REMARKS ON THE ACOUSTIC LIMIT FOR THE BOLTZMANN EQUATION

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Abstract. We improve in three ways the results of [6] that establish the acoustic limit for DiPerna-Lions solutions of Boltzmann equation. First, we enlarge the class of collision kernels treated to that found in [12], thereby treating all classical collision kernels to which the DiPerna-Lions theory applies. Second, we improve the scaling of the kinetic density fluctuations with Knudsen number from $O(\epsilon^m)$ for some $m > \frac{1}{2}$ to $O(\epsilon^{\frac{1}{2}})$. Third, we extend the results from periodic domains to bounded domains with a Maxwell reflection boundary condition, deriving the impermeable boundary condition for the acoustic system.

1. Introduction

In this note we establish the acoustic limit starting from DiPerna-Lions renormalized solutions of the Boltzmann equation considered over a smooth bounded spatial domain $\Omega \subset \mathbb{R}^{\mathrm{D}}$. The acoustic system is the linearization about the homogeneous state of the compressible Euler system. After a suitable choice of units and Galilean frame, it governs the fluctuations in mass density $\rho(x,t)$, bulk velocity $u(x,t)$, and temperature $\theta(x,t)$ over $\Omega \times \mathbb{R}_+$ by the initial-value problem

$$
\begin{aligned}
\partial_t \rho + \nabla_x \cdot u &= 0\,, & \rho(x,0) &= \rho^{\mathrm{in}}(x)\,,\\
\partial_t u + \nabla_x(\rho + \theta) &= 0\,, & u(x,0) &= u^{\mathrm{in}}(x)\,,\\
\tfrac{\mathrm{D}}{2}\partial_t \theta + \nabla_x \cdot u &= 0\,, & \theta(x,0) &= \theta^{\mathrm{in}}(x)\,,
\end{aligned}
\tag{1.1}
$$

subject to the impermeable boundary condition

$$
u \cdot \mathrm{n} = 0\,, \quad \text{on } \partial\Omega\,,
\tag{1.2}
$$

where $\mathrm{n}(x)$ is the unit outward normal at $x \in \partial\Omega$. This is one of the simplest fluid dynamical systems imaginable, being essentially the wave equation.

The acoustic system (1.1, 1.2) can be formally derived from the Boltzmann equation for kinetic densities $F(v,x,t)$ over $\mathbb{R}^{\mathrm{D}} \times \Omega \times \mathbb{R}_+$ that are close to the global Maxwellian

$$
M(v) = \frac{1}{(2\pi)^{\frac{\mathrm{D}}{2}}} \exp\big(-\tfrac{1}{2}|v|^2\big)\,.
\tag{1.3}
$$

We consider families of kinetic densities in the form $F_\epsilon(v,x,t) = M(v) G_\epsilon(v,x,t)$ where the relative kinetic densities $G_\epsilon(v,x,t)$ over $\mathbb{R}^{\mathrm{D}} \times \Omega \times \mathbb{R}_+$ are governed by the rescaled Boltzmann initial-value problem

$$
\partial_t G_\epsilon + v \cdot \nabla_x G_\epsilon = \frac{1}{\epsilon} \mathcal{Q}(G_\epsilon, G_\epsilon)\,, \qquad G_\epsilon(v,x,0) = G_\epsilon^{\mathrm{in}}(v,x)\,.
\tag{1.4}
$$

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Here the Knudsen number $\epsilon > 0$ is the ratio of the mean free path to a macroscopic length scale and the collision operator $\mathcal{Q}(G_\epsilon, G_\epsilon)$ is given by

$$\mathcal{Q}(G_\epsilon, G_\epsilon) = \iint_{\mathbb{S}^{\mathrm{D}-1}\times\mathbb{R}^{\mathrm{D}}} \left(G'_{\epsilon 1}G'_\epsilon - G_{\epsilon 1}G_\epsilon\right) b(\omega, v_1 - v)\,\mathrm{d}\omega\, M_1 \mathrm{d}v_1\,, \tag{1.5}$$

where the collision kernel $b(\omega, v_1 - v)$ is positive almost everywhere while $G_{\epsilon 1}$, G'_ϵ, and $G'_{\epsilon 1}$ denote $G_\epsilon(\,\cdot\,, x, t)$ evaluated at v_1, $v' = v + \omega\omega\cdot(v_1 - v)$, and $v'_1 = v - \omega\omega\cdot(v_1 - v)$ respectively.

We impose a Maxwell reflection boundary condition on $\partial\Omega$ of the form

$$\mathbf{1}_{\Sigma_+} G_\epsilon \circ \mathrm{R} = (1-\alpha)\,\mathbf{1}_{\Sigma_+} G_\epsilon + \alpha\,\mathbf{1}_{\Sigma_+}\sqrt{2\pi}\,\big\langle \mathbf{1}_{\Sigma_+}\, v\cdot n\, G_\epsilon\big\rangle\,. \tag{1.6}$$

Here $\alpha \in [0,1]$ is the Maxwell accommodation coefficient, $(G_\epsilon \circ \mathrm{R})(v, x, t) = G_\epsilon(\mathrm{R}(x)v, x, t)$ where $\mathrm{R}(x) = I - 2\mathrm{n}(x)\mathrm{n}(x)^T$ is the specular reflection matrix at a point $x \in \partial\Omega$, $\mathbf{1}_{\Sigma_+}$ is the indicator function of the so-called outgoing boundary set

$$\Sigma_+ = \left\{(v, x) \in \mathbb{R}^{\mathrm{D}} \times \partial\Omega \,:\, v\cdot\mathrm{n}(x) > 0\right\}\,, \tag{1.7}$$

and $\langle\,\cdot\,\rangle$ denotes the average

$$\langle \xi \rangle = \int_{\mathbb{R}^{\mathrm{D}}} \xi(v)\, M(v)\,\mathrm{d}v\,. \tag{1.8}$$

Because $\sqrt{2\pi}\,\big\langle \mathbf{1}_{\Sigma_+}\, v\cdot\mathrm{n}\big\rangle = 1$, it is easy to see from (1.6) that on $\partial\Omega$ one has

$$\begin{aligned} \langle v\cdot\mathrm{n}\, G_\epsilon\rangle &= \big\langle \mathbf{1}_{\Sigma_+}\, v\cdot\mathrm{n}\,\big(G_\epsilon - G_\epsilon\circ\mathrm{R}\big)\big\rangle \\ &= \alpha\left\langle \mathbf{1}_{\Sigma_+}\, v\cdot\mathrm{n}\left(G_\epsilon - \sqrt{2\pi}\,\big\langle \mathbf{1}_{\Sigma_+}\, v\cdot\mathrm{n}\, G_\epsilon\big\rangle\right)\right\rangle = 0\,. \end{aligned} \tag{1.9}$$

Fluid regimes are those in which the Knudsen number ϵ is small. The acoustic system (1.1, 1.2) can be derived from (1.4, 1.6) for families of solutions $G_\epsilon(v, x, t)$ that are scaled so that

$$G_\epsilon = 1 + \delta_\epsilon g_\epsilon\,, \qquad G^{\mathrm{in}}_\epsilon = 1 + \delta_\epsilon g^{\mathrm{in}}_\epsilon\,, \tag{1.10}$$

where

$$\delta_\epsilon \to 0 \qquad \text{as} \qquad \epsilon \to 0\,, \tag{1.11}$$

and the fluctuations g_ϵ and g^{in}_ϵ converge in the sense of distributions to $g \in L^\infty(\mathrm{d}t; L^2(M\mathrm{d}v\,\mathrm{d}x))$ and $g^{\mathrm{in}} \in L^2(M\mathrm{d}v\,\mathrm{d}x)$ respectively as $\epsilon \to 0$. One finds that g has the infinitesimal Maxwellian form

$$g = \rho + v\cdot u + \left(\tfrac{1}{2}|v|^2 - \tfrac{\mathrm{D}}{2}\right)\theta\,, \tag{1.12}$$

where $(\rho, u, \theta) \in L^\infty(\mathrm{d}t; L^2(\mathrm{d}x; \mathbb{R}\times\mathbb{R}^{\mathrm{D}}\times\mathbb{R}))$ solve (1.1, 1.2) with initial data given by

$$\rho^{\mathrm{in}} = \langle g^{\mathrm{in}}\rangle\,, \qquad u^{\mathrm{in}} = \langle v\, g^{\mathrm{in}}\rangle\,, \qquad \theta^{\mathrm{in}} = \big\langle \big(\tfrac{1}{\mathrm{D}}|v|^2 - 1\big)\, g^{\mathrm{in}}\big\rangle\,. \tag{1.13}$$

The formal derivation leading to (1.1) closely follows that in [6], so its details will not be given here. The boundary condition (1.2) is obtained by noticing that (1.9) implies $\langle v\cdot\mathrm{n}\, g_\epsilon\rangle = 0$, then passing to the limit in this to get $\langle v\cdot\mathrm{n}\, g\rangle = 0$, and finally using (1.12) to obtain (1.2).

The program initiated in [1, 2, 3] seeks to justify fluid dynamical limits for Boltzmann equations in the setting of DiPerna-Lions renormalized solutions [5], which are the only temporally global, large data solutions available. The main obstruction to carrying out this program is that DiPerna-Lions solutions are not known to satisfy many properties that one formally expects for solutions of the Boltzmann equation. For example, they are not known to satisfy the formally expected local conservations laws of momentum and energy. Moreover, their regularity is poor. The justification of fluid dynamical limits in this setting is therefore not easy.

The acoustic limit was first established in this kind of setting in [4] over a periodic domain. There idea introduced there was to pass to the limit in approximate local conservations laws which are satified by DiPerna-Lions solutions. One then shows that the so-called conservation defects vanish as the Knudsen number ϵ vanishes, thereby establishing the local conservation laws in the limit. This was done in [4] using only relative entropy estimates, which restricted the result to collision kernels that are bounded and to fluctuations scaled so that

$$\delta_\epsilon \to 0 \quad \text{and} \quad \frac{\delta_\epsilon}{\epsilon} |\log(\delta_\epsilon)| \to 0 \quad \text{as} \quad \epsilon \to 0\,, \tag{1.14}$$

which is far from the formally expected optimal scaling (1.11).

In [6] the local conservation defects were removed using new dissipation rate estimates. This allowed the treatment of collision kernels that for some $C_b < \infty$ and $\beta \in [0,1)$ satisfied

$$\int_{\mathbb{S}^{D-1}} b(\omega, v_1 - v)\,\mathrm{d}\omega \le C_b \big(1 + |v_1 - v|^2\big)^\beta\,, \tag{1.15}$$

and of fluctuations scaled so that

$$\delta_\epsilon \to 0 \quad \text{and} \quad \frac{\delta_\epsilon}{\epsilon^{1/2}} |\log(\delta_\epsilon)|^{\beta/2} \to 0 \quad \text{as} \quad \epsilon \to 0\,. \tag{1.16}$$

The above class of collision kernels includes all classical kernels that are derived from Maxwell or hard potentials and that satisfy a weak small deflection cutoff. The scaling given by (1.16) is much less restrictive than that given by (1.16), but is far from the formally expected optimal scaling (1.11). Finally, only periodic domains are treated in [6].

Here we improve the result of [6] in three ways. First, we apply estimates from [12] to treat a broader class of collision kernels that includes those derived from soft potentials. Second, we improve the scaling of the fluctuations to $\delta_\epsilon = O(\epsilon^{1/2})$. Finally, we treat domains with a boundary and use new estimates to derive the boundary condition (1.2) in the limit.

We use the L^1 velocity averaging theory of Golse and Saint-Raymond [7] through the nonlinear compactness estimate of [12] to improve the scaling of the fluctuations to $\delta_\epsilon = O(\epsilon^{1/2})$. Without it we would only be able to improve the scaling to $\delta_\epsilon = o(\epsilon^{1/2})$. This is the first time the L^1 averaging theory has played any role in an acoustic limit theorem, albeit for a modest improvement in the scaling of our result. We remark that averaging theory plays no role in establishing the Stokes limit with its formally expected optimal scaling of $\delta_\epsilon = o(\epsilon)$ [12].

We treat domains with boundary in the setting of Mischler [15], who extended the DiPerna-Lions theory to bounded domains with a Maxwell reflection boundary condition. He showed that these boundary conditions are satisfied in a *renormalized* sense. This means we cannot deduce that $\langle v \cdot \mathrm{n}\, g_\epsilon \rangle \to 0$ as $\epsilon \to 0$ to derive the boundary condition (1.2), as we did in our formal argument. Masmoudi and Saint-Raymond [14] developed estimates to obtain boundary conditions in the Stokes limit. However neither these estimates nor their recent extension to the Navier-Stokes limit [11] can handle the acoustic limit. Rather, we develop new boundary *a priori* estimates to obtain a weak form of the boundary condition (1.2) in this limit. In doing so, we treat a broader class of collision kernels than was done in [14].

Finally, we remark that fully establishing the acoustic limit with its formally expected optimal scaling of the fluctuation size (1.11) is still open. This gap must be bridged before one can hope to fully establish the compressible Euler limit starting from DiPerna-Lions solutions to the Boltzmann equation. In contrast, optimal scaling can be obtained within the framework of classical solutions by using the nonlinear energy method developed by Guo. This has been done recently by the first author of this paper with Guo and Jang [9, 10].

Our paper is organized as follows. Section 2 gives its framework. Section 3 states and proves our main result modulo two steps. Section 4 removes the conservation defects. Section 5 establishes the limit boundary mass-flux term.

2. Framework

For the most part we will use the notation of [12]. Here we present only what is needed to state our theorem. For more complete introductions to the Boltzmann equation, see [6, 12].

Let $\Omega \subset \mathbb{R}^{\mathrm{D}}$ be a bounded domain with smooth boundary $\partial\Omega$. Let $\mathrm{n}(x)$ denote the outward unit normal vector at $x \in \partial\Omega$ and $\mathrm{d}\sigma_x$ denote the Lebesgue measure on $\partial\Omega$. The phase space domain associated with Ω is $\mathcal{O} = \mathbb{R}^{\mathrm{D}} \times \Omega$, which has boundary $\partial\mathcal{O} = \mathbb{R}^{\mathrm{D}} \times \partial\Omega$. Let Σ_+ and Σ_- denote the outgoing and incoming subsets of $\partial\mathcal{O}$ defined by

$$\Sigma_\pm = \{(v, x) \in \partial\mathcal{O} \,:\, \pm v \cdot \mathrm{n}(x) > 0\} \,.$$

The global Maxwellian $M(v)$ given by (1.3) corresponds to the spatially homogeneous fluid state with density and temperature equal to 1 and bulk velocity equal to 0. The boundary condition (1.6) corresponds to a wall temperature of 1, so that $M(v)$ is the unique equilibrium of the fluid. Associated with the initial data G_ϵ^{in} we have the normalization

$$\int_\Omega \langle G_\epsilon^{\mathrm{in}} \rangle \,\mathrm{d}x = 1 \,. \tag{2.1}$$

2.1. Assumptions on the Collision Kernel.

The kernel $b(\omega, v_1 - v)$ associated with the collision operator (1.5) is positive almost everywhere. The Galilean invariance of the collisional physics implies that b has the classical form

$$b(\omega, v_1 - v) = |v_1 - v| \, \Sigma(|\omega \cdot n|, |v_1 - v|) \,, \tag{2.2}$$

where $n = (v_1 - v)/|v_1 - v|$ and Σ is the specific differential cross-section. We make five additional technical assumptions regarding b that are adopted from [12].

Our *first technical assumption* is that the collision kernel b satisfies the requirements of the DiPerna-Lions theory. That theory requires that b be locally integrable with respect to $\mathrm{d}\omega \, M_1 \mathrm{d}v_1 \, M \mathrm{d}v$, and that it moreover satisfies

$$\lim_{|v|\to\infty} \frac{1}{1 + |v|^2} \int_K \overline{b}(v_1 - v) \,\mathrm{d}v_1 = 0 \quad \text{for every compact } K \subset \mathbb{R}^{\mathrm{D}} \,, \tag{2.3}$$

where $\overline{b}$ is defined by

$$\overline{b}(v_1 - v) \equiv \int_{\mathbb{S}^{\mathrm{D}-1}} b(\omega, v_1 - v) \,\mathrm{d}\omega \,. \tag{2.4}$$

Galilean symmetry (2.2) implies that $\overline{b}$ is a function of $|v_1 - v|$ only.

Our *second technical assumption* regarding b is that the attenuation coefficient a, which is defined by

$$a(v) \equiv \int_{\mathbb{R}^{\mathrm{D}}} \overline{b}(v_1 - v) \, M_1 \mathrm{d}v_1 = \iint_{\mathbb{S}^{\mathrm{D}-1} \times \mathbb{R}^{\mathrm{D}}} b(\omega, v_1 - v) \,\mathrm{d}\omega \, M_1 \mathrm{d}v_1 \,, \tag{2.5}$$

is bounded below as

$$C_a \big(1 + |v|^2\big)^{\beta_a} \le a(v) \quad \text{for some constants } C_a > 0 \text{ and } \beta_a \in \mathbb{R} \,. \tag{2.6}$$

Galilean symmetry (2.2) implies that a is a function of $|v|$ only.

Our *third technical assumption* regarding b is that there exists $s \in (1, \infty]$ and $C_b \in (0, \infty)$ such that

$$\left(\int_{\mathbb{R}^{\mathrm{D}}} \left| \frac{\overline{b}(v_1 - v)}{a(v_1)\, a(v)} \right|^s a(v_1)\, M_1 \mathrm{d}v_1 \right)^{\frac{1}{s}} \leq C_b \,. \tag{2.7}$$

Because this bound is uniform in v, we may take C_b to be the supremum over v of the left-hand side of (2.7).

Our *fourth technical assumption* regarding b is that the operator

$$\mathcal{K}^+ : L^2(aM\mathrm{d}v) \to L^2(aM\mathrm{d}v) \quad \text{is compact}\,, \tag{2.8}$$

where

$$\mathcal{K}^+ \tilde{g} = \frac{1}{2a} \iint_{\mathbb{S}^{\mathrm{D}-1} \times \mathbb{R}^{\mathrm{D}}} \big(\tilde{g}' + \tilde{g}_1' \big)\, b(\omega, v_1 - v)\, \mathrm{d}\omega\, M_1 \mathrm{d}v_1 \,.$$

We remark that $\mathcal{K}^+ : L^2(aM\mathrm{d}v) \to L^2(aM\mathrm{d}v)$ is always bounded [12, 13] with $\|\mathcal{K}^+\| \leq 1$.

Our *fifth technical assumption* regarding b is that for every $\delta > 0$ there exists C_δ such that $\overline{b}$ satisfies

$$\frac{\overline{b}(v_1 - v)}{1 + \delta \, \dfrac{\overline{b}(v_1 - v)}{1 + |v_1 - v|^2}} \leq C_\delta \big(1 + a(v_1)\big)\big(1 + a(v)\big) \quad \text{for every } v_1, v \in \mathbb{R}^{\mathrm{D}} \,. \tag{2.9}$$

The above assumptions are satisfied by all the classical collision kernels with a weak small deflection cutoff that derive from a repulsive intermolecular potential of the form c/r^k with $k > 2\frac{\mathrm{D}-1}{\mathrm{D}+1}$. This includes all the classical collision kernels to which the DiPerna-Lions theory applies [12, 13]. Kernels that satisfy (1.15) clearly satisfy (2.3). If they moreover satisfy (2.6) with $\beta_a = \beta$ then they also satisfy (2.7) and (2.9).

Because the kernel b satisfies (2.3), it can be normalized so that

$$\iint_{\mathbb{S}^{\mathrm{D}-1} \times \mathbb{R}^{\mathrm{D}} \times \mathbb{R}^{\mathrm{D}}} b(\omega, v_1 - v)\, \mathrm{d}\omega\, M_1\, \mathrm{d}v_1\, M\, \mathrm{d}v = 1 \,.$$

Because $\mathrm{d}\mu = b(\omega, v_1 - v)\, \mathrm{d}\omega\, M_1 \mathrm{d}v_1\, M \mathrm{d}v$ is a positive unit measure on $\mathbb{S}^{\mathrm{D}-1} \times \mathbb{R}^{\mathrm{D}} \times \mathbb{R}^{\mathrm{D}}$, we denote by $\langle\!\langle \Xi \rangle\!\rangle$ the average over this measure of any integrable function $\Xi = \Xi(\omega, v_1, v)$

$$\langle\!\langle \Xi \rangle\!\rangle = \iiint_{\mathbb{S}^{\mathrm{D}-1} \times \mathbb{R}^{\mathrm{D}} \times \mathbb{R}^{\mathrm{D}}} \Xi(\omega, v_1, v)\, \mathrm{d}\mu \,. \tag{2.10}$$

2.2. **DiPerna-Lions-Mischler Theory.** As in [4, 6, 12], we will work in the framework of DiPerna-Lions solutions to the scaled Boltzmann equation on the phase space $\mathcal{O} = \mathbb{R}^{\mathrm{D}} \times \Omega$

$$\begin{aligned} \partial_t G_\epsilon + v \cdot \nabla_x G_\epsilon &= \frac{1}{\epsilon} \mathcal{Q}(G_\epsilon, G_\epsilon) \qquad &&\text{on} \quad \mathcal{O} \times \mathbb{R}_+ \,, \\ G_\epsilon(v, x, 0) &= G_\epsilon^{\mathrm{in}}(v, x) &&\text{on} \quad \mathcal{O} \,, \end{aligned} \tag{2.11}$$

with the Maxwell reflection boundary condition (1.6) which can be expressed as

$$\gamma_- G_\epsilon = (1 - \alpha) L(\gamma_+ G_\epsilon) + \alpha \langle \gamma_+ G_\epsilon \rangle_{\partial\Omega} \quad \text{on} \quad \Sigma_- \times \mathbb{R}_+ \,, \tag{2.12}$$

where $\gamma_\pm G_\epsilon$ denote the traces of G_ϵ on the outgoing and incoming sets $\Sigma_\pm$. Here the local reflection operator L is defined to act on any $|v \cdot \mathrm{n}| M \mathrm{d}v\, \mathrm{d}\sigma_x$-measurable function ϕ over $\partial\mathcal{O}$ by

$$L\phi(v, x) = \phi(\mathrm{R}(x)v, x) \quad \text{for almost every } (v, x) \in \partial\mathcal{O} \,,$$

where $\mathrm{R}(x)v = v - 2v\cdot\mathrm{n}(x)\mathrm{n}(x)$ is the specular reflection of v, while the diffuse reflection operator is defined as

$$\langle\phi\rangle_{\partial\Omega} = \sqrt{2\pi}\int_{v\cdot\mathrm{n}(x)>0}\phi(v,x)\,v\cdot\mathrm{n}(x)\,M\mathrm{d}v\,.$$

DiPerna-Lions theory requires that both the equation and boundary conditions in (2.11) should be understood in the renormalized sense, see (3.7) and (3.11). These solutions were initially constructed by DiPerna and Lions [5] over the whole space $\mathbb{R}^{\mathrm{D}}$ for any initial data satisfying natural physical bounds. For bounded domain case, Mischler [15] recently developed a theory to treat the Maxwell reflection boundary condition (2.12).

The DiPerna-Lions theory does not yield solutions that are known to solve the Boltzmann equation in the usual sense of weak solutions. Rather, it gives the existence of a global weak solution to a class of formally equivalent initial value problems that are obtained by multiplying (2.11) by $\Gamma'(G_\epsilon)$, where Γ' is the derivative of an admissible function Γ:

$$(\partial_t + v\cdot\nabla_x)\Gamma(G_\epsilon) = \frac{1}{\epsilon}\Gamma'(G_\epsilon)\mathcal{Q}(G_\epsilon,G_\epsilon) \quad \text{on} \quad \mathcal{O}\times\mathbb{R}_+\,. \tag{2.13}$$

Here a function $\Gamma:[0,\infty)\to\mathbb{R}$ is called admissible if it is continuously differentiable and for some $C_\Gamma<\infty$ its derivative satisfies

$$|\Gamma'(Z)| \leq \frac{C_\Gamma}{\sqrt{1+Z}} \quad \text{for every } Z\in[0,\infty)\,.$$

The solutions are nonnegative and lie in $C([0,\infty);\text{w-}L^1(M\mathrm{d}v\,\mathrm{d}x))$, where the prefix "w-" on a space indicates that the space is endowed with its weak topology.

Mischler [15] extended DiPerna-Lions theory to domains with a boundary on which the Maxwell reflection boundary condition (2.12) is imposed. This required the proof of a so-called trace theorem that shows that the restriction of G_ϵ to $\partial\mathcal{O}\times\mathbb{R}_+$, denoted γG_ϵ, makes sense. In particular, Mischler showed that γG_ϵ lies in the set of all $|v\cdot\mathrm{n}|M\mathrm{d}v\,\mathrm{d}\sigma_x\,\mathrm{d}t$-measurable functions over $\partial\mathcal{O}\times\mathbb{R}_+$ that are finite almost everywhere, which we denote $L^0(|v\cdot\mathrm{n}|M\mathrm{d}v\,\mathrm{d}\sigma_x\,\mathrm{d}t)$. He then defines $\gamma_\pm G_\epsilon = \mathbf{1}_{\Sigma_\pm}\gamma G_\epsilon$. He proves the following.

Theorem 2.1. (DiPerna-Lions-Mischler Renormalized Solutions [5, 15]) *Let b be a collision kernel that satisfies the assumptions in Section 2.1. Fix $\epsilon>0$. Let G_ϵ^{in} be any initial data in the entropy class*

$$E(M\mathrm{d}v\,\mathrm{d}x) = \left\{G_\epsilon^{\mathrm{in}}\geq 0 \,:\, H(G_\epsilon^{\mathrm{in}})<\infty\right\}, \tag{2.14}$$

where the relative entropy functional is given by

$$H(G) = \int_\Omega\langle\eta(G)\rangle\,\mathrm{d}x \quad \textit{with} \quad \eta(G) = G\log(G) - G + 1\,.$$

Then there exists a $G_\epsilon\geq 0$ in $C([0,\infty);w\text{-}L^1(M\mathrm{d}v\,\mathrm{d}x))$ with $\gamma G_\epsilon\geq 0$ in $L^0(|v\cdot\mathrm{n}|M\mathrm{d}v\,\mathrm{d}\sigma_x\,\mathrm{d}t)$ such that:

- *G_ϵ satisfies the global entropy inequality*

$$H(G_\epsilon(t)) + \int_0^t\left[\frac{1}{\epsilon}R(G_\epsilon(s)) + \frac{\alpha}{\sqrt{2\pi}}\mathcal{E}(\gamma_+G_\epsilon(s))\right]\mathrm{d}s \leq H(G_\epsilon^{\mathrm{in}}) \quad \textit{for every } t>0\,, \tag{2.15}$$

where the entropy dissipation rate functional is given by

$$R(G) = \frac{1}{4}\int_\Omega\left\langle\!\!\left\langle \log\left(\frac{G_1'G'}{G_1G}\right)\left(G_1'G' - G_1G\right)\right\rangle\!\!\right\rangle\mathrm{d}x\,, \tag{2.16}$$

and the so-called Darrozès-Guiraud information is given by

$$\mathcal{E}(\gamma_+ G) = \int_{\partial\Omega} \left[\big\langle \eta(\gamma_+ G) \big\rangle_{\partial\Omega} - \eta\big(\langle \gamma_+ G \rangle_{\partial\Omega} \big) \right] \, \mathrm{d}\sigma_x \,; \tag{2.17}$$

- G_ϵ *satisfies*

$$\begin{aligned} &\int_\Omega \langle \Gamma(G_\epsilon(t_2))\, Y \rangle \, \mathrm{d}x - \int_\Omega \langle \Gamma(G_\epsilon(t_1))\, Y \rangle \, \mathrm{d}x + \int_{t_1}^{t_2} \int_{\partial\Omega} \langle \Gamma(\gamma G_\epsilon)\, Y \, (v \cdot \mathrm{n}) \rangle \, \mathrm{d}\sigma_x \, \mathrm{d}t \\ &\qquad - \int_{t_1}^{t_2} \int_\Omega \langle \Gamma(G_\epsilon)\, v \cdot \nabla_x Y \rangle \, \mathrm{d}x \, \mathrm{d}t = \frac{1}{\epsilon} \int_{t_1}^{t_2} \int_\Omega \langle \Gamma'(G_\epsilon)\, \mathcal{Q}(G_\epsilon, G_\epsilon)\, Y \rangle \, \mathrm{d}x \, \mathrm{d}t \,, \end{aligned} \tag{2.18}$$

for every admissible function Γ, *every* $Y \in C^1 \cap L^\infty(\mathbb{R}^{\mathrm{D}} \times \bar{\Omega})$, *and every* $[t_1, t_2] \subset [0, \infty]$;
- G_ϵ *satisfies*

$$\gamma_- G_\epsilon = (1 - \alpha) L(\gamma_+ G_\epsilon) + \alpha \langle \gamma_+ G_\epsilon \rangle_{\partial\Omega} \quad \textit{almost everywhere on} \quad \Sigma_- \times \mathbb{R}_+ \,. \tag{2.19}$$

Remark. Because the trace γG_ϵ is only known to exist in $L^0(|v \cdot \mathrm{n}| M \mathrm{d}v \, \mathrm{d}\sigma_x \, \mathrm{d}t)$ rather than in $L^1_{loc}(\mathrm{d}t; L^1(|v \cdot \mathrm{n}| M \mathrm{d}v \, \mathrm{d}\sigma_x))$, we cannot conclude from the boundary condition (2.19) that

$$\langle v \, \gamma G_\epsilon \rangle \cdot \mathrm{n} = 0 \quad \text{on} \quad \partial\Omega \,. \tag{2.20}$$

Indeed, we cannot even conclude that the boundary mass-flux $\langle v \, \gamma G_\epsilon \rangle \cdot \mathrm{n}$ is defined on $\partial\Omega$. Moreover, in contrast to DiPerna-Lions theory over the whole space or periodic domains, it is not asserted in [15] that G_ϵ satisfies the weak form of the local mass conservation law

$$\int_\Omega \chi \, \langle G_\epsilon(t_2) \rangle \, \mathrm{d}x - \int_\Omega \chi \, \langle G_\epsilon(t_1) \rangle \, \mathrm{d}x - \int_{t_1}^{t_2} \int_\Omega \nabla_x \chi \cdot \langle v \, G_\epsilon \rangle \, \mathrm{d}x \, \mathrm{d}t = 0 \qquad \forall \chi \in C^1(\overline{\Omega}) \,. \tag{2.21}$$

If this were the case, it would allow a great simplification the proof of our main result. Rather, we will employ the boundary condition (2.19) inside an approximation to (2.21) that has a well-defined boundary flux.

Remark. As was shown in [3], the condition $H(G) < \infty$ found in our definition of the entropy class (2.14) is equivalent to the conditon

$$\iint_{\mathbb{R}^{\mathrm{D}} \times \Omega} \big(1 + |v|^2 + |\log(G)| \big) \, G \, M \mathrm{d}v \, \mathrm{d}x < \infty \,,$$

which is used by Mischler and others. By presenting it as we do, it is clear that the entropy class is simply those kinetic densities G whose relative entropy with respect to M is finite.

3. Main Result

3.1. **Main Theorem.** We will consider families G_ϵ of DiPerna-Lions renormalized solutions to (2.11) such that $G_\epsilon^{\mathrm{in}} \geq 0$ satisfies the entropy bound

$$H(G_\epsilon^{\mathrm{in}}) \leq C^{\mathrm{in}} \delta_\epsilon^2 \tag{3.1}$$

for some $C^{\mathrm{in}} < \infty$ and $\delta_\epsilon > 0$ that satisfies the scaling $\delta_\epsilon \to 0$ as $\epsilon \to 0$.

The value of $H(G)$ provides a natural measure of the proximity of G to the equilibrium $G = 1$. We define the families g_ϵ^{in} and g_ϵ of fluctuations about $G = 1$ by the relations

$$G_\epsilon^{\mathrm{in}} = 1 + \delta_\epsilon g_\epsilon^{\mathrm{in}} \,, \qquad G_\epsilon = 1 + \delta_\epsilon g_\epsilon \,. \tag{3.2}$$

One easily sees [3] that H asymptotically behaves like half the square of the L^2-norm of these fluctuations as $\epsilon \to 0$. Hence, the entropy bound (3.1) combined with the entropy inequality (2.15) is consistent with these fluctuations being of order 1. Just as the relative entropy H

controls the fluctuations g_ϵ, the dissipation rate R given by (2.16) controls the scaled collision integrals defined by

$$q_\epsilon = \frac{1}{\sqrt{\epsilon}\delta_\epsilon}\big(G'_{\epsilon 1}G'_\epsilon - G_{\epsilon 1}G_\epsilon\big)\,.$$

Here we only state the weak acoustic limit theorem because the corresponding strong limit theorem is analogous to that stated in [6] and its proof based on the weak limit theorem and relative entropy convergence is essentially the same.

Theorem 3.1. (Weak Acoustic Limit Theorem) *Let b be a collision kernel that satisfies the assumptions in Section 2.1. Let G_ϵ^{in} be a family in the entropy class $E(M\mathrm{d}v\,\mathrm{d}x)$ that satisfies the normalization* (2.1) *and the entropy bound* (3.1) *for some $C^{\mathrm{in}} < \infty$ and $\delta_\epsilon > 0$ satisfies the scaling*

$$\delta_\epsilon = O\big(\sqrt{\epsilon}\big)\,.$$

Assume, moreover, that for some $(\rho^{\mathrm{in}}, u^{\mathrm{in}}, \theta^{\mathrm{in}}) \in L^2(\mathrm{d}x; \mathbb{R}\times\mathbb{R}^{\mathrm{D}}\times\mathbb{R})$ the family of fluctuations g_ϵ^{in} defined by (3.2) *satisfies*

$$(\rho^{\mathrm{in}}, u^{\mathrm{in}}, \theta^{\mathrm{in}}) = \lim_{\epsilon\to 0}\big(\langle g_\epsilon^{\mathrm{in}}\rangle\,,\,\langle v\, g_\epsilon^{\mathrm{in}}\rangle\,,\,\big\langle\big(\tfrac{1}{\mathrm{D}}|v|^2 - 1\big)\, g_\epsilon^{\mathrm{in}}\big\rangle\big) \quad \textit{in the sense of distributions}\,. \tag{3.3}$$

Let G_ϵ be any family of DiPerna-Lions-Mischler renormalized solutions to the Boltzmann equation (2.11) *that have G_ϵ^{in} as initial values.*

Then, as $\epsilon \to 0$, the family of fluctuations g_ϵ defined by (3.2) *satisfies*

$$g_\epsilon \to \rho + v\cdot u + (\tfrac{1}{2}|v|^2 - \tfrac{\mathrm{D}}{2})\theta \quad \textit{in } w\text{-}L^1_{loc}(\mathrm{d}t; w\text{-}L^1((1+|v|^2)M\mathrm{d}v\,\mathrm{d}x))\,, \tag{3.4}$$

where $(\rho, u, \theta) \in C([0,\infty); L^2(\mathrm{d}x; \mathbb{R}\times\mathbb{R}^{\mathrm{D}}\times\mathbb{R}))$ is the unique solution to the acoustic system (1.1) *that satisfies the impermeable boundary condition* (1.2) *and has initial data $(\rho^{\mathrm{in}}, u^{\mathrm{in}}, \theta^{\mathrm{in}})$ obtain from* (3.3). *In addition, ρ satisfies*

$$\int_\Omega \rho\,\mathrm{d}x = 0\,. \tag{3.5}$$

This result improves upon the acoustic limit result in [6] in three ways. First, its assumption on the collision kernel b is the same as [12], so it treats a broader class of cut-off kernels than was treated in [6]. In particular, it treats kernals derived from soft potentials. Second, its scaling assumption is $\delta_\epsilon = O(\sqrt{\epsilon})$, which is certainly better than the scaling assumption (1.16) used in [6]. This assumption is still a long way from that required by the formal derivation of the acoustic system, which is $\delta_\epsilon \to 0$ as $\epsilon \to 0$. Our more restrictive requirement arises from the way in which we remove the local conservation law defects of the DiPerna-Lions solutions. Third, we derive a weak form of the boundary condition $u\cdot\mathrm{n} = 0$. It is the first time such a boundary condition for the acoustic system is derived from the Boltzmann equation with the Maxwell reflection boundary condition.

3.2. **Proof of the Main Theorem.** In order to derive the fluid equations with boundary conditions, we need to pass to the limit in approximate local conservation laws built from the renormalized Boltzmann equation (2.13). We choose the renormalization used in [12] — namely,

$$\Gamma(Z) = \frac{Z-1}{1+(Z-1)^2}\,. \tag{3.6}$$

After dividing by δ_ϵ, equation (2.13) becomes

$$\partial_t \tilde{g}_\epsilon + v\cdot\nabla_x \tilde{g}_\epsilon = \frac{1}{\sqrt{\epsilon}}\,\Gamma'(G_\epsilon)\iint_{\mathbb{S}^{\mathrm{D}-1}\times\mathbb{R}^{\mathrm{D}}} q_\epsilon\, b(\omega\,, v_1 - v)\,\mathrm{d}\omega\, M_1 \mathrm{d}v_1\,, \tag{3.7}$$

where $\tilde{g}_\epsilon = \Gamma(G_\epsilon)/\delta_\epsilon$. By introducing $N_\epsilon = 1+\delta_\epsilon^2 g_\epsilon^2$, we can write

$$\tilde{g}_\epsilon = \frac{g_\epsilon}{N_\epsilon}\,, \qquad \Gamma'(G_\epsilon) = \frac{2}{N_\epsilon^2} - \frac{1}{N_\epsilon}\,. \tag{3.8}$$

When moment of the renormalized Boltzmann equation (3.7) is formally taken with respect to any $\zeta \in \mathrm{span}\{1\,, v_1\,, \cdots\,, v_{\mathrm{D}}\,, |v|^2\}$, one obtains

$$\partial_t\langle \zeta\,\tilde{g}_\epsilon\rangle + \nabla_x\cdot\langle v\,\zeta\,\tilde{g}_\epsilon\rangle = \frac{1}{\sqrt{\epsilon}}\left\langle\!\!\left\langle \zeta\,\Gamma'(G_\epsilon)\,q_\epsilon\right\rangle\!\!\right\rangle. \tag{3.9}$$

This fails to be a local conservation law because the so-called *conservation defect* on the right-hand side is generally nonzero. We will show that this defect vanishes as $\epsilon\to 0$, while the left-hand side converges to the local conservation law corresponding to ζ. More precisely, it can be shown that every DiPerna-Lions solution satisfies (3.9) in the sense that for every $\chi\in C^1(\Omega)$ and every $[t_1, t_2]\subset[0,\infty)$ it satisfies

$$\begin{aligned}\int_\Omega \chi\,\langle\zeta\,\tilde{g}_\epsilon(t_2)\rangle\,\mathrm{d}x - \int_\Omega \chi\,\langle\zeta\,\tilde{g}_\epsilon(t_1)\rangle\,\mathrm{d}x + \int_{t_1}^{t_2}\int_{\partial\Omega}\chi\,\langle v\,\zeta\,\gamma\tilde{g}_\epsilon\rangle\cdot\mathrm{n}\,\mathrm{d}\sigma_x\,\mathrm{d}t&\\ - \int_{t_1}^{t_2}\int_\Omega \nabla_x\chi\cdot\langle v\,\zeta\,\tilde{g}_\epsilon\rangle\,\mathrm{d}x\,\mathrm{d}t = \int_{t_1}^{t_2}\int_\Omega \chi\,\frac{1}{\sqrt{\epsilon}}\left\langle\!\!\left\langle \zeta\,\Gamma'(G_\epsilon)\,q_\epsilon\right\rangle\!\!\right\rangle\mathrm{d}x\,\mathrm{d}t\,.&\end{aligned} \tag{3.10}$$

Moreover, from (2.19) the boundary condition is understood in the renormalized sense:

$$\gamma_-\tilde{g}_\epsilon = \frac{(1-\alpha)L\gamma_+ g_\epsilon + \alpha\langle\gamma_+ g_\epsilon\rangle_{\partial\Omega}}{1+\delta_\epsilon^2[(1-\alpha)L\gamma_+ g_\epsilon + \alpha\langle\gamma_+ g_\epsilon\rangle_{\partial\Omega}]^2} \quad \text{on } \Sigma_-\times\mathbb{R}_+\,, \tag{3.11}$$

where the equality holds almost everywhere. We will pass to the limit in the weak form (3.10). The Main Theorem will be proved in two steps: the interior equations will be established first and the boundary condition second.

The acoustic system (1.1) is justified in the interior of Ω by showing that the limit of (3.10) as $\epsilon\to 0$ is the weak form of the acoustic system whenever the test function χ vanishes on $\partial\Omega$. We prove that the conservation defect on the right-hand side of (3.10) vanishes as $\epsilon\to 0$ in Proposition 4.1, which is presented in the next section. The proof of the analogous result in [6] must be modified in order to include the case $\delta_\epsilon = O(\sqrt{\epsilon})$. The convergence of the density and flux terms is proved essentially as in [6], so we omit those arguments here. The upshot is that every converging subsequence of the family of fluctuations g_ϵ satisfies

$$g_\epsilon \to \rho + v\cdot u + (\tfrac{1}{2}|v|^2 - \tfrac{\mathrm{D}}{2})\theta \quad \text{in w-}L^1_{loc}(\mathrm{d}t;\text{w-}L^1((1+|v|^2)M\mathrm{d}v\,\mathrm{d}x))\,,$$

where $(\rho,u,\theta)\in C([0,\infty);\text{w-}L^2(\mathrm{d}x;\mathbb{R}\times\mathbb{R}^{\mathrm{D}}\times\mathbb{R}))$ satisfies for every $[t_1,t_2]\subset[0,\infty)$

$$\int_\Omega \chi\,\rho(t_2)\,\mathrm{d}x - \int_\Omega \chi\,\rho(t_1)\,\mathrm{d}x - \int_{t_1}^{t_2}\int_\Omega \nabla_x\chi\cdot u\,\mathrm{d}x\,\mathrm{d}t = 0 \qquad \forall\chi\in C_0^1(\overline{\Omega})\,, \tag{3.12a}$$

$$\int_\Omega w\cdot u(t_2)\,\mathrm{d}x - \int_\Omega w\cdot u(t_1)\,\mathrm{d}x - \int_{t_1}^{t_2}\int_\Omega \nabla_x\cdot w\,(\rho+\theta)\,\mathrm{d}x\,\mathrm{d}t = 0 \qquad \forall w\in C_0^1(\overline{\Omega};\mathbb{R}^{\mathrm{D}})\,, \tag{3.12b}$$

$$\tfrac{\mathrm{D}}{2}\int_\Omega \chi\,\theta(t_2)\,\mathrm{d}x - \tfrac{\mathrm{D}}{2}\int_\Omega \chi\,\theta(t_1)\,\mathrm{d}x - \int_{t_1}^{t_2}\int_\Omega \nabla_x\chi\cdot u\,\mathrm{d}x\,\mathrm{d}t = 0 \qquad \forall\chi\in C_0^1(\overline{\Omega})\,. \tag{3.12c}$$

This shows that the acoustic system (1.1) is satisfied in the interior of Ω.

The more significant step is to justify the impermeable boundary condition (1.2). Unlike to what is done for the incompressible Stokes [14] and Navier-Stokes [11] limits, here we do not have enough control to pass to the limit in the boundary terms in (3.10) for the local conservation laws of momentum and energy. We can however do so for the local conservation law of mass — i.e. when $\zeta = 1$. Indeed, Proposition 5.1 of Section 5 will show that we can extend (3.12a) to

$$\int_\Omega \chi\, \rho(t_2)\, \mathrm{d}x - \int_\Omega \chi\, \rho(t_1)\, \mathrm{d}x - \int_{t_1}^{t_2}\int_\Omega \nabla_{\!x}\chi \cdot u\, \mathrm{d}x\, \mathrm{d}t = 0 \qquad \forall \chi \in C^1(\overline{\Omega})\,. \tag{3.13}$$

We obtain 3.5 by setting $\chi = 1$ and $t_1 = 0$ above, and using the fact that the family G_ϵ^{in} satisfies the normalization (2.1).

Because for every $\chi \in C^1(\overline{\Omega})$ we can find a sequence $\{\chi_n\} \subset C_0^1(\overline{\Omega})$ such that $\chi_n \to \chi$ in $L^2(\mathrm{d}x)$, it follows from (3.12a) and (3.12c) that

$$\begin{aligned}
\tfrac{\mathrm{D}}{2}\int_\Omega \chi\, \theta(t_2)\, \mathrm{d}x - \tfrac{\mathrm{D}}{2}\int_\Omega \chi\, \theta(t_1)\, \mathrm{d}x &= \lim_{n\to\infty} \tfrac{\mathrm{D}}{2}\int_\Omega \chi_n\, \theta(t_2)\, \mathrm{d}x - \lim_{n\to\infty} \tfrac{\mathrm{D}}{2}\int_\Omega \chi_n\, \theta(t_1)\, \mathrm{d}x \\
&= \lim_{n\to\infty} \int_\Omega \chi_n\, \rho(t_2)\, \mathrm{d}x - \lim_{n\to\infty} \int_\Omega \chi_n\, \rho(t_1)\, \mathrm{d}x \\
&= \int_\Omega \chi\, \rho(t_2)\, \mathrm{d}x - \int_\Omega \chi\, \rho(t_1)\, \mathrm{d}x\,.
\end{aligned}$$

It thereby follows from (3.13) that we can extend (3.12c) to

$$\tfrac{\mathrm{D}}{2}\int_\Omega \chi\, \theta(t_2)\, \mathrm{d}x - \tfrac{\mathrm{D}}{2}\int_\Omega \chi\, \theta(t_1)\, \mathrm{d}x - \int_{t_1}^{t_2}\int_\Omega \nabla_{\!x}\chi \cdot u\, \mathrm{d}x\, \mathrm{d}t = 0 \qquad \forall \chi \in C^1(\overline{\Omega})\,. \tag{3.14}$$

Finally, because for every $w \in C^1(\overline{\Omega}; \mathbb{R}^{\mathrm{D}})$ such that $w \cdot \mathrm{n} = 0$ on $\partial\Omega$ we can find a sequence $\{w_n\} \subset C_0^1(\overline{\Omega}; \mathbb{R}^{\mathrm{D}})$ such that $w_n \to w$ in $L^2(\mathrm{d}x; \mathbb{R}^{\mathrm{D}})$ and $\nabla_{\!x}\cdot w_n \to \nabla_{\!x}\cdot w$ in $L^2(\mathrm{d}x)$, it follows from (3.12b) that

$$\begin{aligned}
\int_\Omega w \cdot u(t_2)\, \mathrm{d}x - \int_\Omega w \cdot u(t_1)\, \mathrm{d}x &= \lim_{n\to\infty}\int_\Omega w_n \cdot u(t_2)\, \mathrm{d}x - \lim_{n\to\infty}\int_\Omega w_n \cdot u(t_1)\, \mathrm{d}x \\
&= \lim_{n\to\infty}\int_{t_1}^{t_2}\int_\Omega \nabla_{\!x}\cdot w_n\, (\rho + \theta)\, \mathrm{d}x\, \mathrm{d}t \\
&= \int_{t_1}^{t_2}\int_\Omega \nabla_{\!x}\cdot w\, (\rho + \theta)\, \mathrm{d}x\, \mathrm{d}t\,.
\end{aligned}$$

But this combined with (3.13) and (3.14) is the weak formulation of the acoustic system (1.1) with the boundary condition (1.2). Because this system has a unique weak solution in $C([0,\infty); \text{w-}L^2(\mathrm{d}x; \mathbb{R}\times\mathbb{R}^{\mathrm{D}}\times\mathbb{R}))$, all converging sequences of the family g_ϵ have this same limit. Moreover, this limit must be the strong solution that lies in $C([0,\infty); L^2(\mathrm{d}x; \mathbb{R}\times\mathbb{R}^{\mathrm{D}}\times\mathbb{R}))$. The family of fluctuations g_ϵ therefore converges as asserted by (3.4). □

Remark. Had we known that G_ϵ satisfies the weak form of the local mass conservation law (2.21) then we could have easily obtained (3.13) by passing to the limit in (2.21). In that case there would be a great simplification in our proof because there would be no need for Proposition 5.1.

4. Removal of the Conservation Defects

The conservation defects in (3.9) have the form

$$\frac{1}{\sqrt{\epsilon}}\left\langle\!\left\langle \zeta\, \Gamma'(G_\epsilon)\, q_\epsilon \right\rangle\!\right\rangle = \frac{1}{\sqrt{\epsilon}}\left\langle\!\!\left\langle \zeta \left(\frac{2}{N_\epsilon^2} - \frac{1}{N_\epsilon}\right) q_\epsilon \right\rangle\!\!\right\rangle .$$

In order to establish local conservation laws, we must show that these defects vanish as $\epsilon \to 0$. This is done with the following proposition.

Proposition 4.1. *For $n = 1$ and $n = 2$, and for every $\zeta \in span\{1, v_1, \cdots, v_{\mathrm{D}}, |v|^2\}$ one has*

$$\frac{1}{\sqrt{\epsilon}}\left\langle\!\!\left\langle \zeta\, \frac{q_\epsilon}{N_\epsilon^n} \right\rangle\!\!\right\rangle \to 0 \quad \text{in } w\text{-}L^1_{loc}(\mathrm{d}t; w\text{-}L^1(\mathrm{d}x)) \ \text{as } \epsilon \to 0 . \tag{4.1}$$

Proof. Similar to the proof of Proposition 8.1 in [12], for $n = 1$, we obtain the decomposition

$$\frac{1}{\sqrt{\epsilon}}\left\langle\!\!\left\langle \zeta\, \frac{q_\epsilon}{N_\epsilon} \right\rangle\!\!\right\rangle = \frac{\delta_\epsilon^2}{\sqrt{\epsilon}}\left\langle\!\!\left\langle \zeta\, \frac{g_{\epsilon 1}^2 q_\epsilon}{N_{\epsilon 1} N_\epsilon} \right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle \zeta\, \frac{\delta_\epsilon^2 (g_{\epsilon 1} + g_\epsilon) q_\epsilon^2}{N'_{\epsilon 1} N'_\epsilon N_{\epsilon 1} N_\epsilon} \right\rangle\!\!\right\rangle - \frac{\delta_\epsilon^2}{\sqrt{\epsilon}}\left\langle\!\!\left\langle \zeta'\, \frac{g'_{\epsilon 1} g'_\epsilon\, q_\epsilon}{N'_{\epsilon 1} N'_\epsilon N_{\epsilon 1} N_\epsilon}\, J_\epsilon \right\rangle\!\!\right\rangle , \tag{4.2}$$

where J_ϵ is given by

$$J_\epsilon = 2 + \delta_\epsilon \big(g'_{\epsilon 1} + g'_\epsilon + g_{\epsilon 1} + g_\epsilon\big) - \delta_\epsilon^2 \big(g'_{\epsilon 1} g'_\epsilon - g_{\epsilon 1} g_\epsilon\big) . \tag{4.3}$$

We can then dominate the integrands of the three terms on the right-hand side of (4.2). Because for every $\zeta \in \mathrm{span}\{1, v_1, \cdots, v_{\mathrm{D}}, |v|^2\}$ there exists a constant $C < \infty$ such that $|\zeta| \le C\sigma$ where $\sigma \equiv 1 + |v|^2$, the integrand of the first term is dominated by

$$\frac{\delta_\epsilon^2}{\sqrt{\epsilon}}\, \sigma\, \frac{g_{\epsilon 1}^2 |q_\epsilon|}{N_{\epsilon 1} N_\epsilon} . \tag{4.4}$$

Because $\frac{\delta_\epsilon |g_{\epsilon 1} + g_\epsilon|}{\sqrt{N'_{\epsilon 1} N'_\epsilon N_{\epsilon 1} N_\epsilon}} \le 2$, the integrand of the second term is dominated by

$$\sigma\, \frac{\delta_\epsilon q_\epsilon^2}{\sqrt{N'_{\epsilon 1} N'_\epsilon N_{\epsilon 1} N_\epsilon}} . \tag{4.5}$$

Finally, because $\frac{|J_\epsilon|}{\sqrt{N'_{\epsilon 1} N'_\epsilon N_{\epsilon 1} N_\epsilon}} \le 8$, the integrand of the third term is dominated by

$$\frac{\delta_\epsilon^2}{\sqrt{\epsilon}}\, \sigma'\, \frac{|g'_{\epsilon 1} g'_\epsilon| |q_\epsilon|}{\sqrt{N'_{\epsilon 1} N'_\epsilon N_{\epsilon 1} N_\epsilon}} . \tag{4.6}$$

Hence, the result (4.1) for the case $n = 1$ will follow once we establish that the terms (4.4), (4.5), and (4.6) vanish as $\epsilon \to 0$.

The result (4.1) for the case $n = 2$ will follow similarly. We start with the decomposition

$$\begin{aligned} \frac{1}{\sqrt{\epsilon}}\left\langle\!\!\left\langle \zeta\, \frac{q_\epsilon}{N_\epsilon^2} \right\rangle\!\!\right\rangle = \frac{\delta_\epsilon^2}{\sqrt{\epsilon}}\left\langle\!\!\left\langle \zeta\, \frac{g_{\epsilon 1}^2 q_\epsilon}{N_{\epsilon 1} N_\epsilon}\left(1 + \frac{1}{N_{\epsilon 1}}\right) \right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle \zeta\, \frac{\delta_\epsilon^2 (g_{\epsilon 1} + g_\epsilon) q_\epsilon^2}{N'_{\epsilon 1} N'_\epsilon N_{\epsilon 1} N_\epsilon}\left(\frac{1}{N'_{\epsilon 1} N'_\epsilon} + \frac{1}{N_{\epsilon 1} N_\epsilon}\right) \right\rangle\!\!\right\rangle \\ - \frac{\delta_\epsilon^2}{\sqrt{\epsilon}}\left\langle\!\!\left\langle \zeta'\, \frac{g'_{\epsilon 1} g'_\epsilon\, q_\epsilon}{N'_{\epsilon 1} N'_\epsilon N_{\epsilon 1} N_\epsilon}\, J_\epsilon \left(\frac{1}{N'_{\epsilon 1} N'_\epsilon} + \frac{1}{N_{\epsilon 1} N_\epsilon}\right) \right\rangle\!\!\right\rangle , \end{aligned}$$

where J_ϵ is given by (4.3). Because the terms in parentheses above are each bounded by 2, we can dominate the three terms on the right-hand side above just as we did the terms on the right-hand side of (4.2) for the case $n = 1$. The result (4.1) for the case $n = 2$ will then also follow once we establish that the terms (4.4), (4.5), and (4.6) vanish as $\epsilon \to 0$.

That term (4.5) vanishes is easy to see. The inequality $n'_{\epsilon 1} n'_{\epsilon} n_{\epsilon 1} n_{\epsilon} \leq 2\sqrt{N'_{\epsilon 1} N'_{\epsilon} N_{\epsilon 1} N_{\epsilon}}$, where $n_{\epsilon} = 1 + \frac{\delta_{\epsilon}}{3} g_{\epsilon}$, along with the estimate

$$\sigma \, \frac{q_{\epsilon}^2}{n'_{\epsilon 1} n'_{\epsilon} n_{\epsilon 1} n_{\epsilon}} = O\left(\left|\log\left(\sqrt{\epsilon}\delta_{\epsilon}\right)\right|\right) \quad \text{in } L^1_{loc}(\mathrm{d}t; L^1(\mathrm{d}\mu\,\mathrm{d}x)) \text{ as } \epsilon \to 0\,,$$

which is proved in Lemma 9.4 of [6], imply that

$$\sigma \, \frac{\delta_{\epsilon} q_{\epsilon}^2}{\sqrt{N'_{\epsilon 1} N'_{\epsilon} N_{\epsilon 1} N_{\epsilon}}} = O\left(\delta_{\epsilon}\left|\log\left(\sqrt{\epsilon}\delta_{\epsilon}\right)\right|\right) \to 0 \quad \text{in } L^1_{loc}(\mathrm{d}t; L^1(\mathrm{d}\mu\,\mathrm{d}x)) \text{ as } \epsilon \to 0\,.$$

The fact that the terms (4.4) and (4.6) vanish as $\epsilon \to 0$ follows from Lemma 4.1, which is proved below. We thereby complete the proof of Proposition 4.1. ☐

Lemma 4.1.

$$\frac{\delta_{\epsilon}^2}{\sqrt{\epsilon}} \, \sigma \, \frac{g_{\epsilon 1}^2 |q_{\epsilon}|}{N_{\epsilon 1} N_{\epsilon}} \to 0 \quad \textit{in } L^1_{loc}(\mathrm{d}t; L^1(\mathrm{d}\mu\,\mathrm{d}x)) \ \textit{as } \epsilon \to 0\,, \tag{4.7}$$

$$\frac{\delta_{\epsilon}^2}{\sqrt{\epsilon}} \, \sigma' \, \frac{|g'_{\epsilon 1} g'_{\epsilon}| |q_{\epsilon}|}{\sqrt{N'_{\epsilon 1} N'_{\epsilon} N_{\epsilon 1} N_{\epsilon}}} \to 0 \quad \textit{in } L^1_{loc}(\mathrm{d}t; L^1(\mathrm{d}\mu\,\mathrm{d}x)) \ \textit{as } \epsilon \to 0\,. \tag{4.8}$$

Proof. The key to proving Lemma 4.1 is the fact that

$$\frac{g_{\epsilon}^2}{\sqrt{N_{\epsilon}}} \quad \text{is relatively compact in } \ w\text{-}L^1_{loc}(\mathrm{d}t; w\text{-}L^1(aM\mathrm{d}v\,\mathrm{d}x))\,. \tag{4.9}$$

This fact follows from Proposition 7.1 of [12], where it plays an essential role in establishing the Navier-Stokes limit. The approach to proving Lemma 4.1 is the same used to prove the analogous result in [6]. There the terms (4.4) and (4.6) were estimated by using the entropy dissipation bound along with the nonlinear estimate

$$\sigma \, \frac{g_{\epsilon}^2}{\sqrt{N_{\epsilon}}} = O(|\log(\delta_{\epsilon})|) \quad \text{in } L^{\infty}(\mathrm{d}t; L^1(M\mathrm{d}v\,\mathrm{d}x)) \text{ as } \epsilon \to 0\,.$$

Here this nonlinear estimate, which originated in [3], is replaced by the new weak compactness result (4.9) from [12], thereby extending the result in [6] to the scaling $\delta_{\epsilon} = \sqrt{\epsilon}$.

The entropy inequality (2.15) and the entropy bound (3.1) combine to bound the entropy dissipation as

$$\frac{1}{\epsilon\delta_{\epsilon}^2} \int_0^{\infty} \int_{\Omega} \left\langle\!\!\left\langle \frac{1}{4} r\left(\frac{\sqrt{\epsilon}\delta_{\epsilon} q_{\epsilon}}{G_{\epsilon 1} G_{\epsilon}}\right) G_{\epsilon 1} G_{\epsilon} \right\rangle\!\!\right\rangle \mathrm{d}x\,\mathrm{d}t \leq C^{\mathrm{in}}\,, \tag{4.10}$$

where the function r is defined over $z > -1$ by $r(z) = z\log(1+z)$. The function r is strictly convex over $z > -1$. The proofs of (4.7) and (4.8) are each based on a delicate use of the classical Young inequality satisfied by r and its Legendre dual r^*, namely, the inequality

$$pz \leq r^*(p) + r(z) \quad \text{for every } p \in \mathbb{R} \text{ and } z > -1\,.$$

For every positive ϱ and y we set

$$p = \frac{\sqrt{\epsilon}\delta_{\epsilon} y}{\varrho} \quad \text{and} \quad z = \frac{\sqrt{\epsilon}\delta_{\epsilon}|q_{\epsilon}|}{G_{\epsilon 1} G_{\epsilon}}\,,$$

and use the fact that $r(|z|) \leq r(z)$ for every $z > -1$ to obtain

$$y|q_{\epsilon}| \leq \frac{\varrho}{\epsilon\delta_{\epsilon}^2}\, r^*\left(\frac{\sqrt{\epsilon}\delta_{\epsilon} y}{\varrho}\right) G_{\epsilon 1} G_{\epsilon} + \frac{\varrho}{\epsilon\delta_{\epsilon}^2}\, r\left(\frac{\sqrt{\epsilon}\delta_{\epsilon} q_{\epsilon}}{G_{\epsilon 1} G_{\epsilon}}\right) G_{\epsilon 1} G_{\epsilon}\,. \tag{4.11}$$

This inequality is the starting point for the proofs of assertions (4.7) and (4.8). These proofs also use the facts, recalled from [3], that r^* is superquadratic in the sense

$$r^*(\lambda p) \leq \lambda^2 r^*(p) \quad \text{for every } p > 0 \text{ and } \lambda \in [0,1]\,, \tag{4.12}$$

and that r^* has the exponential asymptotics $r^*(p) \sim \exp(p)$ as $p \to \infty$.

The proof of assertion (4.7) follows that of Lemma 8.2 in [12]. We use the inequality (4.11) with $y = \frac{\sigma}{4s^*} \frac{\delta_\epsilon^2}{\sqrt{\epsilon}} \frac{g_{\epsilon 1}^2}{N_{\epsilon 1} N_\epsilon}$, where $s^* \in [1,\infty)$ is related to $s \in (1,\infty]$ appearing in (2.7) by the duality relation $\frac{1}{s} + \frac{1}{s^*} = 1$. We then apply the superquadratic property (4.12) with $\lambda = \frac{\delta_\epsilon^3 g_{\epsilon 1}^2}{\varrho N_{\epsilon 1} N_\epsilon}$ and $p = \frac{\sigma}{4s^*}$, where we note that $\lambda \leq 1$ whenever $\delta_\epsilon \leq \varrho$. This leads to the bound

$$\frac{\sigma}{4s^*} \frac{\delta_\epsilon^2}{\sqrt{\epsilon}} \frac{g_{\epsilon 1}^2 |q_\epsilon|}{N_{\epsilon 1} N_\epsilon} \leq \frac{1}{\varrho} \frac{\delta_\epsilon^4}{\epsilon} \frac{g_{\epsilon 1}^4}{N_{\epsilon 1}^2 N_\epsilon^2}\, r^*\!\left(\frac{\sigma}{4s^*}\right) G_{\epsilon 1} G_\epsilon + \frac{\varrho}{\epsilon \delta_\epsilon^2} r\!\left(\frac{\sqrt{\epsilon}\delta_\epsilon q_\epsilon}{G_{\epsilon 1} G_\epsilon}\right) G_{\epsilon 1} G_\epsilon\,. \tag{4.13}$$

The second term on the right-hand side above can be made arbitrarily small in $L^1(\mathrm{d}\mu\,\mathrm{d}x\,\mathrm{d}t)$ by using the entropy dissipation bound (4.10) and picking ϱ small enough. Assertion (4.7) will then follow upon showing that for every $\varrho > 0$ the first term on the right-hand side of (4.13) vanishes as $\epsilon \to 0$.

Because $G_{\epsilon 1} G_\epsilon \leq 2\sqrt{N_{\epsilon 1} N_\epsilon}$ while $N_\epsilon \geq 1$, the first term on the right-hand side of (4.13) is bounded by

$$\frac{2\,\delta_\epsilon^2}{\varrho\,\epsilon} \frac{\delta_\epsilon^2 g_{\epsilon 1}^2}{N_{\epsilon 1}} \frac{g_{\epsilon 1}^2}{\sqrt{N_{\epsilon 1}}}\, r^*\!\left(\frac{\sigma}{4s^*}\right)\,.$$

The first factor above is bounded because $\delta_\epsilon^2 = O(\epsilon)$, while the second is bounded above by 1 and satisfies

$$\frac{\delta_\epsilon^2 g_{\epsilon 1}^2}{N_{\epsilon 1}} \to 0 \quad \text{in measure as } \epsilon \to 0\,.$$

It follows from (4.9) and Lemma 8.1 of [12] that

$$\frac{g_{\epsilon 1}^2}{\sqrt{N_{\epsilon 1}}}\, r^*\!\left(\frac{\sigma}{4s^*}\right) \quad \text{is relatively compact in} \quad w\text{-}L^1_{loc}(\mathrm{d}t; w\text{-}L^1(\mathrm{d}\mu\,\mathrm{d}x))\,.$$

We thereby conclude by the Product Limit Theorem [3] that

$$\frac{2\,\delta_\epsilon^2}{\varrho\,\epsilon} \frac{\delta_\epsilon^2 g_{\epsilon 1}^2}{N_{\epsilon 1}} \frac{g_{\epsilon 1}^2}{\sqrt{N_{\epsilon 1}}}\, r^*\!\left(\frac{\sigma}{4s^*}\right) \to 0 \quad \text{in } L^1_{loc}(\mathrm{d}t; L^1(\mathrm{d}\mu\,\mathrm{d}x))\,.$$

Hence, for every $\varrho > 0$ the first term on the right-hand side of (4.13) vanishes as $\epsilon \to 0$. Assertion (4.7) thereby follows.

The proof of assertion (4.8) similarly follows that of Lemma 8.3 in [12]. We use the inequality (4.11) with $y = \frac{\sigma}{4s^*} \frac{\delta_\epsilon^2}{\sqrt{\epsilon}} \frac{|g'_{\epsilon 1} g'_\epsilon|}{\sqrt{N'_{\epsilon 1} N'_\epsilon N_{\epsilon 1} N_\epsilon}}$ and apply the superquadratic property (4.12) to obtain the bound

$$\frac{\sigma'}{4s^*} \frac{\delta_\epsilon^2}{\sqrt{\epsilon}} \frac{|g'_{\epsilon 1} g'_\epsilon||q_\epsilon|}{\sqrt{N'_{\epsilon 1} N'_\epsilon N_{\epsilon 1} N_\epsilon}} \leq \frac{1}{\varrho} \frac{\delta_\epsilon^4}{\epsilon} \frac{{g'}_{\epsilon 1}^2 {g'}_\epsilon^2}{N'_{\epsilon 1} N'_\epsilon N_{\epsilon 1} N_\epsilon}\, r^*\!\left(\frac{\sigma}{4s^*}\right) G_{\epsilon 1} G_\epsilon + \frac{\varrho}{\epsilon \delta_\epsilon^2}\, r\!\left(\frac{\sqrt{\epsilon}\delta_\epsilon q_\epsilon}{G_{\epsilon 1} G_\epsilon}\right) G_{\epsilon 1} G_\epsilon\,.$$

We then argue as we did to prove assertion (4.7) from (4.13). □

5. Limit of the Boundary Mass-Flux Term

In this section we show that as $\epsilon \to 0$ the boundary term vanishes in the weak form of the approximate local conservation of mass that is obtained by setting $\zeta = 1$ in (3.10). This is the key step in establishing the limiting mass conservation equation (3.13) from (3.10), as the limit for all the other terms are obtained exactly as they were when we established the interior mass conservation equation (3.12a). More specifically, we prove the following.

Proposition 5.1. *For every $[t_1, t_2] \subset [0, \infty)$ and every $\chi \in C^1(\overline{\Omega})$ one has*

$$\lim_{\epsilon \to 0} \int_{t_1}^{t_2} \int_{\partial\Omega} \chi \, \langle v \, \gamma \tilde{g}_\epsilon \rangle \cdot \mathrm{n} \, \mathrm{d}\sigma_x \, \mathrm{d}t = 0 \, .$$

Proof. Denote the boundary mass-flux term as

$$j_\epsilon = \int_{t_1}^{t_2} \int_{\partial\Omega} \chi \, \langle v \, \gamma \tilde{g}_\epsilon \rangle \cdot \mathrm{n} \, \mathrm{d}\sigma_x \, \mathrm{d}t \, .$$

The renormalized boundary condition (3.11) can be expressed as

$$\gamma_- \tilde{g}_\epsilon = L \left(\frac{\widehat{g}_\epsilon}{1 + \delta_\epsilon^2 \widehat{g}_\epsilon^2} \right) , \quad \text{where} \quad \widehat{g}_\epsilon = \gamma_+ \big((1 - \alpha) g_\epsilon + \alpha \langle \gamma_+ g_\epsilon \rangle_{\partial\Omega} \big) \, .$$

It follows that

$$\begin{aligned} j_\epsilon &= \int_{t_1}^{t_2} \int_{\partial\Omega} \chi \, \big\langle v \, (\gamma_+ \tilde{g}_\epsilon + \gamma_- \tilde{g}_\epsilon) \big\rangle \cdot \mathrm{n} \, \mathrm{d}\sigma_x \, \mathrm{d}t \\ &= \int_{t_1}^{t_2} \int_{\partial\Omega} \chi \, \left\langle v \left(\gamma_+ \tilde{g}_\epsilon - \frac{\widehat{g}_\epsilon}{1 + \delta_\epsilon^2 \widehat{g}_\epsilon^2} \right) \right\rangle \cdot \mathrm{n} \, \mathrm{d}\sigma_x \, \mathrm{d}t \\ &= \alpha \int_{t_1}^{t_2} \int_{\partial\Omega} \chi \, \left\langle v \, \gamma_+ \frac{(g_\epsilon - \langle \gamma_+ g_\epsilon \rangle_{\partial\Omega})(1 - \delta_\epsilon^2 g_\epsilon \widehat{g}_\epsilon)}{(1 + \delta_\epsilon^2 g_\epsilon^2)(1 + \delta_\epsilon^2 \widehat{g}_\epsilon^2)} \right\rangle \cdot \mathrm{n} \, \mathrm{d}\sigma_x \, \mathrm{d}t \, . \end{aligned}$$

If $\alpha = 0$ then we are done. If $\alpha > 0$ then set $\gamma_\epsilon = \gamma_+ \big(g_\epsilon - \langle \gamma_+ g_\epsilon \rangle_{\partial\Omega} \big)$, so that

$$j_\epsilon = \alpha \int_{t_1}^{t_2} \int_{\partial\Omega} \chi \int_{v \cdot \mathrm{n} > 0} \gamma_\epsilon \, \frac{1 - \delta_\epsilon^2 \gamma_+ g_\epsilon \, \widehat{g}_\epsilon}{(1 + \delta_\epsilon^2 \gamma_+ g_\epsilon^2)(1 + \delta_\epsilon^2 \widehat{g}_\epsilon^2)} \, v \cdot n \, M \mathrm{d}v \, \mathrm{d}\sigma_x \, \mathrm{d}t \, . \tag{5.1}$$

The idea will now be to control γ_ϵ with the bound on the Darrozès-Guiraud information (2.17) given by the entropy inequality (2.15) and entropy bound (3.1). More specifically, we will show in Lemma 5.1 that

$$\lim_{\epsilon \to 0} \int_{t_1}^{t_2} \int_{\partial\Omega} \chi \int_{v \cdot \mathrm{n} > 0} \frac{\gamma_\epsilon}{(1 + \delta_\epsilon^2 \gamma_+ g_\epsilon^2)(1 + \delta_\epsilon^2 \widehat{g}_\epsilon^2)} \, v \cdot \mathrm{n} \, M \mathrm{d}v \, \mathrm{d}\sigma_x \, \mathrm{d}t = 0 \, , \tag{5.2}$$

and that

$$\lim_{\epsilon \to 0} \int_{t_1}^{t_2} \int_{\partial\Omega} \chi \int_{v \cdot \mathrm{n} > 0} \gamma_\epsilon \, \frac{\delta_\epsilon^2 \gamma_+ g_\epsilon \, \widehat{g}_\epsilon}{(1 + \delta_\epsilon^2 \gamma_+ g_\epsilon^2)(1 + \delta_\epsilon^2 \widehat{g}_\epsilon^2)} \, v \cdot n \, M \mathrm{d}v \, \mathrm{d}\sigma_x \, \mathrm{d}t = 0 \, . \tag{5.3}$$

Propostion 5.1 for the case $\alpha > 0$ will then follow from (5.1-5.3) upon proving Lemma 5.1. ☐

Lemma 5.1. *Let $\alpha > 0$. Then the limits (5.2) and (5.3) hold.*

Proof. Following [14], we employ the decomposition

$$\gamma_\epsilon = \gamma_\epsilon^{(1)} + \gamma_\epsilon^{(2)}\,, \quad \text{where} \quad \gamma_\epsilon^{(1)} = \gamma_\epsilon \mathbf{1}_{\{\gamma_+ G_\epsilon \le 2\langle \gamma_+ G_\epsilon\rangle_{\partial\Omega} \le 4\gamma_+ G_\epsilon\}}\,. \tag{5.4}$$

By arguing as in Lemma 6.1 of [14] extended to our more general class of collision kernels as in Lemma 6 of [11], we obtain

$$\frac{\gamma_\epsilon^{(1)}}{(1+\delta_\epsilon^2 \gamma_+ g_\epsilon^2)^{1/4}} \quad \text{is bounded in} \quad L^2_{loc}(\mathrm{d}t; L^2(|v\cdot \mathrm{n}| M \mathrm{d}v\, \mathrm{d}\sigma_x))\,, \tag{5.5}$$

$$\frac{\gamma_\epsilon^{(1)}}{(1+\delta_\epsilon^2 \langle \gamma_+ g_\epsilon\rangle_{\partial\Omega}^2)^{1/4}} \quad \text{is bounded in} \quad L^2_{loc}(\mathrm{d}t; L^2(|v\cdot \mathrm{n}| M \mathrm{d}v\, \mathrm{d}\sigma_x))\,, \tag{5.6}$$

$$\frac{1}{\delta_\epsilon}\gamma_\epsilon^{(2)} \quad \text{is bounded in} \quad L^1_{loc}(\mathrm{d}t; L^1(|v\cdot \mathrm{n}| M \mathrm{d}v\, \mathrm{d}\sigma_x))\,. \tag{5.7}$$

To prove (5.2) we use the fact that $\langle \gamma_\epsilon \rangle_{\partial\Omega} = 0$ and the decompositon (5.4) to write

$$\begin{aligned}
&\int_{t_1}^{t_2}\int_{\partial\Omega} \chi \int_{v\cdot \mathrm{n}>0} \frac{\gamma_\epsilon}{(1+\delta_\epsilon^2\gamma_+ g_\epsilon^2)(1+\delta_\epsilon^2 \widehat{g}_\epsilon^2)}\, v\cdot \mathrm{n}\, M \mathrm{d}v\, \mathrm{d}\sigma_x\, \mathrm{d}t \\
&= \int_{t_1}^{t_2}\int_{\partial\Omega} \chi \int_{v\cdot \mathrm{n}>0} \left(\frac{\gamma_\epsilon}{(1+\delta_\epsilon^2\gamma_+ g_\epsilon^2)(1+\delta_\epsilon^2 \widehat{g}_\epsilon^2)} - \frac{\gamma_\epsilon}{(1+\delta_\epsilon^2\langle\gamma_+ g_\epsilon\rangle_{\partial\Omega}^2)^2}\right) v\cdot \mathrm{n}\, M \mathrm{d}v\, \mathrm{d}\sigma_x\, \mathrm{d}t \\
&= \int_{t_1}^{t_2}\int_{\partial\Omega} \chi \int_{v\cdot \mathrm{n}>0} \left(\frac{\gamma_\epsilon^{(1)}}{(1+\delta_\epsilon^2\gamma_+ g_\epsilon^2)(1+\delta_\epsilon^2 \widehat{g}_\epsilon^2)} - \frac{\gamma_\epsilon^{(1)}}{(1+\delta_\epsilon^2\langle\gamma_+ g_\epsilon\rangle_{\partial\Omega}^2)^2}\right) v\cdot \mathrm{n}\, M \mathrm{d}v\, \mathrm{d}\sigma_x\, \mathrm{d}t \\
&\quad + \int_{t_1}^{t_2}\int_{\partial\Omega} \chi \int_{v\cdot \mathrm{n}>0} \frac{\gamma_\epsilon^{(2)}}{(1+\delta_\epsilon^2\gamma_+ g_\epsilon^2)(1+\delta_\epsilon^2 \widehat{g}_\epsilon^2)}\, v\cdot \mathrm{n}\, M \mathrm{d}v\, \mathrm{d}\sigma_x\, \mathrm{d}t \\
&\quad - \int_{t_1}^{t_2}\int_{\partial\Omega} \chi \int_{v\cdot \mathrm{n}>0} \frac{\gamma_\epsilon^{(2)}}{(1+\delta_\epsilon^2\langle\gamma_+ g_\epsilon\rangle_{\partial\Omega}^2)^2}\, v\cdot \mathrm{n}\, M \mathrm{d}v\, \mathrm{d}\sigma_x\, \mathrm{d}t\,.
\end{aligned}$$

The last two terms on the right-hand side above vanish as $\epsilon \to 0$ by the bound (5.7).

To show that the first term on the right-hand side above also vanishes as $\epsilon \to 0$, we observe from the bounds (5.5) and (5.6), the two terms in its integrand are relatively compact in w-$L^2_{loc}(\mathrm{d}t;$ w-$L^2(|v\cdot \mathrm{n}| M\mathrm{d}v\, \mathrm{d}\sigma_x))$. Their difference is

$$\begin{aligned}
&\frac{\gamma_\epsilon^{(1)}}{(1+\delta_\epsilon^2\gamma_+ g_\epsilon^2)(1+\delta_\epsilon^2 \widehat{g}_\epsilon^2)} - \frac{\gamma_\epsilon^{(1)}}{(1+\delta_\epsilon^2\langle\gamma_+ g_\epsilon\rangle_{\partial\Omega}^2)^2} \\
&= \gamma_\epsilon^{(1)} \frac{[(1+\delta_\epsilon^2\langle\gamma_+ g_\epsilon\rangle_{\partial\Omega}^2)^2 - (1+\delta_\epsilon^2\gamma_+ g_\epsilon^2)^2] + [(1+\delta_\epsilon^2\gamma_+ g_\epsilon^2) - (1+\delta_\epsilon^2\gamma_+ g_\epsilon^2)(1+\delta_\epsilon^2 \widehat{g}_\epsilon^2)]}{(1+\delta_\epsilon^2\gamma_+ g_\epsilon^2)(1+\delta_\epsilon^2 \widehat{g}_\epsilon^2)(1+\delta_\epsilon^2\langle\gamma_+ g_\epsilon\rangle_{\partial\Omega}^2)^2}\,.
\end{aligned}$$

Noting that $\gamma_+ g_\epsilon - \widehat{g}_\epsilon = \alpha\gamma_\epsilon$, we see that

$$\begin{aligned}
&\frac{\gamma_\epsilon^{(1)}}{(1+\delta_\epsilon^2\gamma_+ g_\epsilon^2)(1+\delta_\epsilon^2 \widehat{g}_\epsilon^2)} - \frac{\gamma_\epsilon^{(1)}}{(1+\delta_\epsilon^2\langle\gamma_+ g_\epsilon\rangle_{\partial\Omega}^2)^2} \\
&= \delta_\epsilon \left(\frac{\gamma^{(1)}}{(1+\delta_\epsilon^2 \gamma g_\epsilon^2)^{1/4}} \right)^2 \mathbf{1}_{\{\gamma_+ G_\epsilon \le 2\langle \gamma_+ G_\epsilon\rangle_{\partial\Omega} \le 4\gamma_+ G_\epsilon\}} (S_\epsilon^{(1)} + S_\epsilon^{(2)})\,,
\end{aligned} \tag{5.8}$$

where

$$S_\epsilon^{(1)} = \frac{\delta_\epsilon(\gamma_+ g_\epsilon + \langle\gamma_+ g_\epsilon\rangle_{\partial\Omega})(2 + \delta_\epsilon^2\gamma_+ g_\epsilon^2 + \delta_\epsilon^2\langle\gamma_+ g_\epsilon\rangle_{\partial\Omega}^2)}{(1+\delta_\epsilon^2\gamma_+ g_\epsilon^2)^{1/2}(1+\delta_\epsilon^2\widehat{g}_\epsilon^2)(1+\delta_\epsilon^2\langle\gamma_+ g_\epsilon\rangle_{\partial\Omega}^2)^2}\,,$$
$$S_\epsilon^{(2)} = -\frac{\alpha\delta_\epsilon(\gamma_+ g_\epsilon + \widehat{g}_\epsilon)(1+\delta_\epsilon^2\gamma_+ g_\epsilon^2)}{(1+\delta_\epsilon^2\gamma_+ g_\epsilon^2)^{1/2}(1+\delta_\epsilon^2\widehat{g}_\epsilon^2)(1+\delta_\epsilon^2\langle\gamma_+ g_\epsilon\rangle_{\partial\Omega}^2)^2}\,.$$

Note that $\gamma_+ G_\epsilon \le 2\langle\gamma_+ G_\epsilon\rangle_{\partial\Omega} \le 4\gamma_+ G_\epsilon$ implies that

$$\delta_\epsilon\gamma_+ g_\epsilon \le 2\delta_\epsilon\langle\gamma_+ g_\epsilon\rangle_{\partial\Omega} + 1\,, \quad \text{and} \quad \delta_\epsilon\langle\gamma_+ g_\epsilon\rangle_{\partial\Omega} \le 2\delta_\epsilon\gamma_+ g_\epsilon + 1\,.$$

It is easy to see that both

$$\mathbf{1}_{\{\gamma_+ G_\epsilon \le 2\langle\gamma_+ G_\epsilon\rangle_{\partial\Omega} \le 4\gamma_+ G_\epsilon\}} S_\epsilon^{(1)} \quad \text{and} \quad \mathbf{1}_{\{\gamma_+ G_\epsilon \le 2\langle\gamma_+ G_\epsilon\rangle_{\partial\Omega} \le 4\gamma_+ G_\epsilon\}} S_\epsilon^{(2)}$$

are bounded in L^∞. Noting that the extra δ_ϵ in front of (5.8), we conclude that for any convergent subsequence,

$$\frac{\gamma_\epsilon^{(1)}}{(1+\delta_\epsilon^2\gamma_+ g_\epsilon^2)(1+\delta_\epsilon^2\widehat{g}_\epsilon^2)} - \frac{\gamma_\epsilon^{(1)}}{(1+\delta_\epsilon^2\langle\gamma_+ g_\epsilon\rangle_{\partial\Omega}^2)^2} \to 0\,, \quad \text{in} \quad \text{w-}L^2_{loc}(\mathrm{d}t; \text{w-}L^2(|v\cdot \mathrm{n}|M\mathrm{d}v\,\mathrm{d}\sigma_x))\,,$$

as $\epsilon \to 0$. This establishes limit (5.2).

To prove limit (5.3) separate $\gamma_\epsilon = \gamma_\epsilon^{(1)} + \gamma_\epsilon^{(2)}$, using the bound (5.7) for $\gamma_\epsilon^{(2)}$, it is easy to estimate that

$$\delta_\epsilon \left\| \frac{\alpha}{\delta_\epsilon}\gamma_\epsilon^{(2)} \right\|_{L^1_{loc}(\mathrm{d}t; L^1(|v\cdot\mathrm{n}|M\mathrm{d}v\,\mathrm{d}\sigma_x))} \frac{|\delta_\epsilon\gamma_+ g_\epsilon\,\delta_\epsilon\gamma_+\widehat{g}_\epsilon|}{(1+\delta_\epsilon^2\gamma_+ g_\epsilon^2)(1+\delta_\epsilon^2\widehat{g}_\epsilon^2)} \le C\tfrac{1}{4}\delta_\epsilon\,.$$

For the $\gamma_\epsilon^{(1)}$ part, from the L^2 bounds (5.5) and (5.6),

$$\frac{\gamma_\epsilon^{(1)}}{\sqrt{1+\delta_\epsilon^2\gamma_+ g_\epsilon^2}} \tag{5.9}$$

is relatively compact in w-$L^1_{loc}(\mathrm{d}t; \text{w-}L^1(|v\cdot\mathrm{n}|M\mathrm{d}v\,\mathrm{d}\sigma_x))$. Use the fact that

$$\frac{\delta_\epsilon\gamma_+ g_\epsilon\,\delta_\epsilon\widehat{g}_\epsilon}{\sqrt{1+\delta_\epsilon^2\gamma_+ g_\epsilon^2}(1+\delta_\epsilon^2\widehat{g}_\epsilon^2)} \tag{5.10}$$

is bounded in L^∞ and goes to 0 a.e. Then again by the Product Limit Theorem of [3], the product of (5.9) and (5.10) goes to 0 in $L^1_{loc}(\mathrm{d}t; L^1(|v\cdot\mathrm{n}|M\mathrm{d}v\,\mathrm{d}\sigma_x))$ as $\epsilon \to 0$. We thereby finish the proof of the Lemma. $\square$

Remark. The most important difference between the acoustic limit and the incompressible limits (Stokes in [14] and Navier-Stokes in [11]) is that the compactness of the renormalized traces $\gamma\tilde{g}_\epsilon$ in the acoustic limit case is not available. The pointwise convergence $\delta_\epsilon\tilde{g}_\epsilon \to 0$ a.e. is also unavailable. (compare Lemma 5.2 in [14].) In contrast, for the incompressible limits the entropy bounds from boundary provide *a priori* estimates on the quantity $\gamma_\epsilon = \gamma_+ g_\epsilon - \mathbf{1}_{\Sigma_+}\langle\gamma_+ g_\epsilon\rangle_{\partial\Omega}$. Specifically, we have the L^2 bound on $\frac{1}{\delta_\epsilon}\frac{\gamma_\epsilon^{(1)}}{n_\epsilon}$ with some renormalizer n_ϵ, see bounds (6.2) and (6.3) in [14]. However, in the acoustic limit, because of the acoustic scaling, we have only the L^2 bound on $\frac{\gamma_\epsilon^{(1)}}{n_\epsilon}$ which is much weaker than in the incompressible limits cases.

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