pi^2}\sum_{j=1}^{+\infty}h_j^2e^j \ge \frac{1}{32\pi^2}\sum_{j=1}^{+\infty}a_j^2e^j \end{aligned}$$

from which we deduce that

$$\|a\|^2_{(e)} = a_0^2 + \sum_{k=1}^{+\infty}a_k^2e^k \lesssim h_0^2 + \|u^\sharp\|^2_{L^2(\mathbb{R}^4\setminus B_1)},$$

and the proof is complete if we show that

$$h_0^2 \lesssim \|u^\sharp\|^2_{L^2(\mathbb{R}^4\setminus B_1)}.$$

This is again a consequence of Proposition 3.5 (see also Remark 3) and Hölder's inequality, in fact for $r \in (1,\, e^{\frac{1}{16}})$

$$h_0 - \sqrt{32\pi^2}u^\sharp(r) \le \frac{1}{2}\int_{|B_1|}^{|B_r|}\frac{[f^{**}(s)]}{\sqrt{s}}\,ds \le \frac{1}{2}\Big(\int_{|B_1|}^{+\infty}[f^{**}(s)]^2\,ds\Big)^{\frac{1}{2}}\Big(\int_{|B_1|}^{|B_{e^{1/16}}|}\frac{1}{s}\,ds\Big)^{\frac{1}{2}} \le \frac{1}{2} \le \frac{h_0}{2},$$

that is $h_0 \lesssim u^\sharp(r)$. Thus

$$\|u^\sharp\|^2_{L^2(\mathbb{R}^4\setminus B_1)} \ge 2\pi^2\int_1^{e^{\frac{1}{16}}}[u^\sharp(r)]^2r^3\,dr \gtrsim h_0^2.$$

□

Now, it is sufficient to follow the proof of [21], Lemma 3.4 to obtain

Lemma 3.9. *For any $h > 1$, we have*

$$\mu_d(h) \sim \frac{e^{\frac{h^2}{2}}}{h}.$$

This last lemma together with Lemma 3.8 completes the proof of Proposition 3.7.

4. Proof of the Main Theorem (Theorem 2.1)

In this Section we will show that Theorem 2.1 holds, to this aim let us fix $u \in W^{2,2}(\mathbb{R}^4)$ with $\|\Delta u\|_2 \le 1$. We define $R_0 = R_0(u) > 0$ as

$$R_0 := \inf\{r > 0 \mid u^\sharp(r) \le 1\} \in [0, +\infty)$$

and the idea is to split the integral we are interested in into two parts

$$\int_{\mathbb{R}^4} \frac{e^{32\pi^2 u^2} - 1}{(1+|u|)^2}\, dx = \int_{\mathbb{R}^4} \frac{e^{32\pi^2 (u^\sharp)^2} - 1}{(1+u^\sharp)^2}\, dx = \int_{B_{R_0}} + \int_{\mathbb{R}^4 \setminus B_{R_0}} \frac{e^{32\pi^2 (u^\sharp)^2} - 1}{(1+u^\sharp)^2}\, dx.$$

Since $R_0 = R_0(u) > 0$, we have to obtain an estimate independent of R_0.

By construction $u^\sharp \le 1$ on $\mathbb{R}^4 \setminus B_{R_0}$, hence we have that

$$\frac{e^{32\pi^2 (u^\sharp)^2} - 1}{(1+u^\sharp)^2} \le 32\pi^2 (u^\sharp)^2 e^{32\pi^2 (u^\sharp)^2} \le 32\pi^2 e^{32\pi^2} (u^\sharp)^2 \quad \text{on } \mathbb{R}^4 \setminus B_{R_0}.$$

Consequently

$$\int_{\mathbb{R}^4 \setminus B_{R_0}} \frac{e^{32\pi^2 (u^\sharp)^2} - 1}{(1+u^\sharp)^2}\, dx \le 32\pi^2 e^{32\pi^2} \int_{\mathbb{R}^4 \setminus B_{R_0}} (u^\sharp)^2\, dx \le 32\pi^2 e^{32\pi^2} \|u^\sharp\|_2^2 \le 32\pi^2 e^{32\pi^2} \|u\|_2^2$$

and without loss of generality we may assume that $R_0 > 0$.

From now on, we will focus our attention in the estimate of the integral on the ball B_{R_0}. In order to simplify the notations, we set

$$f := -\Delta u \quad \text{in } \mathbb{R}^4$$

and

$$\beta := \int_0^{+\infty} [f^{**}(s)]^2\, ds.$$

As a consequence of Proposition 3.1 with $p = 2$ and in view of the assumption $\|\Delta u\|_2 \le 1$, we have

$$\beta \le 4. \tag{4.1}$$

Let $R_1 = R_1(u) > 0$ be such that

$$\int_0^{|B_{R_1}|} [f^{**}(s)]^2\, ds = \beta(1-\varepsilon_0) \quad \text{and} \quad \int_{|B_{R_1}|}^{+\infty} [f^{**}(s)]^2\, ds = \beta\,\varepsilon_0 \tag{4.2}$$

with $\varepsilon_0 \in (0, 1)$ arbitrarily fixed. We point out that $\varepsilon_0 \in (0, 1)$ is fixed independently of u and this forces R_1 to depend on u, so our final estimate has to be independent of R_1 but may instead depend on ε_0.

Applying Proposition 3.5 (see also Remark 3) and Hölder's inequality, we have the following estimate

$$u^\sharp(r_1) - u^\sharp(r_2) \le \frac{\sqrt{2}}{16\pi} \Big(\int_{|B_{r_1}|}^{|B_{r_2}|} [f^{**}(s)]^2\, ds \Big)^{\frac12} \Big(\log \frac{r_2^4}{r_1^4} \Big)^{\frac12} \quad \text{for a.e. } 0 < r_1 \le r_2. \tag{4.3}$$

Hence, from (4.1) and (4.3), it follows that

$$u^\sharp(r_1) - u^\sharp(r_2) \le \Big(\frac{1-\varepsilon_0}{32\pi^2} \Big)^{\frac12} \Big(\log \frac{r_2^4}{r_1^4} \Big)^{\frac12} \quad \text{for a.e. } 0 < r_1 \le r_2 \le R_1, \tag{4.4}$$

$$u^\sharp(r_1) - u^\sharp(r_2) \le \Big(\frac{\varepsilon_0}{32\pi^2} \Big)^{\frac12} \Big(\log \frac{r_2^4}{r_1^4} \Big)^{\frac12} \quad \text{for a.e. } R_1 \le r_1 \le r_2. \tag{4.5}$$

In order to estimate the integral on the ball B_{R_0}, we will distinguish between two cases:

- *First case:* $0 < R_0 \le R_1$,

- *Second case:* $0 < R_1 < R_0$.

The first case is subcritical and easily estimated. In fact in the case $R_0 \le R_1$, applying (4.4), we get for $0 < r \le R_0$

$$u^\sharp(r) \le 1 + \Big(\frac{1-\varepsilon_0}{32\pi^2}\Big)^{\frac{1}{2}} \Big(\log \frac{R_0^4}{r^4}\Big)^{\frac{1}{2}},$$

since $u^\sharp(R_0) = 1$. Thus, recalling that

$$(a+b)^2 \le (1+\varepsilon)a^2 + \Big(1 + \frac{1}{\varepsilon}\Big)b^2 \quad \forall a,\, b \in \mathbb{R},\ \forall\, \varepsilon > 0, \tag{4.6}$$

we obtain

$$[u^\sharp(r)]^2 \le \frac{1-\varepsilon_0^2}{32\pi^2} \log \frac{R_0^4}{r^4} + \Big(1 + \frac{1}{\varepsilon_0}\Big) \qquad \text{for a.e. } 0 < r \le R_0.$$

Hence

$$\begin{aligned}
\int_{B_{R_0}} \frac{e^{32\pi^2 (u^\sharp)^2} - 1}{(1+u^\sharp)^2}\, dx \ &\le\ 2\pi^2 e^{32\pi^2\left(1+\frac{1}{\varepsilon_0}\right)} R_0^{4(1-\varepsilon_0)} \int_0^{R_0} r^{3-4(1-\varepsilon_0)}\, dr = \\
&=\ 2\pi^2 e^{32\pi^2\left(1+\frac{1}{\varepsilon_0}\right)} R_0^4 = e^{32\pi^2\left(1+\frac{1}{\varepsilon_0}\right)} \int_{B_{R_0}} dx \le \\
&\le\ e^{32\pi^2\left(1+\frac{1}{\varepsilon_0}\right)} \int_{B_{R_0}} [u^\sharp]^2\, dx \le e^{32\pi^2\left(1+\frac{1}{\varepsilon_0}\right)} \|u^\sharp\|_2^2 = e^{32\pi^2\left(1+\frac{1}{\varepsilon_0}\right)} \|u\|_2^2.
\end{aligned}$$

Therefore, from now on, we will always assume that $0 < R_1 < R_0$. We can write

$$\int_{B_{R_0}} \frac{e^{32\pi^2 (u^\sharp)^2} - 1}{(1+u^\sharp)^2}\, dx = \int_{B_{R_1}} + \int_{B_{R_0} \setminus B_{R_1}} \frac{e^{32\pi^2 (u^\sharp)^2} - 1}{(1+u^\sharp)^2}\, dx$$

and again the estimate of the integral on $B_{R_0} \setminus B_{R_1}$ is easier. In fact, applying (4.5) and (4.6), we get for $R_1 \le r \le R_0$

$$[u^\sharp(r)]^2 \le \frac{\varepsilon_0(1+\varepsilon)}{32\pi^2} \log \frac{R_0^4}{r^4} + \Big(1 + \frac{1}{\varepsilon}\Big) \quad \forall\, \varepsilon > 0.$$

Consequently, for arbitrarily fixed $\varepsilon > 0$, we have

$$\int_{B_{R_0} \setminus B_{R_1}} \frac{e^{32\pi^2 (u^\sharp)^2} - 1}{(1+u^\sharp)^2}\, dx \le 2\pi^2 e^{32\pi^2\left(1+\frac{1}{\varepsilon}\right)} R_0^{4\,\varepsilon_0(1+\varepsilon)} \int_{R_1}^{R_0} r^{3-4\,\varepsilon_0(1+\varepsilon)}\, dr.$$

Since $\varepsilon_0 < 1$, we can choose $\varepsilon > 0$ so that $1 + \varepsilon < 1/\varepsilon_0$ and with this choice of $\varepsilon > 0$ we obtain

$$\begin{aligned}
\int_{B_{R_0} \setminus B_{R_1}} \frac{e^{32\pi^2 (u^\sharp)^2} - 1}{(1+u^\sharp)^2}\, dx \ &\le\ 2\pi^2 e^{32\pi^2\left(1+\frac{1}{\varepsilon}\right)}\ \frac{R_0^4 - R_0^{4\,\varepsilon_0(1+\varepsilon)} R_1^{4-4\,\varepsilon_0(1+\varepsilon)}}{4 - 4\,\varepsilon_0(1+\varepsilon)} \le \\
&\le\ 2\pi^2 e^{32\pi^2\left(1+\frac{1}{\varepsilon}\right)}\ \frac{R_0^4 - R_1^4}{4 - 4\,\varepsilon_0(1+\varepsilon)} \le \\
&\le\ \frac{e^{32\pi^2\left(1+\frac{1}{\varepsilon}\right)}}{4 - 4\,\varepsilon_0(1+\varepsilon)} \int_{B_{R_0} \setminus B_{R_1}} dx \le \frac{e^{32\pi^2\left(1+\frac{1}{\varepsilon}\right)}}{4 - 4\,\varepsilon_0(1+\varepsilon)} \|u\|_2^2.
\end{aligned}$$

Hence to conclude the proof it remains to estimate the integral on the ball B_{R_1}, assuming that $0 < R_1 < R_0$ and in particular $u^\sharp(R_1) > 1$.

4.1. **Estimate of the integral on the ball B_{R_1} with $u^\sharp(R_1) > 1$.** We recall that $u^\sharp(R_1) > 1$ where $R_1 > 0$ satisfies (4.2). Since (4.1) holds, from the optimal descending growth condition (Proposition 3.6), we deduce that

$$\frac{e^{\frac{32\pi^2}{\varepsilon_0}[u^\sharp(R_1)]^2}}{[u^\sharp(R_1)]^2} R_1^4 \le C \frac{\|u^\sharp\|^2_{L^2(\mathbb{R}^4\setminus B_{R_1})}}{\varepsilon_0^2} \le \frac{C}{\varepsilon_0^2}\|u^\sharp\|_2^2 = \frac{C}{\varepsilon_0^2}\|u\|_2^2. \tag{4.7}$$

Applying (4.6), we can estimate for $0 \le r \le R_1$

$$[u^\sharp(r)]^2 \le (1+\varepsilon)[u^\sharp(r) - u^\sharp(R_1)]^2 + \Big(1 + \frac{1}{\varepsilon}\Big)[u^\sharp(R_1)]^2 \quad \forall \varepsilon > 0,$$

hence

$$\begin{aligned}
\int_{B_{R_1}} \frac{e^{32\pi^2(u^\sharp)^2} - 1}{(1+u^\sharp)^2}\, dx &\le \frac{1}{[u^\sharp(R_1)]^2}\int_{B_{R_1}} e^{32\pi^2(u^\sharp)^2}\, dx \le \\
&\le \frac{e^{32\pi^2\left(1+\frac{1}{\varepsilon}\right)[u^\sharp(R_1)]^2}}{[u^\sharp(R_1)]^2}\int_{B_{R_1}} e^{32\pi^2(1+\varepsilon)[u^\sharp - u^\sharp(R_1)]^2}\, dx \le \\
&\le \frac{e^{\frac{32\pi^2}{\varepsilon_0}[u^\sharp(R_1)]^2}}{[u^\sharp(R_1)]^2}\int_{B_{R_1}} e^{32\pi^2(1+\varepsilon)[u^\sharp - u^\sharp(R_1)]^2}\, dx
\end{aligned}$$

provided that $\varepsilon \ge \frac{\varepsilon_0}{1-\varepsilon_0}$. Therefore in view of (4.7), to conclude the proof it suffices to show that

$$\int_{B_{R_1}} e^{32\pi^2(1+\varepsilon)[u^\sharp - u^\sharp(R_1)]^2}\, dx \le C R_1^4 \tag{4.8}$$

for some $\varepsilon \ge \frac{\varepsilon_0}{1-\varepsilon_0}$. To this aim, we can notice that from Proposition 3.5 it follows that

$$0 \le u^*(r) - u^*(|B_{R_1}|) \le \frac{\sqrt{2}}{16\pi}\int_r^{|B_{R_1}|} \frac{f^{**}(\xi)}{\sqrt{\xi}}\, d\xi \qquad \text{for a.e. } 0 < r < |B_{R_1}|,$$

consequently

$$\int_{B_{R_1}} e^{32\pi^2(1+\varepsilon)[u^\sharp - u^\sharp(R_1)]^2}\, dx \le \int_0^{|B_{R_1}|} e^{\left(\frac{\sqrt{1+\varepsilon}}{2}\int_r^{|B_{R_1}|}\frac{f^{**}(\xi)}{\sqrt{\xi}}\, d\xi\right)^2}\, dr.$$

Now, making the change of variable $r = |B_{R_1}|e^{-t}$, we get

$$\int_{B_{R_1}} e^{32\pi^2(1+\varepsilon)[u^\sharp - u^\sharp(R_1)]^2}\, dx \le |B_{R_1}|\int_0^{+\infty} e^{\left(\frac{\sqrt{1+\varepsilon}}{2}\int_{|B_{R_1}|e^{-t}}^{|B_{R_1}|}\frac{f^{**}(\xi)}{\sqrt{\xi}}\, d\xi\right)^2 - t}\, dt \tag{4.9}$$

and the proof of (4.8) reduces to the application of the following one-dimensional calculus inequality due to J. Moser [28].

Lemma 4.1 ([28]). *There exists a constant $c_0 > 0$ such that for any nonnegative measurable function $\phi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ satisfying*

$$\int_0^{+\infty} \phi^2(s)\, ds \le 1$$

the following inequality holds

$$\int_0^{+\infty} e^{-F(t)}\, dt \le c_0,$$

where

$$F(t) := t - \left(\int_0^t \phi(s)\, ds\right)^2.$$

Let

$$\phi(s) := \sqrt{|B_{R_1}|}\,\frac{\sqrt{1+\varepsilon}}{2} f^{**}(|B_{R_1}|e^{-s})e^{-\frac{s}{2}} \quad \forall s \geq 0.$$

By construction $\phi \geq 0$ and, in view of (4.1) and (4.2),

$$\begin{aligned}\int_0^{+\infty} \phi^2(s)\,ds &= |B_{R_1}|\frac{1+\varepsilon}{4}\int_0^{+\infty} [f^{**}(|B_{R_1}|e^{-s})]^2 e^{-s}\,ds = \\ &= \frac{1+\varepsilon}{4}\int_0^{|B_{R_1}|} [f^{**}(r)]^2\,dr \leq (1+\varepsilon)(1-\varepsilon_0) \leq 1\end{aligned}$$

provided that $\varepsilon \leq \frac{\varepsilon_0}{1-\varepsilon_0}$. In particular, choosing $\varepsilon = \frac{\varepsilon_0}{1-\varepsilon_0}$, we have that $\int_{-\infty}^{+\infty} \phi^2(s)\,ds \leq 1$.

Hence, applying Lemma 4.1, we get

$$\int_0^{+\infty} e^{-F(t)}\,dt \leq c_0,$$

where

$$\begin{aligned}F(t) &= t - \left(\int_0^t \phi(s)\,ds\right)^2 = t - \left(\sqrt{|B_{R_1}|}\,\frac{\sqrt{1+\varepsilon}}{2}\int_0^t f^{**}(|B_{R_1}|e^{-s})e^{-\frac{s}{2}}\,ds\right)^2 = \\ &= t - \left(\frac{\sqrt{1+\varepsilon}}{2}\int_{|B_{R_1}|e^{-t}}^{|B_{R_1}|} \frac{f^{**}(\xi)}{\sqrt{\xi}}\,d\xi\right)^2.\end{aligned}$$

This together with (4.9) ends the proof of (4.8).

Remark 4. It is interesting to notice that with these arguments (Proposition 3.5 and Lemma 4.1) we can obtain an alternative proof of Adams' inequality in its original form, namely of Theorem 1.5, see Section 7.

Remark 5. As we will see in Section 8, it is possible to simplify the proof of Theorem 1.4. Besides, we point out that one can estimate the integral on the ball B_{R_1} in the case when $u^\sharp(R_1) > 1$ following the longer arguments given in [21] and a proper use of Proposition 3.5.

5. Sharpness

In order to prove that the inequality expressed in Theorem 2.1 is sharp, we have to show that the following inequality

$$\int_{\mathbb{R}^4} g(u)\,dx \leq C\|u\|_2^2 \quad \forall u \in W^{2,2}(\mathbb{R}^4) \text{ with } \|\Delta u\|_2 \leq 1 \tag{5.1}$$

fails if we replace the growth

$$g(u) := \frac{e^{32\pi^2 u^2}-1}{(1+|u|)^2}$$

with the higher order growth

$$g(u) := \frac{e^{\beta u^2}-1}{(1+|u|)^p}, \tag{5.2}$$

where either $\beta > 32\pi^2$ and $p = 2$ or $\beta = 32\pi^2$ and $p < 2$. In particular, it suffices to prove that (5.1) fails if we consider an exponential nonlinearity of the form (5.2) with $\beta = 32\pi^2$ and $p < 2$.

To this aim, for fixed $p < 2$, we argue by contradiction assuming that

$$\int_{\mathbb{R}^4} \frac{e^{32\pi^2 u^2}-1}{(1+|u|)^p}\,dx \leq C\|u\|_2^2 \quad \forall u \in W^{2,2}(\mathbb{R}^4) \text{ with } \|\Delta u\|_2 \leq 1. \tag{5.3}$$

We consider the sequence of test functions $\{u_k\}_{k\geq 1}$ introduced in [27]:

$$u_k(r) := \begin{cases} \sqrt{\frac{1}{32\pi^2}\log\frac{1}{R_k}} - \frac{r^2}{\sqrt{8\pi^2 R_k \log\frac{1}{R_k}}} + \frac{1}{\sqrt{8\pi^2\log\frac{1}{R_k}}} & \text{if } 0\leq r\leq \sqrt[4]{R_k} \\ \frac{1}{\sqrt{2\pi^2\log\frac{1}{R_k}}}\log\frac{1}{r} & \text{if } \sqrt[4]{R_k} < r \leq 1 \\ \eta_k & \text{if } r>1 \end{cases}$$

where $\{R_k\}_{k\geq 1}\subset\mathbb{R}^+$, $R_k\downarrow 0^+$ and η_k is a smooth function satisfying for some $R>1$

$$\eta_k\big|_{\partial B_1} = \eta_k\big|_{\partial B_R} = 0,$$

$$\frac{\partial\eta_k}{\partial\nu}\Big|_{\partial B_1} = \frac{1}{\sqrt{2\pi^2\log\frac{1}{R_k}}}, \quad \frac{\partial\eta_k}{\partial\nu}\Big|_{\partial B_R} = 0$$

and η_k, $\Delta\eta_k$ are all $\mathcal{O}\big(1/\sqrt{\log 1/R_k}\big)$. Then, by construction,

$$\|u_k\|_2^2 = \mathcal{O}\left(\frac{1}{\log\frac{1}{R_k}}\right), \quad 1\leq \|\Delta u_k\|_2^2 = 1+\mathcal{O}\left(\frac{1}{\log\frac{1}{R_k}}\right).$$

Let $\tilde u_k := \frac{u_k}{\|\Delta u_k\|_2}$ for any $k\geq 1$. From (5.3), it follows that

$$\int_{\mathbb{R}^4}\frac{e^{32\pi^2\tilde u_k^2}-1}{(1+|\tilde u_k|)^p}\,dx \lesssim \frac{\|u_k\|_2^2}{\|\Delta u_k\|_2^2} \lesssim \frac{1}{\log\frac{1}{R_k}},$$

from which we deduce that

$$\lim_{k\to+\infty}\log\frac{1}{R_k}\int_{\mathbb{R}^4}\frac{e^{32\pi^2\tilde u_k^2}-1}{(1+|\tilde u_k|)^p}\,dx < +\infty. \tag{5.4}$$

On the other hand, for any $k\geq 1$, we can estimate

$$\begin{aligned}\int_{\mathbb{R}^4}\frac{e^{32\pi^2\tilde u_k^2}-1}{(1+|\tilde u_k|)^p}\,dx &\geq \int_{B_{\sqrt[4]{R_k}}}\frac{e^{32\pi^2\tilde u_k^2}-1}{(1+|\tilde u_k|)^p}\,dx \gtrsim \int_{B_{\sqrt[4]{R_k}}}\frac{e^{32\pi^2\tilde u_k^2}}{|\tilde u_k|^p}\,dx \gtrsim \\ &\gtrsim e^{\left(\frac{1}{\|\Delta u_k\|_2^2}-1\right)\log\frac{1}{R_k}}\left(\log\frac{1}{R_k}\right)^{-\frac{p}{2}},\end{aligned}$$

since $u_k\geq\sqrt{\frac{1}{32\pi^2}\log\frac{1}{R_k}}$ on $B_{\sqrt[4]{R_k}}$. Therefore

$$\lim_{k\to+\infty}\log\frac{1}{R_k}\int_{\mathbb{R}^4}\frac{e^{32\pi^2\tilde u_k}-1}{(1+|\tilde u_k|)^p}\,dx \gtrsim \lim_{k\to+\infty}\left(\log\frac{1}{R_k}\right)^{1-\frac{p}{2}} = +\infty,$$

which contradicts (5.4).

6. From Adams' inequality with the exact growth condition to Adams' inequality in $W^{2,2}(\mathbb{R}^4)$

In this section we will show that our Adams' inequality with the exact growth condition (Theorem 2.1) implies Adams' inequality in $W^{2,2}(\mathbb{R}^4)$ (Theorem 1.7). More precisely we will prove the following result

Theorem 6.1. *There exists a constant $C>0$ such that for any domain $\Omega\subseteq\mathbb{R}^4$*

$$\sup_{u\in W_0^{2,2}(\Omega),\,\|\Delta u\|_2^2+\|u\|_2^2\leq 1}\int_\Omega (e^{32\pi^2u^2}-1)\,dx \leq C \tag{6.1}$$

and this inequality is sharp.

Since

$$\|u\|^2_{W^{2,\,2}} := \|\Delta u\|_2^2 + \|\nabla u\|_2^2 + \|u\|_2^2 \geq \|\Delta u\|_2^2 + \|u\|_2^2 \quad \forall u \in W_0^{2,\,2}(\Omega),$$

inequality (6.1) implies inequality (1.7). Also, we point out that the sharpness of inequality (6.1) can be proved using the same arguments as in Section 5 or, equivalently, it follows from the sharpness of (1.7).

In order to deduce inequality (6.1) from Theorem 2.1, we will follow the arguments introduced in [21] to show that the Trudinger-Moser inequality with the exact growth condition (Theorem 1.4) implies the Trudinger-Moser inequality in $W^{1,\,2}(\mathbb{R}^2)$ (Theorem 1.3). Firstly, we recall that as a direct consequence of our Adams' inequality with the exact growth condition we have inequality (2.3) and in particular

$$\int_{\mathbb{R}^4} (e^{u^2} - 1)\, dx \leq C\|u\|_2^2 \quad \forall u \in W^{2,\,2}(\mathbb{R}^4) \text{ with } \|\Delta u\|_2^2 \leq 1. \tag{6.2}$$

Using the power series expansion of the exponential function and the Stirling's formula, from (6.2), we can deduce the existence of a constant $C > 0$ such that for any integer $k \geq 1$

$$\|u^2\|_k \leq Ck\|u\|_2^{\frac{2}{k}} \quad \forall u \in W^{2,\,2}(\mathbb{R}^4) \text{ with } \|\Delta u\|_2^2 \leq 1.$$

This Sobolev estimate can be extended to nonintegers $p \geq 1$ applying an interpolation inequality (see e.g. [9], Chapitre IV.2, Remarque 2). Hence, there exists a constant $C_1 > 0$ such that for any $p \geq 1$

$$\|u^2\|_p \leq C_1 p\|u\|_2^{\frac{2}{p}} \quad \forall u \in W^{2,\,2}(\mathbb{R}^4) \text{ with } \|\Delta u\|_2^2 \leq 1. \tag{6.3}$$

Proof of Theorem 6.1. Let $u \in W^{2,\,2}(\mathbb{R}^4) \setminus \{0\}$ be such that $\|\Delta u\|_2^2 + \|u\|_2^2 \leq 1$. Then for some $\theta \in (0,\,1)$, we have

$$\|u\|_2^2 = \theta \quad \text{and} \quad \|\Delta u\|_2^2 \leq 1 - \theta.$$

We distinguish two cases:

- the case $\theta \geq \frac{1}{2}$,
- and the case $\theta < \frac{1}{2}$.

If $\theta \geq \frac{1}{2}$, we define $\tilde{u} := \sqrt{2}u$ so that

$$\|\tilde{u}\|_2^2 \leq 2 \quad \text{and} \quad \|\Delta \tilde{u}\|_2^2 \leq 1,$$

then the desired inequality follows from (2.3) applied to $\tilde{u}$ with $\alpha = 16\pi^2$.

If $\theta < \frac{1}{2}$, then we introduce the set

$$A := \{x \in \mathbb{R}^4 \;\mid\; |u(x)| \geq 1\}$$

and, we split the integral into two parts

$$\int_{\mathbb{R}^4} (e^{32\pi^2 u^2} - 1)\, dx = \int_{\mathbb{R}^4 \setminus A} (e^{32\pi^2 u^2} - 1)\, dx + \int_A (e^{32\pi^2 u^2} - 1)\, dx.$$

The integral on $\mathbb{R}^4 \setminus A$ can be easily estimated using the definition of A and recalling that $|e^x - 1| \leq |x|e^{|x|}$ $\forall x \in \mathbb{R}$. Hence, to complete the proof, we have to estimate the integral on A. To this aim, we can notice that

$$\begin{aligned}\int_A (e^{32\pi^2 u^2} - 1)\, dx &\leq \left(\int_A \frac{[e^{32\pi^2 u^2} - 1]^p}{(1+|u|)^2}\, dx\right)^{\frac{1}{p}} \left(\int_A (1+|u|)^{\frac{2}{p-1}}\, dx\right)^{\frac{p-1}{p}} \leq \\ &\leq 4\left(\int_A \frac{e^{32\pi^2 p u^2} - 1}{(1+|u|)^2}\, dx\right)^{\frac{1}{p}} \left(\int_A |u|^{\frac{2}{p-1}}\, dx\right)^{\frac{p-1}{p}} =: 4 I_u^{\frac{1}{p}} \|u^2\|_{\frac{1}{p-1}}^{\frac{1}{p}},\end{aligned}$$

where $p > 1$ and we simply applied Hölder's inequality. So, choosing $p := \frac{1}{1-\theta}$, we have

$$\int_A (e^{32\pi^2 u^2} - 1)\, dx \le 4 I_u^{1-\theta} \|u^2\|^{1-\theta}_{\frac{1-\theta}{\theta}}\,. \tag{6.4}$$

Using the Sobolev estimate (6.3), we obtain

$$\|u^2\|^{1-\theta}_{\frac{1-\theta}{\theta}} \le C_1 \Big(\frac{1-\theta}{\theta}\Big)^{1-\theta} \|u\|_2^{2\theta} = C_1 \Big(\frac{1-\theta}{\theta}\Big)^{1-\theta} \theta^\theta. \tag{6.5}$$

On the other hand, if we define $\tilde u := \frac{u}{\sqrt{1-\theta}}$, so that

$$\|\tilde u\|_2^2 = \frac{\|u\|_2^2}{1-\theta} = \frac{\theta}{1-\theta} \quad \text{and} \quad \|\Delta \tilde u\|_2^2 = \frac{\|\Delta u\|_2^2}{1-\theta} \le 1,$$

then

$$I_u \le \frac{1}{1-\theta} \int_{\mathbb{R}^4} \frac{e^{32\pi^2 \tilde u^2} - 1}{(1+|\tilde u|)^2}\, dx \le C \frac{\theta}{(1-\theta)^2} \tag{6.6}$$

as a consequence of Theorem 2.1.

Combining (6.5) and (6.6) with (6.4), we deduce that

$$\int_A (e^{32\pi^2 u^2} - 1)\, dx \lesssim \frac{\theta^\theta}{(1-\theta)^{1-\theta}}$$

and the right hand side of this last inequality is uniformly bounded for $0 < \theta < \frac{1}{2}$. □

7. Appendix A: An Alternative Proof of Adams' Inequality

The arguments introduced in Section 4.1 in order to prove an Adams type inequality with the exact growth condition (Theorem 2.1) lead to an alternative proof of Adams' inequality in its original form (Theorem 1.5).

We recall that Adams' approach to the problem is to express u as the Riesz potential of its Laplacian and then apply the following result

Theorem 7.1 ([1], Theorem 2)**.** *There exists a constant $c_0 > 0$ such that for all $f \in L^2(\mathbb{R}^4)$ with support contained in $\Omega \subset \mathbb{R}^4$*

$$\int_\Omega e^{\frac{2}{\pi^2}\left|\frac{I_2 * f(x)}{\|f\|_2}\right|^2} \le c_0$$

where $I_2 * f$ is the Riesz potential of order 2, namely

$$I_2 * f(x) := \int_{\mathbb{R}^4} \frac{f(y)}{|x-y|^2}\, dy.$$

The reason why it is convenient to write u in terms of Riesz potential is that one cannot use directly the idea of decreasing rearrangement $u^\sharp$ to treat the higher order case, because no inequality of the type (1.5) is known to hold for higher order derivatives. To avoid this problem, Adams applied a result of R. O'Neil [31] on nonincreasing rearrangements for convolution integrals, if $h := g * f$ then

$$h^{**}(t) \le t g^{**}(t) f^{**}(t) + \int_t^{+\infty} g^*(s) f^*(s)\, ds. \tag{7.1}$$

Hence, a change of variables reduces the estimate to the following one-dimensional calculus inequality

Lemma 7.2 ([1], Lemma 1)**.** *Let $a : \mathbb{R}\times[0,+\infty) \to \mathbb{R}$ be a nonnegative measurable function such that a.e.*

$$a(s,t) \le 1 \quad \text{when } 0 < s < t\,, \qquad \text{and} \qquad \sup_{t>0}\left(\int_{-\infty}^{0} + \int_{t}^{+\infty} a^2(s,t)\,ds\right)^{\frac{1}{2}} =: b < +\infty.$$

Then there exists a constant $c_0 = c_0(b)$ such that for any $\phi : \mathbb{R} \to \mathbb{R}$ satisfying $\phi \ge 0$ and

$$\int_{-\infty}^{+\infty} \phi^2(s)\,ds \le 1\,,$$

the following inequality holds

$$\int_0^{+\infty} e^{-F(t)}\,dt \le c_0$$

where

$$F(t) := t - \left(\int_{-\infty}^{+\infty} a(s,t)\phi(s)\,ds\right)^2 .$$

We point out that the one-dimensional calculus inequality of Moser, Lemma 4.1, corresponds to the particular case

$$a(s,t) = \begin{cases} 1 & \text{when } 0 < s < t, \\ 0 & \text{otherwise.} \end{cases}$$

Alternatively, it is possible to prove Theorem 1.5 avoiding to consider the Riesz potential's representation formula and using the sharp estimate given by Proposition 3.5 instead of O'Neil's lemma (7.1). Moreover, arguing in this way, it suffices to merely apply the one-dimensional inequality due to J. Moser, namely Lemma 4.1, instead of the refined version due to D. R. Adams, namely Lemma 7.2.

In fact, let $\Omega \subset \mathbb{R}^4$ be a bounded domain and let $u \in W^{2,2} \cap W_0^{1,2}(\Omega)$ be such that $\|\Delta u\|_{L^2(\Omega)} \le 1$. Our aim is to show that

$$\int_\Omega e^{32\pi^2 u^2}\,dx \le C|\Omega| \tag{7.2}$$

for some constant $C > 0$ independent of u.

As a consequence of Proposition 3.1, if we set $f := -\Delta u$ on Ω then

$$\int_0^{|\Omega|} [f^{**}(r)]^2\,dr \le 4\int_0^{|\Omega|} [f^*(r)]^2\,dr = 4\|\Delta u\|^2_{L^2(\Omega)} \le 4. \tag{7.3}$$

Moreover, from Proposition 3.5 it follows that

$$0 \le u^*(r) \le \frac{\sqrt{2}}{16\pi}\int_r^{|\Omega|} \frac{f^{**}(\xi)}{\sqrt{\xi}}\,d\xi \quad \text{for a.e. } 0 < r \le |\Omega|.$$

Hence, we can estimate

$$\int_\Omega e^{32\pi^2 u^2}\,dx = \int_0^{|\Omega|} e^{32\pi^2[u^*]^2}\,dr \le \int_0^{|\Omega|} e^{\left(\frac{1}{2}\int_r^{|\Omega|} \frac{f^{**}(\xi)}{\sqrt{\xi}}\,d\xi\right)^2}\,dr$$

and, making the change of variable $r = |\Omega|e^{-t}$, we get

$$\int_\Omega e^{32\pi^2 u^2}\,dx \le |\Omega|\int_0^{+\infty} e^{\left(\frac{1}{2}\int_{|\Omega|e^{-t}}^{|\Omega|} \frac{f^{**}(\xi)}{\sqrt{\xi}}\,d\xi\right)^2 - t}\,dt. \tag{7.4}$$

Now, let

$$\phi(s) := \frac{\sqrt{|\Omega|}}{2} f^{**}(|\Omega|e^{-s})e^{-\frac{s}{2}} \quad \forall s \ge 0,$$

so that $\phi \geq 0$ and, in view of (7.3),

$$\int_0^{+\infty} \phi^2(s)\,ds = \frac{|\Omega|}{4}\int_0^{+\infty} [f^{**}(|\Omega|e^{-s})]^2 e^{-s}\,ds = \frac{1}{4}\int_0^{|\Omega|} [f^{**}(r)]^2\,dr \leq 1.$$

Defining

$$\begin{aligned} F(t) \; &:= \; t - \left(\int_0^t \phi(s)\,ds\right)^2 = t - \left(\frac{\sqrt{|\Omega|}}{2}\int_0^t f^{**}(|\Omega|e^{-s})e^{-\frac{s}{2}}\,ds\right)^2 = \\ &= \; t - \left(\frac{1}{2}\int_{|\Omega|e^{-t}}^{|\Omega|} \frac{f^{**}(\xi)}{\sqrt{\xi}}\,d\xi\right)^2 \end{aligned}$$

and recalling (7.4), the proof of (7.2) reduces to showing that

$$\int_0^{+\infty} e^{-F(t)}\,dt \leq c_0$$

which is nothing but Lemma 4.1.

Remark 6. These arguments simplify Adams' proof and at the same time allow to obtain a stronger inequality, i.e. an Adams-type inequality for the Sobolev space

$$W^{2,2} \cap W_0^{1,2}(\Omega) \supsetneq W_0^{2,2}(\Omega).$$

8. Appendix B: An Alternative Proof of the Trudinger-Moser Inequality with the Exact Growth Condition

In [21], the authors proved the following exponential version of the radial Sobolev inequality.

Theorem 8.1 ([21], Theorem 3.1)**.** *There exists a constant $C > 0$ such that for any non-negative radially decreasing function $u \in W^{1,2}(\mathbb{R}^2)$ satisfying*

$$u(R) > 1 \quad \text{and} \quad \|\nabla u\|^2_{L^2(\mathbb{R}^2\setminus B_R)} \leq K \quad \text{for some } R,\, K > 0,$$

we have

$$\frac{e^{\frac{4\pi}{K}u^2(R)}}{u^2(R)}R^2 \leq C\frac{\|u\|^2_{L^2(\mathbb{R}^2\setminus B_R)}}{K^2}\,.$$

It is interesting to notice that this optimal descending growth condition is strong enough to lead us to reduce the proof of the Trudinger-Moser inequality with the exact growth condition (Theorem 1.4) to a simple application of the Trudinger-Moser inequality in its original form (Theorem 1.1).

In order to prove that there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}^2} \frac{e^{4\pi u^2}-1}{(1+|u|)^2}\,dx \leq C\|u\|_2^2 \quad \forall u \in W^{1,2}(\mathbb{R}^2) \text{ with } \|\nabla u\|_2 \leq 1\,, \tag{8.1}$$

the authors in [21] introduced an enlightening method of proof that allows to reduce (8.1) to an estimate of an integral on a ball. We point out that in Section 4 we followed exactly this method in order to reduce the proof of Theorem 2.1 to a similar estimate, but for the convenience of the reader we recall the sketch of the proof of (8.1) in [21].

Exploiting the properties of rearrangements of functions, it suffices to show that (8.1) holds for any non-negative and radially decreasing function $u \in W^{1,2}(\mathbb{R}^2)$ with $\|\nabla u\|_2 \leq 1$. So let u be such a function and let $R_1 = R_1(u) > 0$ be such that

$$\int_{B_{R_1}} |\nabla u|^2\,dx = 2\pi\int_0^{R_1} u_r^2 r\,dr \leq 1-\varepsilon_0 \quad \text{and} \quad \int_{\mathbb{R}^2\setminus B_{R_1}} |\nabla u|^2\,dx = 2\pi\int_{R_1}^{+\infty} u_r^2 r\,dr \leq \varepsilon_0,$$

where $\varepsilon_0 \in (0,\,1)$ is arbitrarily fixed independently of u.

Applying Hölder's inequality, we have the following estimate

$$u(r_1) - u(r_2) \le \int_{r_1}^{r_2} -u_r,\, dr \le \Big(\int_{r_1}^{r_2} u_r r\, dr\Big)^{\frac{1}{2}} \Big(\log \frac{r_2}{r_2}\Big)^{\frac{1}{2}} \qquad \text{for a.e. } 0 < r_1 \le r_2$$

and, in view of the choice of R_1, we get:

$$u(r_1) - u(r_2) \le \Big(\frac{1-\varepsilon_0}{2\pi}\Big)^{\frac{1}{2}} \Big(\log \frac{r_2}{r_2}\Big)^{\frac{1}{2}} \qquad \text{for a.e. } 0 < r_1 \le r_2 \le R_1,$$

$$u(r_1) - u(r_2) \le \Big(\frac{\varepsilon_0}{2\pi}\Big)^{\frac{1}{2}} \Big(\log \frac{r_2}{r_2}\Big)^{\frac{1}{2}} \qquad \text{for a.e. } R_1 \le r_1 \le r_2.$$

Since we are in the same framework as in Section 4, it is easy to see that the difficult part of the proof is to show that

$$\int_{B_{R_1}} \frac{e^{4\pi u^2}-1}{(1+u)^2}\, dx \le C\|u\|_2^2 \tag{8.2}$$

in the case when $u(R_1) > 1$.

To obtain (8.2), the authors in [21] argue as follows. Applying the monotone convergence theorem, one can see that

$$\int_{B_{R_1}} \frac{e^{4\pi u^2}-1}{(1+u)^2}\, dx \le C \lim_{N\to+\infty} \sum_{j=0}^{N} \frac{e^{\frac{4\pi}{\varepsilon_0}u^2(r_j)}}{u^2(r_j)} r_j^2,$$

where $C > 0$ and $\{r_j\}_{j\ge0}$ is a monotone decreasing sequence of positive real numbers such that $r_j \downarrow 0$ as $j \to +\infty$ and $r_0 := R_1$, more precisely $r_j := R_1 e^{-2\pi j}$ for any $j \ge 0$. Thus, in view of the optimal descending growth condition (Theorem 8.1), the proof of (8.2) is complete if one shows the existence of a constant $C > 0$ such that for any $N \ge 1$

$$\sum_{j=0}^{N} \frac{e^{\frac{4\pi}{\varepsilon_0}u^2(r_j)}}{u^2(r_j)} r_j^2 \le C \frac{e^{\frac{4\pi}{\varepsilon_0}}u^2(R_1)}{u^2(R_1)} R_1^2. \tag{8.3}$$

But to succeed in the proof of this last inequality, the authors in [21] need to apply a really delicate procedure. It is interesting to notice that, to obtain (8.2), in fact we can avoid the passage through (8.3) merely using the optimal descending growth condition (Theorem 8.1) together with the Trudinger-Moser inequality in its original form (Theorem 1.1).

To do this, we start following the argument introduced by B. Ruf in [33] and we define for $0 \le r \le R_1$

$$v(r) := u(r) - u(R_1),$$

so that $v \in W_0^{1,2}(B_{R_1})$ and $\|\nabla v\|_{L^2(B_{R_1})} = \|\nabla u\|_{L^2(B_{R_1})} \le 1 - \varepsilon_0$. In this way, for arbitrarily fixed $\varepsilon > 0$, we can estimate for $0 \le r \le R_1$

$$u^2(r) = [v(r) + u(R_1)]^2 \le (1+\varepsilon)v^2(r) + \Big(1 + \frac{1}{\varepsilon}\Big)u^2(R_1)$$

and thus

$$\int_{B_{R_1}} \frac{e^{4\pi u^2}-1}{(1+u)^2}\, dx \le \frac{e^{4\pi\left(1+\frac{1}{\varepsilon}\right)u^2(R_1)}}{u^2(R_1)} \int_{B_{R_1}} e^{4\pi w^2}\, dx, \tag{8.4}$$

where $w := \sqrt{1+\varepsilon}\,v$ on B_{R_1}.

Firstly, we can notice that in view of the choice of R_1 and as a consequence of the optimal descending growth condition (Theorem 8.1) we have

$$\frac{e^{4\pi\left(1+\frac{1}{\varepsilon}\right)u^2(R_1)}}{u^2(R_1)}R_1^2 \leq C\frac{\|u\|^2_{L^2(\mathbb{R}^2\setminus B_{R_1})}}{\varepsilon_0^2} \tag{8.5}$$

provided that $\varepsilon \geq \frac{\varepsilon_0}{1+\varepsilon_0}$. On the other hand, by construction $w \in W_0^{1,2}(B_{R_1})$ and

$$\|\nabla w\|^2_{L^2(B_{R_1})} = (1+\varepsilon)\|\nabla v\|^2_{L^2(B_{R_1})} \leq (1+\varepsilon)(1-\varepsilon_0) \leq 1$$

provided that $0 < \varepsilon \leq \frac{\varepsilon_0}{1+\varepsilon_0}$, hence with this choice of ε we have

$$\int_{B_{R_1}} e^{4\pi w^2}\,dx \leq CR_1^2 \tag{8.6}$$

as a consequence of the Trudinger-Moser inequality (Theorem 1.1).

In conclusion, choosing $\varepsilon = \frac{\varepsilon_0}{1+\varepsilon_0}$ we have that both the inequalities (8.5) and (8.6) hold, and this allows us to deduce the desired estimate (8.2) from (8.4).

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Courant Institute, 251 Mercer Street, New York, NY 10012-1185, USA,
E-mail address: masmoudi@courant.nyu.edu

Università degli Studi di Milano, via Cesare Saldini 50, 20133 Milano, Italy,
E-mail address: federica.sani@unimi.it