

UNIFORM REGULARITY FOR THE NAVIER-STOKES EQUATION WITH NAVIER BOUNDARY CONDITION

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ABSTRACT. We prove that there exists an interval of time which is uniform in the vanishing viscosity limit and for which the Navier-Stokes equation with Navier boundary condition has a strong solution. This solution is uniformly bounded in a conormal Sobolev space and has only one normal derivative bounded in L^∞ . This allows to get the vanishing viscosity limit to the incompressible Euler system from a strong compactness argument.

1. INTRODUCTION

We consider the incompressible Navier-Stokes equation

$$(1.1) \quad \partial_t u + u \cdot \nabla u + \nabla p = \varepsilon \Delta u, \quad \nabla \cdot u = 0, \quad x \in \Omega,$$

in a domain Ω of \mathbb{R}^3 . The velocity u is a three-dimensional vector field on Ω and the pressure p of the fluid is a scalar function. We add on the boundary the Navier (slip) boundary condition

$$(1.2) \quad u \cdot n = 0, \quad (Su \cdot n)_\tau = -\alpha u_\tau, \quad x \in \partial\Omega$$

where n stands for the outward unit normal to Ω , S is the strain tensor,

$$Su = \frac{1}{2}(\nabla u + \nabla u^t)$$

and for some vector field v on $\partial\Omega$, v_τ stands for the tangential part of v : $v_\tau = v - (v \cdot n)n$.

The parameter $\varepsilon > 0$ is the inverse of the Reynolds number whereas α is another coefficient which measures the tendency of the fluid to slip on the boundary. This type of boundary condition is often used to model rough boundaries, we refer for example to [3], [8] (see also [28] for a derivation from the Maxwell boundary condition of the Boltzmann equation through a hydrodynamic limit). It is known that when ε tends to zero a weak solution of (1.1), (1.2) converges towards a solution of the Euler equation, we refer to [1], [7], [20], [15]. In particular, in the three-dimensional case, in [15], it is proven by a modulated energy type approach that for sufficiently smooth solution of the Euler equation, an L^2 convergence holds. The situation for this problem is thus very different from the case of no-slip boundary conditions which is widely open for the Navier-Stokes equation except in the analytic case [34] (see also [19] for some necessary condition to get convergence and [23] for some special case).

Here, we are interested in the existence of strong solutions of (1.1), (1.2) with uniform bounds on an interval of time independent of $\varepsilon \in (0, 1]$ and in a topology sufficiently strong to deduce by a strong compactness argument that the solution converges strongly to a solution of the Euler equation

$$(1.3) \quad \partial_t u + u \cdot \nabla u + \nabla p = 0, \quad \nabla \cdot u = 0$$

with the boundary condition $u \cdot n = 0$ on $\partial\Omega$. Note that for such an argument to succeed, we need to work in a functional space where both (1.1) and (1.3) are well-posed.

Let us recall that there are two classical ways to study the vanishing viscosity limit by compactness arguments. The first one consists in trying to pass to the limit weakly in the Leray solution of the Navier-Stokes system. However, there is a lack of compactness and one cannot pass to the

limit in the nonlinear term. It is indeed an open problem to characterize the weak limit of any sequence of the Navier-Stokes system when the viscosity goes to zero even in the whole space case (see [22, 25]). The second way consists in trying to work with strong solutions in Sobolev spaces. In the case of the whole space (or the case there is no boundary) this approach yields a uniform time of existence and the convergence towards a solution of the Euler system (see [35, 18, 26]). The problem is that due to the presence of a boundary the time of existence T_ε depends on the viscosity and one often cannot prove that it stays bounded away from zero. Nevertheless, in domain with boundaries, for some special type of Navier boundary conditions or boundaries, some uniform H^3 (or $W^{2,p}$, with p large enough) estimates and a uniform time of existence for Navier-Stokes when the viscosity goes to zero have been recently obtained (see [39, 5, 4, 6]). As we shall see below, for these special boundary conditions, the main part of the boundary layer vanishes which allows this uniform control in some limited regularity Sobolev space.

Here, our approach can be seen as intermediate between these two cases since we shall get strong solutions but controlling many tangential derivatives and only one normal derivative. This control is compatible with the presence of a boundary layer when the viscosity goes to zero.

To understand, the difficulties in the presence of boundaries, one can use formal boundary layer expansions. The solution u^ε of (1.1), (1.2) is expected to have the following expansion

$$(1.4) \quad u^\varepsilon(t, x) = u(t, x) + \sqrt{\varepsilon} V(t, y, z/\sqrt{\varepsilon}) + \mathcal{O}(\varepsilon)$$

(we assume that $(y, z) \in \Omega = \mathbb{R}^2 \times (0, +\infty)$ to simplify this heuristic part) where V is a smooth profile which is fastly decreasing in its last variable. Note that the rigorous constuction of such expansions have been performed in [16] where it was also proven that the remainder is indeed $\mathcal{O}(\varepsilon)$ in L^2 . With such an expansion, we immediately get that in the simplest space where the 3-D Euler equation is well-posed, namely H^s , $s > 5/2$ the norm of u^ε cannot be uniformly bounded because of the profile V . For some special Navier boundary conditions considered in [39, 5, 4, 6], the leading profile V vanishes and hence uniform H^3 or $W^{2,p}$, $p > 3$ estimates have been obtained. Nevertheless, as pointed out in [16], in the generic case, V does not vanish.

We shall prove in this paper that in the general case, we can indeed achieve the above program by working in anisotropic conormal Sobolev spaces. Again, because of (1.4), we can hope a uniform control of one normal derivative of the solution in L^∞ and thus a control of the Lipschitz norm of the solution hence it seems reasonable to be able to recover in the limit the well-posedness of the Euler equation. The situation is thus also different from the case of "non-characteristic" Dirichlet condition where boundary layers are of size ε but of amplitude 1. In this situation, one can prove in some stable cases the L^2 convergence, but since strong compactness in the normal variable cannot be expected, the proof uses in a crucial way the construction of an asymptotic expansion and the control of the remainder. We refer for example to [38], [10], [9], [32], [11, 24], [12], [31], [27], [30]. The drawbacks of this approach are that it requires the a priori knowledge of the well-posedness of the limit problem and that it requires the solution of the limit problem to be smoother than the one of the viscous problem (which is not very natural). Finally, let us mention that for some problems where only the normal viscosity vanishes, it is also possible to use weak compactness arguments, [33].

Our aim here is to prove that in a situation where the formal expansion is under the form (1.4), one can get strong solutions of the viscous and the inviscid problem in the same appropriate functional framework and justify the vanishing viscosity limit by a strong compactness argument. In some sense, we want to use on a boundary layer problem the same approach that is classically used in singular oscillatory limits (as the compressible-incompressible limit, see [21], [29] for example) where the existence of a strong solution on an interval of time independent of the small parameter is first proven and the convergence studied in a second step. To go further in the analogy, we can

think of boundary layer problems with formal expansions as (1.4) as analogous to well-prepared problems.

We consider a domain $\Omega \subset \mathbb{R}^3$ such that there exists a covering of Ω under the form $\Omega \subset \Omega_0 \cup_{i=1}^n \Omega_i$ where $\overline{\Omega_0} \subset \Omega$ and in each Ω_i , there exists a function ψ_i such that $\Omega \cap \Omega_i = \{(x = (x_1, x_2, x_3), x_3 > \psi_i(x_1, x_2)) \cap \Omega_i$ and $\partial\Omega \cap \Omega_i = \{x_3 = \psi_i(x_1, x_2)\} \cap \Omega_i$. We say that Ω is \mathcal{C}^m if the functions ψ_i are \mathcal{C}^m .

To define Sobolev conormal spaces, we consider $(Z_k)_{1 \leq k \leq N}$ a finite set of generators of vector fields that are tangent to $\partial\Omega$ and we set

$$H_{co}^m(\Omega) = \{f \in L^2(\Omega), \quad Z^I \in L^2(\Omega), \quad |I| \leq m\}$$

where for $I = (k_1, \dots, k_m)$,

$$Z^I = Z_{k_1} \cdots Z_{k_m}.$$

We also set

$$\|f\|_m^2 = \sum_{|I| \leq m} \|Z^I f\|_{L^2}^2.$$

For a vector field, u , we shall say that u is in $H_{co}^m(\Omega)$ if each of its components are in H_{co}^m and thus

$$\|u\|_m^2 = \sum_{i=1}^3 \sum_{|I| \leq m} \|Z^I u_i\|_{L^2}^2.$$

In the same way, we set

$$\|u\|_{k,\infty} = \sum_{|I| \leq m} \|Z^I u\|_{L^\infty}$$

and we say that $u \in W_{co}^{k,\infty}$ if $\|u\|_{k,\infty}$ is finite.

Throughout the paper, we shall denote by $\|\cdot\|_{W^{k,\infty}}$ the usual Sobolev norm and use the notations $\|\cdot\|$ and (\cdot, \cdot) for the L^2 norms and scalar products.

Note that the $\|\cdot\|_m$ norm yields inside Ω a control of the standard H^m norm, whereas close to the boundary, there is no control of the normal derivatives. The use of conormal Sobolev spaces has a long history in (hyperbolic) boundary value problems, we refer for example to [14], [36], [2], [13, 17] and references therein.

Let us set $E^m = \{u \in H_{co}^m, \nabla u \in H_{co}^{m-1}\}$. Our main result is the following:

Theorem 1.1. *Let m be an integer satisfying $m > 6$ and Ω be a \mathcal{C}^{m+2} domain. Consider $u_0 \in E^m$ such that $\nabla u_0 \in W_{co}^{1,\infty}$ and $\nabla \cdot u_0 = 0$, $u_0 \cdot n_{\partial\Omega} = 0$. Then, there exists $T > 0$ such that for every $\varepsilon \in (0, 1)$ and α , $|\alpha| \leq 1$, there exists a unique $u^\varepsilon \in \mathcal{C}([0, T], E^m)$ such that $\|\nabla u^\varepsilon\|_{1,\infty}$ is bounded on $[0, T]$ solution of (1.1), (1.2) with initial data u_0 . Moreover, there exists $C > 0$ independent of ε and α such that*

$$(1.5) \quad \sup_{[0, T]} \left(\|u(t)\|_m + \|\nabla u(t)\|_{m-1} + \|\nabla u(t)\|_{1,\infty} \right) + \varepsilon \int_0^T \|\nabla^2 u(s)\|_{m-1}^2 ds \leq C.$$

Note that the uniqueness part is obvious since we work with functions with Lipschitz regularity. The fact that we need to control $\|\nabla u\|_{1,\infty}$ and not only the Lipschitz norm is classical in characteristic hyperbolic problems when one tries to work with the minimal normal regularity, we refer for example to [13]. The same remark holds for the required regularity, the same restriction on m holds in the case of general characteristic hyperbolic problems studied in [13]. It is maybe possible to improve this by using more precisely the structure of the incompressible equations. The fact that we need to control $m - 1$ conormal derivatives for $\partial_n u$ and not only $m - 2$ is linked to the control of the pressure in our incompressible framework. The regularity of the domain that we require, is also mainly due to the estimate of the pressure, this is the classical regularity in order to estimate the

pressure in the Euler equation (see [37] for example). Another important remark is that in proving Theorem 1.1, we get a uniform existence time for the solution of (1.1), (1.2) without using that there exists a solution of the Euler equation. In particular, we shall get by passing to the limit that the Euler equation is well-posed in the same functional framework. We hope to be able to use this approach on more complicated problems where it is much easier to prove the local well-posedness for the viscous problem than for the inviscid one. Finally, it is also possible to prove that in the case that the initial data is H^s and satisfies some suitable compatibility conditions, we can deduce from the estimate (1.5) and the regularity result for the Stokes problem that u is in the standard H^s Sobolev space on $[0, T]$. Nevertheless, higher order normal derivatives will not be uniformly bounded in ε .

The main steps of the proof of Theorem 1.1 are the following. We shall first get a conormal energy estimate in H_{co}^m for the velocity u which is valid as long as the Lipschitz norm of the solution is controlled. The second step is to estimate $\|\partial_n u\|_{m-1}$. In order to get this estimate by an energy method, $\partial_n u$ is not a convenient quantity since it does not vanish on the boundary. Nevertheless, we observe that $\partial_n u \cdot n$ can be immediately controlled thanks to the control of u in H_{co}^m and the incompressibility condition. Moreover, due to the Navier condition (1.2), it is convenient to study $\eta = (Sun + \alpha u)_\tau$. Indeed it vanishes on the boundary and gives a control of $(\partial_n u)_\tau$. We shall thus prove a control of $\|\eta\|_{m-1}$ by performing energy estimates on the equation solved by η . This estimate will be valid as long as $\|\nabla u(t)\|_{1,\infty}$ remains bounded. The third step is to estimate the pressure. Indeed, since the conormal fields Z_i do not commute with the gradient, the pressure is not transparent in the estimates. We shall prove that the pressure can be split into two parts, the first one has the same regularity as in the Euler equation and the second part is linked to the Navier condition. Finally, the last step is to estimate $\|\nabla u(t)\|_{1,\infty}$ and actually $\|(\partial_n u)_\tau\|_{1,\infty}$ since the other terms can be controlled by Sobolev embedding. To perform this estimate we shall again choose an equivalent quantity which satisfies an homogeneous Dirichlet condition and solves at leading order a convection diffusion equation. The estimate will be obtained by using the fundamental solution of an approximate equation.

Once Theorem 1.1 is obtained, we can easily get the inviscid limit:

Theorem 1.2. *Let m be an integer satisfying $m > 6$ and Ω be a C^{m+2} domain. Consider $u_0 \in E^m$ such that $\nabla u_0 \in W_{co}^{1,\infty}$, $\nabla \cdot u_0 = 0$, $u_0 \cdot n_{\partial\Omega} = 0$ and u^ε the solution of (1.1), (1.2) with initial value u_0 given by Theorem 1.1. Then there exists a unique solution to the Euler system (1.3), $u \in L^\infty(0, T, E^m)$ such that $\|\nabla u\|_{1,\infty}$ is bounded on $[0, T]$ and such that*

$$\sup_{[0, T]} (\|u^\varepsilon - u\|_{L^2} + \|u^\varepsilon - u\|_{L^\infty}) \rightarrow 0$$

when ε tends to zero.

We shall obtain Theorem 1.2 by a classical strong compactness argument. Note that the L^∞ convergence was not obtained in [15], [16]. It does not seem possible to get such a convergence thanks to a modulated energy type argument.

Note that if Ω is not bounded the above convergences hold on every compact of $\bar{\Omega}$.

The paper is organized as follows: in section 3, we shall first explain the main steps of the proof of Theorem 1.1 in the simpler case where Ω is the half-space $\mathbb{R}^2 \times (0, +\infty)$. This allows to present the analytical part of the proof without complications coming from the geometry of the domain. The general case will be treated in section 4. Finally section 6 is devoted to the proof of Theorem 1.2.

2. A FIRST ENERGY ESTIMATE

In this section, we first recall the basic a priori L^2 energy estimate which holds for (1.1), (1.2).

Proposition 2.1. Consider a (smooth) solution of (1.1), (1.2), then we have for every $\varepsilon > 0$ and $\alpha \in \mathbb{R}$,

$$\frac{d}{dt} \left(\frac{1}{2} \|u\|^2 \right) + 2\varepsilon \|Su\|^2 + 2\alpha \varepsilon |u_\tau|_{L^2(\partial\Omega)}^2 = 0.$$

Proof. By using (1.1), we obtain:

$$\frac{d}{dt} \left(\frac{1}{2} \|u\|^2 \right) = (\varepsilon \Delta u, u) - (\nabla p, u) - (u \cdot \nabla u, u)$$

where (\cdot, \cdot) stands for the L^2 scalar product. Next, thanks to integration by parts and the boundary condition (1.2), we find

$$\begin{aligned} (\nabla p, u) &= \int_{\partial\Omega} p u \cdot n - \int_{\Omega} p \nabla \cdot u = 0, \\ (u \cdot \nabla u, u) &= \int_{\partial\Omega} \frac{|u|^2}{2} u \cdot n = 0, \\ (\varepsilon \Delta u, u) &= 2\varepsilon (\nabla \cdot Su, u) = -2\varepsilon \|Su\|^2 + 2\varepsilon \int_{\partial\Omega} ((Su) \cdot n) \cdot u \, d\sigma. \end{aligned}$$

Finally, we get from the boundary condition (1.2) that

$$\int_{\partial\Omega} (Su \cdot n) \cdot u = \int_{\partial\Omega} ((Su) \cdot n)_\tau \cdot u_\tau = -\alpha \int_{\partial\Omega} |u_\tau|^2 \, d\sigma.$$

□

Remark 2.2. Note that if Ω is a Lipschitz domain, we get from the Korn inequality that for some $C_\Omega > 0$, we have for every H^1 vector field u which is tangent to the boundary that

$$\|\nabla u\|^2 \leq C_\Omega (\|Su\|^2 + \|u\|^2).$$

Consequently, we deduce from Proposition 2.1 that

$$(2.1) \quad \frac{d}{dt} \left(\frac{1}{2} \|u\|^2 \right) + \varepsilon c_\Omega \|\nabla u\|^2 + \alpha \varepsilon |u_\tau|_{L^2(\partial\Omega)}^2 \leq \varepsilon C_\Omega \|u\|^2.$$

If $\alpha \geq 0$, this always provides a good energy estimate.

Remark 2.3. Even if $\alpha \leq 0$, we get from the trace Theorem that there exists $C > 0$ independent of ε such that

$$|u_\tau|_{L^2(\partial\Omega)}^2 \leq C \|\nabla u\| \|u\| + \|u\|^2$$

and hence, we find by using the Young inequality

$$(2.2) \quad ab \leq \delta a^2 + \frac{1}{4\delta} b^2, \quad a, b \geq 0, \delta > 0$$

that

$$(2.3) \quad \frac{d}{dt} \left(\frac{1}{2} \|u\|^2 \right) + \frac{\varepsilon}{2} c_\Omega \|\nabla u\|^2 + \varepsilon |u_\tau|_{L^2(\partial\Omega)}^2 \leq 2C^2 \varepsilon (\alpha^2 + 1) \|u\|^2.$$

Consequently, if α is such that $\varepsilon \alpha^2 \leq 1$, we still get a uniform L^2 estimate from the Gronwall Lemma.

3. THE CASE OF A HALF-SPACE: $\Omega = \mathbb{R}_+^3$

In order to avoid complications due to the geometry of the domain in the obtention of higher order energy estimates, we shall first give the proof of Theorem 1.1 in the case where Ω is the half space $\Omega = \mathbb{R}^2 \times (0, +\infty)$. We shall use the notation $x = (y, z)$, $z > 0$ for a point x in Ω . To define the conormal Sobolev spaces, it suffices to use $Z_i = \partial_i$, $i = 1, 2$ and $Z_3 = \varphi(z)\partial_z$ where $\varphi(z)$ is any smooth bounded function such that $\varphi(0) = 0$, $\varphi'(0) \neq 0$ and $\varphi(z) > 0$ for $z > 0$ (for example, $\varphi(z) = z(1+z)^{-1}$ fits). Consequently, we have

$$\|u\|_m^2 = \sum_{|\alpha| \leq m} \|Z^\alpha u\|_{L^2}^2, \quad \|u\|_{k,\infty}^2 = \sum_{|\alpha| \leq k} \|Z^\alpha u\|_{L^\infty}^2$$

where $Z^\alpha = Z_1^{\alpha_1} Z_2^{\alpha_2} Z_3^{\alpha_3} u$.

Throughout this section, we shall focus on the proof of a priori estimates for a sufficiently smooth solution of (1.1), (1.2) in order to get (1.5). We use the symbol \lesssim for $\leq C$ where C is a positive number which may change from line to line but which is independent of ε and α for $\varepsilon \in (0, 1)$ and $|\alpha| \leq 1$.

The aim of this section is to prove the following a priori estimate in the case of the half space which is the crucial part towards the proof of Theorem 1.1.

Theorem 3.1. *For $m > 6$, there exists $C > 0$ independent of $\varepsilon \in (0, 1]$ and α , $|\alpha| \leq 1$ such that for every sufficiently smooth solution defined on $[0, T]$ of (1.1), (1.2) in $\Omega = \mathbb{R}^2 \times (0, +\infty)$, we have the a priori estimate*

$$N_m(t) \leq C \left(N_m(0) + (1+t + \varepsilon^3 t^2) \int_0^t (N_m(s) + N_m(s)^2) ds \right), \quad \forall t \in [0, T]$$

where

$$N_m(t) = \|u(t)\|_m^2 + \|\nabla u(t)\|_{m-1}^2 + \|\nabla u\|_{1,\infty}^2.$$

3.1. Conormal energy estimate.

Proposition 3.2. *For every $m \geq 0$, a smooth solution of (1.1), (1.2) satisfies the estimate*

$$\begin{aligned} & \frac{d}{dt} \|u(t)\|_m^2 + c_0 \varepsilon \int_0^t \|\nabla u\|_m^2 \\ & \lesssim \|\nabla p\|_{m-1} \|u\|_m + (1 + \|u\|_{W^{1,\infty}}) (\|u\|_m^2 + \|\partial_z u\|_{m-1}^2) \end{aligned}$$

for some $c_0 > 0$ independent of ε .

Proof of Proposition 3.2. In the proof, we shall use the notation $x = (y, z) \in \mathbb{R}^{d-1} \times (0, +\infty)$, $u = (u_h, u_3) \in \mathbb{R}^{d-1} \times \mathbb{R}$.

The case $m = 0$ just follows from Proposition 2.1 and Remark 2.3, and the term containing the pressure does not show up. By induction, let us assume that it is proven for $k \leq m-1$. By applying Z^α to (1.1) for $|\alpha| = m$, we find

$$(3.1) \quad \partial_t Z^\alpha u + u \cdot \nabla Z^\alpha u + \nabla Z^\alpha p = \varepsilon \Delta Z^\alpha u + \mathcal{C}$$

where the term \mathcal{C} involving commutators can be written as

$$\mathcal{C} = \sum_{i=1}^3 \mathcal{C}_i$$

where

$$(3.2) \quad \mathcal{C}_1 = -[Z^\alpha, u \cdot \nabla]u, \quad \mathcal{C}_2 = -[Z^\alpha, \nabla]p, \quad \mathcal{C}_3 = \varepsilon[Z^\alpha, \Delta]u.$$

From the divergence free condition in (1.1), we get

$$(3.3) \quad \nabla \cdot Z^\alpha u = \mathcal{C}_d, \quad \mathcal{C}_d = -[Z^\alpha, \nabla \cdot]u.$$

Finally, let us notice that from the boundary condition (1.2) which reads explicitly in the case of a half-space

$$(3.4) \quad u_3 = 0, \quad \partial_z u_h = 2\alpha u_h, \quad x \in \partial\Omega,$$

we get

$$(3.5) \quad Z^\alpha u_3 = 0, \quad \partial_z Z^\alpha u_h = 2\alpha Z^\alpha u_h + \mathcal{C}_b, \quad \mathcal{C}_b = -([\partial_z, Z^\alpha]u_h)_{/\partial\Omega}, \quad x \in \partial\Omega.$$

As in the proof of Proposition 2.1 and Remark 2.3, we get from the standard energy estimate and the boundary condition (3.5):

$$(3.6) \quad \begin{aligned} & \frac{d}{dt} \left(\frac{1}{2} \|Z^\alpha u\|^2 \right) + \varepsilon \|\nabla Z^\alpha u\|^2 + \varepsilon |Z^\alpha u_h|_{L^2(\partial\Omega)}^2 \\ & \lesssim |(\mathcal{C}, Z^\alpha u)| + |(Z^\alpha p, \mathcal{C}_d)| + \varepsilon |\mathcal{C}_b|_{L^2(\partial\Omega)} |Z^\alpha u_h|_{L^2(\partial\Omega)} + \|u\|_m^2. \end{aligned}$$

Indeed, since $\nabla \cdot u = 0$, $u_3 = 0$, and $Z^\alpha u_3 = 0$ on $\partial\Omega$, we get that

$$(u \cdot \nabla Z^\alpha u, Z^\alpha u) = 0, \quad (\nabla Z^\alpha p, Z^\alpha u) = -(Z^\alpha p, \mathcal{C}_d).$$

Note that when $\partial\Omega$ is not flat, the boundary condition (1.2) does not imply that $Z^\alpha u \cdot n = 0$ on $\partial\Omega$ thus a boundary term shows up in the integration by parts and hence the estimate for the term involving the pressure will be worse (see the next section).

To estimate the last term in the right-hand side of (3.6), we can use as in Remark 2.3 the trace theorem and the Young inequality to get

$$\varepsilon |\mathcal{C}_b|_{L^2(\partial\Omega)} |Z^\alpha u_h|_{L^2(\partial\Omega)} \leq \frac{1}{2} \varepsilon \|\nabla Z^\alpha u\|^2 + C \|u\|_m^2 + C \varepsilon |\mathcal{C}_b|_{L^2(\partial\Omega)}^2$$

and hence, we find

$$(3.7) \quad \frac{d}{dt} \left(\frac{1}{2} \|Z^\alpha u\|^2 \right) + \frac{1}{2} \varepsilon \|\nabla Z^\alpha u\|^2 + \varepsilon |Z^\alpha u_h|_{L^2(\partial\Omega)}^2 \lesssim \|u\|_m^2 + |(\mathcal{C}, Z^\alpha u)| + \varepsilon |\mathcal{C}_b|_{L^2(\partial\Omega)}^2 + |(Z^\alpha p, \mathcal{C}_d)|.$$

To conclude, we need to estimate the commutators. First, since $[Z_3, \nabla \cdot]u = -\varphi' \partial_z u_3 = \varphi' \nabla_h \cdot u_h$ (thanks to the divergence free condition) and $[Z_i, \nabla \cdot] = 0$ for $i = 1, \dots, d-1$, we easily get that for $m \geq 1$,

$$(3.8) \quad \|\mathcal{C}_d\| \lesssim \|u\|_m$$

and hence, we obtain

$$(3.9) \quad |(Z^\alpha p, \mathcal{C}_d)| \lesssim \|u\|_m \|\nabla p\|_{m-1}.$$

We also get that

$$([\partial_z, Z_i]u_h)_{/\partial\Omega} = 0, \quad ([\partial_z, Z_3]u_h)_{/\partial\Omega} = -(\varphi' \partial_z u_h)_{/\partial\Omega}$$

since φ vanishes on the boundary. Therefore, from (3.4), we get

$$([\partial_z, Z_3]u_h)_{/\partial\Omega} = -2\alpha(\varphi' u_h)_{/\partial\Omega}.$$

By using this last property and the fact that φ vanishes on the boundary, we find

$$|\mathcal{C}_b|_{L^2(\partial\Omega)} \lesssim |u_{/\partial\Omega}|_{H^{m-1}(\partial\Omega)}$$

and hence, we get from the trace theorem that

$$(3.10) \quad \varepsilon |\mathcal{C}_b|_{L^2(\partial\Omega)}^2 \lesssim \varepsilon \|\partial_z u\|_{m-1} \|u\|_{H^{m-1}(\partial\Omega)}.$$

It remains to estimate \mathcal{C} . First, we observe that

$$(3.11) \quad \|\mathcal{C}_2\| \lesssim \|\nabla p\|_{m-1}.$$

Next, since we have

$$[Z_i, \Delta] = 0, \quad [Z_3, \Delta]u = -2\varphi' \partial_{zz}u - \varphi'' \partial_z u,$$

we also get by using repeatedly this property that

$$|(\mathcal{C}_3, Z^\alpha u)| \lesssim \tilde{\mathcal{C}}_3 + \varepsilon \|\partial_z u\|_{m-1} \|u\|_m + \|u\|_m^2$$

where $\tilde{\mathcal{C}}_3$ is given by

$$\tilde{\mathcal{C}}_3 = \sum_{\beta, 0 \leq |\beta| \leq m-1} \varepsilon |(c_\beta \partial_{zz} Z_3^\beta u, Z^\alpha u)|$$

for some harmless functions c_β depending on derivatives of φ . To estimate $\tilde{\mathcal{C}}_3$, we use integration by parts. If $\beta \neq 0$, $|\beta| \neq 1$, since φ vanishes on the boundary, we immediately get that

$$\varepsilon |(c_\beta \partial_{zz} Z_3^\beta u, Z^\alpha u)| \lesssim \varepsilon (\|\partial_z u\|_m + \|u\|_m) \|\partial_z u\|_{m-1}.$$

For $\beta = 0$ or $|\beta| = 1$, there is an additional term on the boundary, we have

$$|(c_\beta \partial_{zz} Z_3^\beta u, Z^\alpha u)| \lesssim \varepsilon (\|\partial_z u\|_m + \|u\|_m) \|\partial_z u\|_{m-1} + \varepsilon |\partial_z u_h|_{L^2(\partial\Omega)} |Z^\alpha u|_{L^2(\partial\Omega)}.$$

From the boundary condition (3.4) and the trace theorem, we also find

$$\varepsilon |\partial_z u_h|_{L^2(\partial\Omega)} |Z^\alpha u|_{L^2(\partial\Omega)} \lesssim \varepsilon |u|_{L^2(\partial\Omega)} |Z^\alpha u|_{L^2(\partial\Omega)} \lesssim \varepsilon \|\partial_z u\|_m \|u\|_m.$$

We have consequently proven that

$$(3.12) \quad |(\mathcal{C}_3, Z^\alpha u)| \lesssim \varepsilon \|\partial_z u\|_m (\|\partial_z u\|_{m-1} + \|u\|_m) + \|u\|_m^2 + \|\partial_z u\|_{m-1}^2.$$

It remains to estimate \mathcal{C}_1 . By an expansion, we find that \mathcal{C}_1 is under the form

$$(3.13) \quad \mathcal{C}_1 = \sum_{\beta+\gamma=\alpha, \beta \neq 0} c_{\beta,\gamma} Z^\beta u \cdot Z^\gamma \nabla u + u \cdot [Z^\alpha, \nabla]u.$$

To estimate the last term, we first observe that

$$(3.14) \quad \|u \cdot [Z^\alpha, \nabla]u\| \lesssim \sum_{|\beta| \leq m-1} \|u_3 \partial_z Z^\beta u\|$$

and then that because of the first boundary condition in (3.4) we have

$$|u_3(t, x)| \leq \varphi(z) \|u_3\|_{W^{1,\infty}}.$$

This yields

$$(3.15) \quad \|u \cdot [Z^\alpha, \nabla]u\| \lesssim \|u_3\|_{W^{1,\infty}} \|u\|_m.$$

To estimate the other terms, we can use the following generalized Sobolev-Gagliardo-Nirenberg inequality, we refer for example to [13] for the proof:

Lemma 3.3. *For $u, v \in L^\infty \cap H_{co}^k$, we have*

$$(3.16) \quad \|Z^{\alpha_1} u Z^{\alpha_2} v\| \lesssim \|u\|_{L^\infty} \|v\|_k + \|v\|_{L^\infty} \|u\|_k, \quad |\alpha_1| + |\alpha_2| = k.$$

For $\beta \neq 0$, this immediately yields

$$(3.17) \quad \begin{aligned} \|c_{\beta,\gamma} Z^\beta u \cdot Z^\gamma \nabla u\| &\lesssim \|Z^\beta u_h \cdot Z^\gamma \nabla_h u\| + \|Z^\beta u_3 \cdot Z^\gamma \partial_z u\| \\ &\lesssim \|Zu\|_{L^\infty} \|u\|_m + \|Zu\|_{L^\infty} \|\partial_z u\|_{m-1} + \|\partial_z u\|_{L^\infty} \|Zu_3\|_{m-1} \\ &\lesssim \|\nabla u\|_{L^\infty} (\|u\|_m + \|\partial_z u\|_{m-1}) \end{aligned}$$

and hence, we find the estimate

$$(3.18) \quad \|\mathcal{C}_1\| \lesssim \|\nabla u\|_{L^\infty} (\|u\|_m + \|\partial_z u\|_{m-1}).$$

From (3.7) and (3.9), (3.10), (3.11), (3.12), (3.18) and the remark (2.3), we find

$$\begin{aligned} & \frac{d}{dt} \left(\frac{1}{2} \|u\|_m^2 \right) + \frac{1}{2} \varepsilon \|\nabla u\|_m^2 \lesssim \varepsilon \|\partial_z u\|_m (\|\partial_z u\|_{m-1} + \|u\|_m) \\ & + \|u\|_m^2 + \|\nabla p\|_{m-1} \|u\|_m + (1 + \|u\|_{W^{1,\infty}}) (\|u\|_m^2 + \|\partial_z u\|_{m-1}^2). \end{aligned}$$

To get the result, it suffices to use the Young inequality to absorb the term $\varepsilon \|\partial_z u\|_m$ in the left hand side. This ends the proof of Proposition 3.2.

3.2. Normal derivative estimates. In this section, we shall provide an estimate for $\|\partial_z u\|_{m-1}$.

A first useful remark is that because of the divergence free condition we have

$$(3.19) \quad \|\partial_z u_3\|_{m-1} \leq \|u\|_m.$$

Consequently, it suffices to estimate $\partial_z u_h$. Let us introduce the vorticity

$$\omega = \operatorname{curl} u = \begin{pmatrix} \partial_2 u_3 - \partial_3 u_2 \\ \partial_3 u_1 - \partial_1 u_3 \\ \partial_1 u_2 - \partial_2 u_1 \end{pmatrix}$$

which solves

$$(3.20) \quad \partial_t \omega + u \cdot \nabla \omega - \omega \cdot \nabla u = \varepsilon \Delta \omega, \quad x \in \Omega.$$

On the boundary, we find thanks to (3.4) that

$$\omega_h = 2\alpha u_h^\perp, \quad x \in \partial\Omega.$$

This leads us to introduce the unknown

$$\eta = \omega_h - 2\alpha u_h^\perp.$$

Indeed, the main advantages of this quantity is that on the boundary, we have

$$(3.21) \quad \eta = 0, \quad x \in \partial\Omega$$

and that we have the estimate

$$(3.22) \quad \|\partial_z u_h\|_{m-1} \lesssim \|u\|_m + \|\eta\|_{m-1}.$$

Consequently, we shall estimate in this section $\|\eta\|_{m-1}$. We have the following result:

Proposition 3.4. *For every $m \geq 1$, every smooth solution of (1.1), (1.2), satisfies the following estimate :*

$$\begin{aligned} & \frac{d}{dt} \|\eta(t)\|_{m-1}^2 + c_0 \varepsilon \|\nabla \eta\|_{m-1}^2 \\ & \lesssim \|\nabla p\|_{m-1} \|\eta\|_{m-1} + (1 + \|u\|_{2,\infty} + \|\partial_z u\|_{1,\infty}) (\|\eta\|_{m-1}^2 + \|u\|_m^2) \end{aligned}$$

Proof of Proposition 3.4. From the definition of η , we find that it solves the equation

$$(3.23) \quad \partial_t \eta + u \cdot \nabla \eta - \varepsilon \Delta \eta = \omega \cdot \nabla u_h + 2\alpha \nabla_h^\perp p$$

with the boundary condition (3.21). By a standard L^2 energy estimate, we find

$$\frac{d}{dt} \frac{1}{2} \|\eta(t)\|^2 + \varepsilon \|\nabla \eta\|^2 \lesssim \left(\|\nabla p\| \|\eta\| + \|\omega \cdot \nabla u_h\| \|\eta\| \right).$$

Furthermore, by using that

$$\|\omega \cdot \nabla u_h\| \lesssim \|\nabla u\|_{L^\infty} \|\omega\| \lesssim \|\nabla u\|_{L^\infty} (\|\eta\| + \|u\|_1),$$

we find the result for $m = 1$.

Now, let us assume that Proposition (3.4) is proven for $k \leq m - 2$. We shall now estimate $\|\eta\|_{m-1}$. By applying Z^α for $|\alpha| = m - 1$ to (3.23), we find

$$(3.24) \quad \partial_t Z^\alpha \eta + u \cdot \nabla Z^\alpha \eta - \varepsilon \Delta Z^\alpha \eta = Z^\alpha (\omega \cdot \nabla u_h) + 2\alpha Z^\alpha \nabla_h^\perp p + \mathcal{C}$$

where \mathcal{C} is the commutator:

$$\mathcal{C} = \mathcal{C}_1 + \mathcal{C}_2, \quad \mathcal{C}_1 = [Z^\alpha, u \cdot \nabla] \eta, \quad \mathcal{C}_2 = -\varepsilon [Z^\alpha, \Delta] \eta.$$

Since $Z^\alpha \eta$ vanishes on the boundary, the standard L^2 energy estimate for (3.24) yields

$$(3.25) \quad \begin{aligned} & \frac{d}{dt} \frac{1}{2} \|\eta(t)\|_{m-1}^2 + \varepsilon \|\nabla \eta\|_{m-1}^2 \\ & \lesssim \|\nabla p\|_{m-1} \|\eta\|_{m-1} + \|\omega \cdot \nabla u_h\|_{m-1} \|\eta\|_{m-1} + |(\mathcal{C}, Z^\alpha \eta)|. \end{aligned}$$

To estimate the terms in the right-hand side, we first write thanks to Lemma 3.3 that

$$(3.26) \quad \begin{aligned} \|\omega \cdot \nabla u_h\|_{m-1} & \lesssim \|\omega\|_{L^\infty} (\|u_h\|_m + \|\partial_z u_h\|_{m-1}) + \|\nabla u_h\|_{L^\infty} \|\omega\|_{m-1} \\ & \lesssim \|\nabla u\|_{L^\infty} (\|u_h\|_m + \|\eta\|_{m-1}). \end{aligned}$$

Note that we have again used (3.22) to get the last line.

Next, we need to estimate the commutator \mathcal{C} . As for (3.12), we first get from integration by parts since $Z^\alpha \eta$ vanishes on the boundary that

$$(3.27) \quad |(\mathcal{C}_2, Z^\alpha \eta)| \lesssim \varepsilon \|\partial_z \eta\|_{m-1} (\|\partial_z \eta\|_{m-2} + \|\eta\|_{m-1}) + \|\eta\|_{m-1}^2.$$

It remains to estimate \mathcal{C}_1 which is the most difficult term. We can again write

$$\mathcal{C}_1 = \sum_{\beta+\gamma=\alpha, \beta \neq 0} c_{\beta, \gamma} Z^\beta u \cdot Z^\gamma \nabla \eta + u \cdot [Z^\alpha, \nabla] \eta.$$

To estimate the last term, we first observe that

$$\|u \cdot [Z^\alpha, \nabla] \eta\| \lesssim \sum_{k \leq m-2} \|u_3 \partial_z Z_3^k \eta\|$$

and by using again that

$$(3.28) \quad |u_3(t, x)| \leq \varphi(z) \|u_3\|_{W^{1, \infty}},$$

we find

$$\|u \cdot [Z^\alpha, \nabla] \eta\| \lesssim \|u_3\|_{W^{1, \infty}} \|\eta\|_{m-1}.$$

To estimate the other terms in the commutator, we write

$$\|c_{\beta, \gamma} Z^\beta u \cdot Z^\gamma \nabla \eta\| \lesssim \|Z^\beta u_h \cdot Z^\gamma \nabla_h \eta\| + \|Z^\beta u_3 Z^\gamma \partial_z \eta\|.$$

Thanks to Lemma 3.3, we have since $\beta \neq 0$ that

$$\|Z^\beta u_h \cdot Z^\gamma \nabla_h \eta\| \lesssim \|\nabla u\|_{L^\infty} \|\eta\|_{m-1} + \|\eta\|_{L^\infty} \|Z u\|_{m-2} \lesssim \|\nabla u\|_{L^\infty} (\|\eta\|_{m-1} + \|u\|_m).$$

The remaining term is the most involved. We want to get an estimate for which $\partial_z \eta$ does not appear. Indeed, due to the expected behaviour in the boundary layer (1.4), one cannot hope an estimate which is uniform in ε for $\|\partial_z \eta\|_{L^\infty}$ or $\|\partial_z \eta\|_m$. We first write

$$Z^\beta u_3 Z^\gamma \partial_z \eta = \frac{1}{\varphi(z)} Z^\beta u_3 \varphi(z) Z^\gamma \partial_z \eta$$

and then we can expand this term as a sum of terms under the form

$$c_{\tilde{\beta}, \tilde{\gamma}} Z^{\tilde{\beta}} \left(\frac{1}{\varphi(z)} u_3 \right) Z^{\tilde{\gamma}} (\varphi \partial_z \eta)$$

where $\tilde{\beta} + \tilde{\gamma} \leq m - 1$, $|\tilde{\gamma}| \neq m - 1$ and $c_{\tilde{\beta}, \tilde{\gamma}}$ is some smooth bounded coefficient.

Indeed, we first notice that $Z^\alpha \varphi$ has the same properties than φ , thus the commutator $[\varphi, Z^\gamma]$ can be expanded under the form $\tilde{\varphi}_{\tilde{\gamma}} Z^{\tilde{\gamma}}$ with $|\tilde{\gamma}| < |\gamma|$ where $\tilde{\varphi}_{\tilde{\gamma}}$ have the same properties as φ . Then, we can write

$$\tilde{\varphi}_{\tilde{\gamma}} Z^{\tilde{\gamma}} = \frac{\tilde{\varphi}_{\tilde{\gamma}}}{\varphi} \left(Z^{\tilde{\gamma}}(\varphi \cdot) + [\varphi, Z^{\tilde{\gamma}}] \right)$$

where the coefficient $\tilde{\varphi}_{\tilde{\gamma}}/\varphi$ is smooth and bounded. Finally, we reiterate the process to express the commutators $[\varphi, Z^{\tilde{\gamma}}]$. Hence, after a finite number of steps, we indeed get that $[\varphi, Z^\gamma]$ can be expanded as a sum of terms under the form $c_{\tilde{\gamma}} Z^{\tilde{\gamma}}(\varphi \cdot)$ where $c_{\tilde{\gamma}}$ is smooth and bounded. In a similar way, we note that $Z^\alpha(1/\varphi)$ has the same properties as $1/\varphi$ and hence, by the same argument, we get that the commutator $[1/\varphi, Z^\beta]$ can be expanded as a sum of terms under the form $c_{\tilde{\beta}} Z^{\tilde{\beta}}(\frac{1}{\varphi} \cdot)$.

If $\tilde{\beta} = 0$, and hence $|\tilde{\gamma}| \leq m - 2$, we have

$$\|Z^{\tilde{\beta}}(\frac{1}{\varphi(z)} u_3) Z^{\tilde{\gamma}} Z_3 \eta\| \lesssim \|\frac{1}{\varphi(z)} u_3\|_{L^\infty} \|\eta\|_{m-1}.$$

Moreover, since u_3 vanishes on the boundary, we have

$$\|\frac{1}{\varphi(z)} u_3\|_{L^\infty} \lesssim \|u\|_{W^{1,\infty}}.$$

We have thus proven that for $\tilde{\beta} = 0$

$$\|Z^{\tilde{\beta}}(\frac{1}{\varphi(z)} u_3) Z^{\tilde{\gamma}} Z_3 \eta\| \lesssim \|u\|_{W^{1,\infty}} \|\eta\|_{m-1}.$$

Next, for $\tilde{\beta} \neq 0$, we can use Lemma 3.3 to get

$$\|Z^{\tilde{\beta}}(\frac{1}{\varphi(z)} u_3) Z^{\tilde{\gamma}} Z_3 \eta\| \lesssim \|Z(\frac{1}{\varphi(z)} u_3)\|_{L^\infty} \|Z_3 \eta\|_{m-2} + \|Z(\frac{1}{\varphi(z)} u_3)\|_{m-2} \|Z \eta\|_{L^\infty}.$$

And hence, since $Z^\alpha u_3$ vanishes on the boundary, we get from the Hardy inequality that

$$\|Z(\frac{1}{\varphi(z)} u_3)\|_{m-2} \lesssim \|\partial_z u_3\|_{m-1}.$$

Indeed, For $i = 1, 2$, we directly get that

$$\|Z_i(\frac{1}{\varphi(z)} u_3)\|_{m-2} = \|\frac{1}{\varphi(z)} Z_i u_3\|_{m-2} \lesssim \|\partial_z u_3\|_{m-1}.$$

For $i = 3$, since $Z_3(\frac{1}{\varphi})$ have the same properties as $1/\varphi$, we have

$$\|Z_3(\frac{1}{\varphi(z)} u_3)\|_{m-2} \lesssim \|\frac{1}{\varphi(z)} Z_3 u_3\|_{m-2} + \|\frac{1}{\varphi(z)} u_3\|_{m-2}$$

and hence the Hardy inequality yields

$$\|Z_3(\frac{1}{\varphi(z)} u_3)\|_{m-2} \lesssim \|\partial_z Z_3 u_3\|_{m-2} + \|\partial_z u_3\|_{m-2} \lesssim \|\partial_z u_3\|_{m-1}.$$

By using again the divergence free condition, we thus get that

$$\|Z(\frac{1}{\varphi(z)} u_3)\|_{m-2} \lesssim \|\partial_z u_3\|_{m-1} \lesssim \|u\|_m$$

Consequently, we obtain that

$$\|Z^{\tilde{\beta}}(\frac{1}{\varphi(z)} u_3) Z^{\tilde{\gamma}} Z_3 \eta\| \lesssim (\|u\|_{2,\infty} + \|Z \eta\|_{L^\infty}) (\|\eta\|_{m-1} + \|u\|_m).$$

We have thus proven that

$$(3.29) \quad \|\mathcal{C}_1\| \lesssim (\|u\|_{2,\infty} + \|u\|_{W^{1,\infty}} + \|Z \eta\|_{L^\infty}) (\|\eta\|_{m-1} + \|u\|_m).$$

To end the proof of Proposition 3.4, it suffices to collect (3.25), (3.26) (3.27) and (3.29).

3.3. Pressure estimates. It remains to estimate the pressure and the L^∞ norms in the right hand side of the estimates of Propositions 3.4 and 3.2

The aim of this section is to give the estimate of $\|\nabla p\|_{m-1}$.

Proposition 3.5. *For every $m \geq 2$, there exists $C > 0$ such that for every $\varepsilon \in (0, 1]$, a smooth solution of (1.1), (1.2) on $[0, T]$ satisfies the estimate*

$$\|\nabla p(t)\|_{m-1} \leq C \left(\varepsilon \|\nabla u(t)\|_{m-1} + (1 + \|u(t)\|_{W^{1,\infty}}) (\|u(t)\|_m + \|\partial_z u(t)\|_{m-1}) \right), \quad \forall t \in [0, T].$$

Note that by combining Proposition 3.5, Proposition 3.2, Proposition 3.4 and (3.19), (3.22), we find that

$$(3.30) \quad \begin{aligned} & \|u(t)\|_m^2 + \|\partial_z u(t)\|_{m-1}^2 + \varepsilon \int_0^t (\|\nabla u\|_m^2 + \|\nabla^2 u\|_{m-1}^2) \\ & \lesssim \|u_0\|_m^2 + \|\partial_z u_0\|_{m-1}^2 + \int_0^t (1 + \|u\|_{2,\infty} + \|\partial_z u\|_{1,\infty}) (\|\partial_z u\|_{m-1}^2 + \|u\|_m^2). \end{aligned}$$

In particular, we see from this estimate that it only remains to control $\|u\|_{2,\infty} + \|\partial_z u\|_{1,\infty}$.

The proof of Proposition 3.5 relies on the following estimate for the Stokes problem in a half-space. Consider the system

$$(3.31) \quad \partial_t u - \varepsilon \Delta u + \nabla p = F, \quad \nabla \cdot u = 0, \quad z > 0,$$

with the Navier boundary condition (1.2) which reads

$$(3.32) \quad u_3 = 0, \quad \partial_z u_h = 2\alpha u_h, \quad z = 0$$

where F is some given source term.

We have the following estimates for the Stokes problem

Theorem 3.6. *For every $m \geq 2$, there exists $C > 0$ such that for every $t \geq 0$, we have the estimate*

$$\|\nabla p\|_{m-1} \leq C \left(\|F\|_{m-1} + \|\nabla \cdot F\|_{m-2} + \varepsilon \|\nabla u\|_{m-1} + \|u\|_{m-1} \right).$$

The proof can be obtained from standard elliptic regularity results. Nevertheless, in the case of a half-space, the proof follows easily from explicit computations in the Fourier side. We shall thus sketch the proof for the sake of completeness.

Proof of Theorem 3.6. By taking the divergence of (3.31), we get that p solves

$$\Delta p = \nabla \cdot F, \quad z > 0.$$

Note that in this proof, the time will be only a parameter, for notational convenience, we shall not write down explicitly that all the involved functions depend on it.

From the third component of the velocity equation, we get that

$$(3.33) \quad \partial_z p(y, 0) = \varepsilon \partial_{zz} u_3(y, 0) + \varepsilon \Delta_h u_3(y, 0) - \partial_t u_3(y, 0) + F_3(y, 0).$$

From, the boundary condition for the velocity, we have that

$$\Delta_h u_3(y, 0) = 0, \quad \partial_t u_3(y, 0) = 0.$$

Moreover by applying ∂_z to the divergence free condition, we get that

$$\partial_{zz} u_3(y, 0) = -\nabla_h \cdot \partial_z u_h(y, 0)$$

and hence from the second boundary condition in (3.32), we obtain

$$\partial_{zz} u_3(y, 0) = -2\alpha \nabla_h \cdot u_h.$$

Consequently, we can use (3.33) to express the pressure on the boundary and we obtain the following elliptic equation with Neumann boundary condition for the pressure:

$$(3.34) \quad \Delta p = \nabla \cdot F, \quad z > 0, \quad \partial_z p(y, 0) = -2\alpha\varepsilon \nabla_h \cdot u_h(y, 0) + F_3(y, 0).$$

Note that we can express p as $p = p_1 + p_2$ where p_1 solves

$$(3.35) \quad \Delta p_1 = \nabla \cdot F, \quad z > 0, \quad \partial_z p_1(y, 0) = F_3(y, 0)$$

and p_2 solves

$$(3.36) \quad \Delta p_2 = 0, \quad z > 0, \quad \partial_z p_2(y, 0) = 2\alpha\varepsilon \nabla_h \cdot u_h(y, 0).$$

The meaning of this decomposition is that p_1 corresponds to the gradient part of the usual Leray-Hodge decomposition of the vector field F whereas p_2 is purely determined by the Navier boundary condition. The desired estimates for p_1 and p_2 can be obtained from standard elliptic theory. In the case of our very simple geometry, the proof is very easy thanks to the explicit representation of the solutions in Fourier space.

To estimate p_1 , we can use an explicit representation of the solution in Fourier space (we refer for example to the appendix of [27]). By taking the Fourier transform in the (x_1, x_2) variable, we get that \hat{p}_1 solves

$$(3.37) \quad \partial_{zz} \hat{p}_1 - |\xi|^2 \hat{p}_1 = i\xi \cdot \hat{F}_h + \partial_z \hat{F}_3, \quad z > 0, \quad \partial_z \hat{p}_1(\xi, 0) = \hat{F}_3(\xi, 0).$$

Consequently the resolution of this ordinary differential equation gives

$$\hat{p}_1(\xi, z) = \int_0^{+\infty} G_\xi(z, z') \hat{F}(\xi, z') dz'$$

where $G_\xi(z, z')$ is defined as

$$G_\xi(z, z') = \begin{cases} -\left(e^{-|\xi|z'} \frac{\cosh(|\xi|z)}{|\xi|} i\xi, e^{-|\xi|z'} \cosh(|\xi|z) \right), & z < z', \\ -\left(e^{-|\xi|z} \frac{\cosh(|\xi|z')}{|\xi|} i\xi, -e^{-|\xi|z} \sinh(|\xi|z') \right), & z > z'. \end{cases}$$

Note that the product $G_\xi \hat{F}$ has to be understood as the product of a $(1, 3)$ matrix and a $(3, 1)$ matrix.

In particular, we obtain that

$$\partial_z \hat{p}_1(\xi, z) = \int_0^{+\infty} K_\xi(z, z') \hat{F}(\xi, z') dz' + \hat{F}_3(\xi, z)$$

where $K_\xi(z, z')$ is defined by

$$K_\xi(z, z') = \begin{cases} \partial_z G_\xi(z, z'), & z < z' \\ \partial_z G_\xi(z, z'), & z > z'. \end{cases}$$

Since

$$\sup_{z, \xi} (\|K_\xi(z, \cdot)\|_{L^1(0, +\infty)} + |\xi| \|G_\xi(z, \cdot)\|_{L^1(0, +\infty)}) < +\infty$$

and

$$\sup_{z', \xi} (\|K_\xi(\cdot, z')\|_{L^1(0, +\infty)} + |\xi| \|G_\xi(\cdot, z')\|_{L^1(0, +\infty)}) < +\infty,$$

we get by using the Schur Lemma that

$$\|\partial_z \hat{p}_1(\xi, \cdot)\|_{L^2(0, +\infty)} + |\xi| \|\hat{p}_1(\xi, \cdot)\|_{L^2(0, +\infty)} \leq C \|\hat{F}(\xi, \cdot)\|_{L^2(0, +\infty)},$$

where C does not depend on ξ . Hence, by using the Bessel identity we obtain from the previous estimate that

$$\|\nabla p_1\|_{L^2} \lesssim \|F\|_{L^2}.$$

In a similar way, we get by multiplication in the Fourier side that

$$\|\nabla_h^k p_1\| \lesssim \|F\|_k, \quad \forall k \leq m-1.$$

Moreover, by using (3.37), we also obtain that

$$\|\partial_{zz} p_1\|_k \lesssim \|\nabla \cdot F\|_k, \quad \forall k \leq m-2.$$

Consequently, since $[\partial_{zz}, Z_3] = \varphi'' \partial_z + 2\varphi' \partial_{zz}$, the result for p_1 follows easily by applying $Z_3^{\alpha_3}$ to (3.37) and by induction on α_3 . This yields finally

$$(3.38) \quad \|\nabla p_1\|_{m-1} \lesssim \|F\|_{m-1} + \|\nabla \cdot F\|_{m-2}.$$

which is the desired estimate for p_1 .

Let us turn to the estimate of p_2 . Again, by using the Fourier transform, we can solve explicitly (3.36). We obtain that

$$(3.39) \quad \hat{p}_2(\xi, z) = 2i\alpha\varepsilon \frac{\xi}{|\xi|} \cdot \hat{u}_h(\xi, 0) e^{-|\xi|z}.$$

From the Bessel identity, this yields

$$\|\nabla p_2\|_{m-1} \lesssim \varepsilon |\alpha| \|u_h(\cdot, 0)\|_{H^{m-\frac{1}{2}}(\mathbb{R}^2)}$$

and hence from the Trace Theorem, we obtain

$$(3.40) \quad \|\nabla p_2\|_{m-1} \lesssim \varepsilon |\alpha| \|\nabla u_h\|_{m-1}^{\frac{1}{2}} \|u_h\|_{m-1}^{\frac{1}{2}}.$$

Consequently, we can collect (3.38), (3.40) to get the result. This ends the proof of Theorem 3.6.

It remains the:

Proof of Proposition 3.5. We can first use Theorem 3.6 with $F = -u \cdot \nabla u$ to get

$$\|\nabla p\|_{m-1} \lesssim \|u \cdot \nabla u\|_{m-1} + \|\nabla u \cdot \nabla u\|_{m-2} + \varepsilon \|\nabla u\|_{m-1} + \|u\|_{m-1}.$$

Since, by using again Lemma 3.3, we have

$$\begin{aligned} \|u \cdot \nabla u\|_{m-1} &\lesssim \|u\|_{W^{1,\infty}} (\|u\|_{m-1} + \|\nabla u\|_{m-1}) \lesssim \|u\|_{W^{1,\infty}} (\|u\|_m + \|\partial_z u\|_{m-1}), \\ \|\nabla u \cdot \nabla u\|_{m-2} &\lesssim \|\nabla u\|_{L^\infty} \|\nabla u\|_{m-2}, \end{aligned}$$

the proof of Proposition 3.5 follows.

3.4. L^∞ estimates. In this section, we shall provide the L^∞ estimates which are needed to estimate the right-hand sides in the estimates of Propositions 3.2, 3.4. Let us set

$$(3.41) \quad Q_m(t) = \|u(t)\|_m^2 + \|\eta(t)\|_{m-1}^2 + \|\eta\|_{1,\infty}^2$$

Proposition 3.7. *For $m_0 > 1$, we have*

$$(3.42) \quad \|u\|_{W^{1,\infty}} \lesssim \|u\|_{m_0+2} + \|\eta\|_{m_0+1} + \|\eta\|_{L^\infty} \leq Q_m^{\frac{1}{2}}(t), \quad m \geq m_0 + 2$$

$$(3.43) \quad \|u\|_{2,\infty} \lesssim \|u\|_{m_0+3} + \|\eta\|_{m_0+2} \leq Q_m^{\frac{1}{2}}(t), \quad m \geq m_0 + 3,$$

$$(3.44) \quad \|\nabla u\|_{1,\infty} \lesssim \|u\|_{m_0+3} + \|\eta\|_{m_0+3} + \|\eta\|_{1,\infty} \lesssim Q_m^{\frac{1}{2}}(t), \quad m \geq m_0 + 3$$

From this proposition and (3.30), we see that we shall only need to estimate $\|\eta\|_{1,\infty}$ in order to conclude.

Proof. We easily get (3.42), (3.43) and (3.44) from the anisotropic Sobolev embedding :

$$(3.45) \quad \|f\|_{L^\infty}^2 \lesssim \| |f|_{H^{m_0}(\partial\Omega)} \|_{L^\infty}^2 \lesssim \|\partial_z f\|_{m_0} \|f\|_{m_0} + \|f\|_{m_0}^2,$$

where we use the notation

$$\| |f|_{H^{m_0}(\partial\Omega)} \|_{L^\infty} = \sup_z |f(\cdot, z)|_{H^{m_0}(\mathbb{R}^2)},$$

the divergence free condition which provides

$$|\partial_z u_3(t, x)| \leq |\nabla_h u_h(t, x)|$$

and the fact that by definition of η , we have

$$|\partial_z u_h(t, x)| \lesssim |\nabla_h u_3(t, x)| + |u_h(t, x)| + |\eta(t, x)|.$$

□

We shall next estimate $\|\eta\|_{L^\infty}$ and $\|Z\eta\|_{L^\infty}$. Note that we cannot estimate these two quantities by using (3.45). Indeed, we do not expect $\partial_z \eta \sim \partial_{zz} u$ to be uniformly bounded in conormal spaces in the boundary layer (recall that u is expected to behave as $\sqrt{\varepsilon} U(z/\sqrt{\varepsilon}, y)$ as shown in [16]). Consequently, we need to use more carefully the properties of the equation for η to get these needed L^∞ estimates directly. This is the aim of the following proposition.

Proposition 3.8. *We have, for $m > 6$, the estimate:*

$$\|\eta(t)\|_{1,\infty}^2 \lesssim Q(0) + (1 + t + \varepsilon^3 t^2) \int_0^t (Q_m(s)^2 + Q_m(s)) ds.$$

Proof of Proposition 3.8. The estimate of $\|\eta\|_{L^\infty}$ is a consequence of the maximum principle for the transport-diffusion equation (3.23). Let us set

$$(3.46) \quad F = \omega \cdot \nabla u_h + 2\alpha \nabla_h^\perp p$$

so that (3.23) reads

$$(3.47) \quad \partial_t \eta + u \cdot \nabla \eta = \varepsilon \Delta \eta + F.$$

We obtain that

$$\|\eta(t)\|_{L^\infty} \leq \|\eta_0\|_{L^\infty} + \int_0^t \|F\|_{L^\infty}$$

and hence from the Cauchy-Schwarz inequality that

$$(3.48) \quad \|\eta(t)\|_{L^\infty}^2 \leq \|\eta_0\|_{L^\infty}^2 + t \int_0^t \|F\|_{L^\infty}^2$$

Next, we want to get a similar estimate for $Z_i \eta$. The main difficulty is the estimate of $Z_3 \eta$ since the commutator of this vector field with the Laplacian involves two derivatives in the normal variable.

Let $\chi(z)$ be a smooth compactly supported function which takes the value one in the vicinity of 0 and is supported in $[0, 1]$. We can write

$$\eta = \chi \eta + (1 - \chi) \eta := \eta^b + \eta^{int}$$

where η^{int} is supported away from the boundary and η^b is compactly supported in z .

Since $1 - \chi$ and $\partial_z \chi$ vanish in the vicinity of the boundary, and that our conormal H^m norm is equivalent to the usual H^s norm away from the boundary, we can write thanks to the usual Sobolev embedding that

$$\|\eta^{int}\|_{1,\infty} \lesssim \|\kappa u\|_{H^{s_0}}, \quad s_0 > 2 + \frac{3}{2}$$

for some κ supported away from the boundary and hence we get that

$$(3.49) \quad \|\eta^{int}(t)\|_{1,\infty} \lesssim \|u\|_m \lesssim Q_m(t)^{\frac{1}{2}}, \quad m \geq 4.$$

Consequently, it only remains to estimate η^b . We first notice that η^b solves the equation

$$(3.50) \quad \partial_t \eta^b + u \cdot \nabla \eta^b = \varepsilon \Delta \eta^b + \chi F + \mathcal{C}^b,$$

in the half-space $z > 0$ with homogeneous Dirichlet boundary condition, where \mathcal{C}^b is the commutator

$$\mathcal{C}^b = -2\varepsilon \partial_z \chi \partial_z \eta - \varepsilon \partial_{zz} \chi \eta + u_3 \partial_z \chi$$

Note that again since $\partial_z \chi$ and $\partial_{zz} \chi$ are supported away from the boundary, we have from the usual Sobolev embedding that

$$\|\mathcal{C}^b\|_{1,\infty} \lesssim \|\kappa u\|_{W^{3,\infty}} \lesssim \|\kappa u\|_{H^{s_0}}, \quad s_0 > 3 + \frac{3}{2}$$

and hence that

$$(3.51) \quad \|\mathcal{C}^b\|_{1,\infty} \lesssim \|u\|_m \leq Q_m^{\frac{1}{2}}, \quad m \geq 5.$$

A crucial estimate towards the proof of Proposition (3.8) is the following:

Lemma 3.9. *Consider ρ a smooth solution of*

$$(3.52) \quad \partial_t \rho + u \cdot \nabla \rho = \varepsilon \partial_{zz} \rho + \mathcal{S}, \quad z > 0, \quad \rho(t, y, 0) = 0$$

for some smooth divergence free vector field u such that $u \cdot n = u_3$ vanishes on the boundary. Assume that ρ and \mathcal{S} are compactly supported in z . Then, we have the estimate:

$$\|\rho(t)\|_{1,\infty} \lesssim \|\rho_0\|_{1,\infty} + \int_0^t \left((\|u\|_{2,\infty} + \|\partial_z u\|_{1,\infty}) (\|\rho\|_{1,\infty} + \|\rho\|_{m_0+3}) + \|\mathcal{S}\|_{1,\infty} \right)$$

for $m_0 > 2$.

Let us first explain how we can use the result of Lemma 3.9 to conclude. By applying Lemma 3.9 to (3.50) with $\mathcal{S} = \chi F + \mathcal{C}^b + \varepsilon \Delta_y \eta^b$ (where Δ_y is the Laplacian acting only on the y variable), we immediately get that

$$(3.53) \quad \|\eta^b(t)\|_{1,\infty} \lesssim \|\eta_0\|_{1,\infty} + \int_0^t \left((\|u\|_{2,\infty} + \|\nabla u\|_{1,\infty}) (\|\eta\|_{1,\infty} + \|\eta\|_{m_0+3}) + \|\mathcal{C}^b\|_{1,\infty} + \|F\|_{1,\infty} + \varepsilon \|\Delta_y \eta^b\|_{1,\infty} \right)$$

Note that $\|\mathcal{C}^b\|_{1,\infty}$ is well controlled thanks to (3.51) and that thanks to Lemma 3.7, we have

$$(3.54) \quad \|F\|_{1,\infty} \lesssim \|\nabla_h p\|_{1,\infty} + \|\omega\|_{1,\infty} \|\nabla u\|_{1,\infty} \lesssim \|\nabla_h p\|_{1,\infty} + Q_m.$$

From the anisotropic Sobolev embedding (3.45), we note that

$$\|\nabla_h p\|_{1,\infty} \lesssim \|\nabla p\|_{m-1}$$

for $m-1 \geq m_0+2 \geq 5$. Finally, we also notice that thanks to a new use of (3.45), we have that

$$\begin{aligned} \left(\varepsilon \int_0^t \|\Delta_y \eta^b\|_{1,\infty} \right)^2 &\lesssim \varepsilon^2 \left(\int_0^t \|\nabla^2 u\|_{m-1}^{\frac{1}{2}} Q_m^{\frac{1}{4}} \right)^2 + \varepsilon^2 t \int_0^t Q_m \\ &\lesssim \varepsilon^2 t \left(\int_0^t \|\nabla^2 u\|_{m-1}^2 \right)^{\frac{1}{2}} \left(\int_0^t Q_m \right)^{\frac{1}{2}} + \varepsilon^2 t \int_0^t Q_m \\ &\lesssim \varepsilon \int_0^t \|\nabla^2 u\|_{m-1}^2 + (\varepsilon^2 t + \varepsilon^3 t^2) \int_0^t Q_m \end{aligned}$$

for $m \geq m_0 + 4$. Consequently, we get from (3.53) and (3.49) that

$$\begin{aligned} \|\eta(t)\|_{1,\infty}^2 &\lesssim \|\eta_0\|_{1,\infty} + Q_m(t) + \varepsilon \int_0^t \|\nabla^2 u\|_{m-1}^2 + t \int_0^t (Q_m(s)^2 + \|\nabla p(s)\|_{m-1}^2) ds \\ &\quad + (1+t+\varepsilon^3 t^2) \int_0^t Q_m ds. \end{aligned}$$

Finally, we get from this last estimate and (3.30) that

$$\|\eta(t)\|_{1,\infty}^2 \lesssim Q(0) + (1+t+\varepsilon^3 t^2) \int_0^t (Q_m(s) + Q_m(s)^2) ds.$$

This ends the proof of Proposition 3.8.

It remains to prove Lemma 3.9

Proof of Lemma 3.9. The estimate of $\|\rho\|_{L^\infty}$ and $\|\partial_i \rho\|_{L^\infty} = \|Z_i \rho\|_{L^\infty}$, $i = 1, 2$ also follow easily from the maximum principle. Indeed, we get that $\partial_i \rho$ solves the equation

$$\partial_t \partial_i \rho + u \cdot \nabla \partial_i \rho = \varepsilon \partial_{zz} \partial_i \rho + \partial_i \mathcal{S} - \partial_i u \cdot \nabla \rho$$

still with an homogeneous Dirichlet boundary condition. Consequently, by using again the maximum principle, we find

$$(3.55) \quad \|\nabla_h \rho\|_{L^\infty} \leq \|\eta_0\|_{1,\infty} + \int_0^t (\|\mathcal{S}\|_{1,\infty} + \|\partial_i u \cdot \nabla \rho\|_{L^\infty}).$$

To estimate the last term in the above expression, we write again

$$(3.56) \quad \|\partial_i u \cdot \nabla \rho\|_{L^\infty} \lesssim \|u\|_{1,\infty} \|\rho\|_{1,\infty} + \|\partial_z \partial_i u_3\|_{L^\infty} \|Z_3 \rho\|_{L^\infty} \lesssim \|u\|_{2,\infty} \|\rho\|_{1,\infty}.$$

by a new use of the fact that u is divergence free.

It remains to estimate $\|Z_3 \rho\|_{L^\infty}$ which is the most difficult term. We cannot use the same method as previously due to the bad commutator between Z_3 and the Laplacian. We shall use a more precise description of the solution of (3.50). We shall first rewrite the equation (3.50) as

$$\partial_t \rho + z \partial_z u_3(t, y, 0) \partial_z \rho + u_h(t, y, 0) \cdot \nabla_h \rho - \varepsilon \partial_{zz} \rho = \mathcal{S} - R := G$$

where

$$R = (u_h(t, x) - u_h(t, y, 0)) \cdot \nabla_h \rho + (u_3(t, x) - z \partial_z u_3(t, y, 0)) \partial_z \rho.$$

The idea will be to use an exact representation of the Green's function of the operator in the left-hand side to perform the estimate.

Let $S(t, \tau)$ be the C^0 evolution operator generated by the left hand side of the above equation. This means that $f(t, y, z) = S(t, \tau) f_0(y, z)$ solves the equation

$$\partial_t f + z \partial_z u_3(t, y, 0) \partial_z f + u_h(t, y, 0) \cdot \nabla_h f - \varepsilon \partial_{zz} f = 0, \quad z > 0, t > \tau, \quad f(t, y, 0) = 0.$$

with the initial condition $f(\tau, y, z) = f_0(y, z)$. Then we have the following estimate:

Lemma 3.10. *There exists $C > 0$ such that*

$$\|z \partial_z S(t, \tau) f_0\|_{L^\infty} \leq C (\|f_0\|_{L^\infty} + \|z \partial_z f_0\|_{L^\infty}), \quad \forall t \geq \tau \geq 0.$$

We shall postpone the proof of the Lemma until the end of the section.

By using Duhamel formula, we deduce that

$$(3.57) \quad \rho(t) = S(t, \tau) \rho_0 + \int_0^t S(t, \tau) G(\tau) d\tau.$$

Consequently, by using Lemma 3.10, we obtain

$$\|Z_3 \rho\|_{L^\infty} \lesssim \left(\|\rho_0\|_{L^\infty} + \|z \partial_z \rho_0\|_{L^\infty} + \int_0^t (\|G\|_{L^\infty} + \|z \partial_z G\|_{L^\infty}) \right).$$

Since ρ and G are compactly supported, we obtain

$$(3.58) \quad \|Z_3\rho\|_{L^\infty} \lesssim \left(\|\rho_0\|_{1,\infty} + \int_0^t \|G\|_{1,\infty} \right).$$

It remains to estimate the right hand side. First, let us estimate the term involving R . Since $u_3(t, y, 0) = 0$, we have

$$\|R\|_{L^\infty} \lesssim \|u_h\|_{L^\infty} \|\nabla_h \rho\|_{L^\infty} + \|\partial_z u_3\|_{L^\infty} \|Z_3\rho\|_{L^\infty} \lesssim \|u\|_{1,\infty} \|\rho\|_{1,\infty}.$$

Note that we have used again the divergence free condition to get the last estimate. Next, in a similar way, we get

$$\|ZR\|_{L^\infty} \lesssim \|u\|_{2,\infty} \|\rho\|_{1,\infty} + \left\| (u_h(t, x) - u_h(t, y, 0)) \cdot Z\nabla_h \rho \right\|_{L^\infty} + \left\| (u_3(t, x) - z\partial_z u_3(t, y, 0)) Z\partial_z \rho \right\|_{L^\infty}$$

By using the Taylor formula and the fact that ρ is compactly supported in z , this yields

$$\|ZR\|_{L^\infty} \lesssim \|u\|_{2,\infty} \|\rho\|_{1,\infty} + \|\partial_z u_h\|_{L^\infty} \|\varphi(z) Z\nabla_h \rho\|_{L^\infty} + \|\partial_{zz} u_3\|_{L^\infty} \|\varphi^2(z) Z\partial_z \rho\|_{L^\infty}.$$

Consequently, by using the divergence free condition, we get

$$\|R\|_{L^\infty} \lesssim (\|u\|_{2,\infty} + \|\partial_z u\|_{1,\infty}) (\|\rho\|_{1,\infty} + \|\varphi(z)\rho\|_{2,\infty}).$$

The additional factor φ in the last term is crucial to close our estimate. Indeed, by the Sobolev embedding (3.45), we have that for $|\alpha| = 2$

$$\|\varphi Z^\alpha \eta\|_{L^\infty} \lesssim \|Z^\alpha \eta\|_{m_0} + \|\partial_z(\varphi Z^\alpha \eta)\|_{m_0}$$

and hence we obtain by definition of Z_3 that

$$(3.59) \quad \|\varphi Z^\alpha \eta\|_{L^\infty} \lesssim \|\eta\|_{m_0+3}, \quad |\alpha| = 2.$$

Consequently, we finally get by using Proposition 3.7 that for $m \geq m_0 + 4$

$$(3.60) \quad \|R(t)\|_{1,\infty} \lesssim (\|u\|_{2,\infty} + \|\partial_z u\|_{1,\infty}) (\|\rho\|_{1,\infty} + \|\rho\|_{m_0+3}).$$

Finally, the proof of Proposition 3.8 follows from the last estimate and (3.58).

It remains to prove Lemma 3.10.

Proof of Lemma 3.10. Let us set $f(t, y, z) = S(t, \tau) f_0(y, z)$, then f solves the equation

$$\partial_t f + z\partial_z u_3(t, y, 0)\partial_z f + u_h(t, y, 0) \cdot \nabla_h f - \varepsilon\partial_{zz} f = 0, \quad z > 0, \quad f(t, y, 0) = 0.$$

We can first transform the problem into a problem in the whole space. Let us define \tilde{f} by

$$(3.61) \quad \tilde{f}(t, y, z) = f(t, y, z), \quad z > 0, \quad \tilde{f}(t, y, z) = -f(t, y, -z), \quad z < 0$$

then \tilde{f} solves

$$(3.62) \quad \partial_t \tilde{f} + z\partial_z u_3(t, y, 0)\partial_z \tilde{f} + u_h(t, y, 0) \cdot \nabla_h \tilde{f} - \varepsilon\partial_{zz} \tilde{f} = 0, \quad z \in \mathbb{R}$$

with the initial condition $\tilde{f}(\tau, y, z) = \tilde{f}_0(y, z)$.

We shall get the estimate by using an exact representation of the solution.

To solve (3.62), we can first define

$$(3.63) \quad g(t, y, z) = f(t, \Phi(t, \tau, y), z)$$

where Φ is the solution of

$$\partial_t \Phi = u_h(t, \Phi, 0), \quad \Phi(\tau, \tau, y) = y.$$

Then, g solves the equation

$$\partial_t g + z\gamma(t, y)\partial_z g - \varepsilon\partial_{zz} g = 0, \quad z \in \mathbb{R}, \quad g(\tau, y, z) = \tilde{f}_0(y, z)$$

where

$$(3.64) \quad \gamma(t, y) = \partial_z u_3(t, \Phi(t, \tau, y), 0)$$

which is a one-dimensional Fokker-Planck type equation (note that now y is only a parameter in the problem). By a simple computation in Fourier space, we find the explicit representation

$$\begin{aligned} g(t, x) &= \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi\varepsilon \int_{\tau}^t e^{2\varepsilon(\Gamma(t)-\Gamma(s))} ds}} \exp\left(-\frac{(z-z')^2}{4\varepsilon \int_{\tau}^t e^{2\varepsilon(\Gamma(t)-\Gamma(s))} ds}\right) \tilde{f}_0(y, e^{-\Gamma(t)} z') dz' \\ &= \int_{\mathbb{R}} k(t, \tau, y, z-z') \tilde{f}_0(y, e^{-\Gamma(t)} z') dz' \end{aligned}$$

where $\Gamma(t) = \int_{\tau}^t \gamma(s, y) ds$ (note that Γ depends on y and τ , we do not write down explicitly this dependence for notational convenience).

Note that k is non-negative and that $\int_{\mathbb{R}} k(t, \tau, y, z) dz = 1$, thus, we immediately recover that

$$\|g\|_{L^\infty} \leq \|\tilde{f}_0\|_{L^\infty}.$$

Next, we observe that we can write

$$z \partial_z k(t, \tau, z-z') = (z-z') \partial_z k - z' \partial_{z'} k(t, \tau, z-z')$$

with

$$\int_{\mathbb{R}} |(z-z') \partial_z k| dz' \lesssim 1$$

and thus by using an integration by parts, we find

$$\|z \partial_z g\|_{L^\infty} \lesssim \|\tilde{f}\|_{L^\infty} + \left\| e^{-\Gamma(t)} \int_{\mathbb{R}} k(t, \tau, y, z') z' \partial_z \tilde{f}_0(y, e^{-\Gamma(t)} z') dz' \right\|_{L^\infty}.$$

By using (3.64), this yields

$$\|z \partial_z g\|_{L^\infty} \lesssim \|\tilde{f}_0\|_{L^\infty} + \|z \partial_z \tilde{f}_0\|_{L^\infty}.$$

By using (3.61) and (3.63), we obtain

$$\|z \partial_z f\|_{L^\infty} \lesssim \|z \partial_z \tilde{f}\|_{L^\infty} \lesssim \|\tilde{f}_0\|_{L^\infty} + \|z \partial_z \tilde{f}_0\|_{L^\infty} \lesssim \|f_0\|_{L^\infty} + \|z \partial_z f_0\|_{L^\infty}.$$

This ends the proof of Lemma 3.10.

3.5. Final a priori estimate. By combining Propositions 3.8, 3.7 and (3.30), the proof of Theorem 3.1 follows.

4. THE CASE OF A GENERAL DOMAIN WITH SMOOTH BOUNDARY

4.1. Notations and conormal spaces. We recall that Ω is a bounded domain of \mathbb{R}^3 and we assume that there exists a covering of Ω under the form

$$(4.1) \quad \Omega \subset \Omega_0 \cup_{i=1}^n \Omega_i$$

where $\overline{\Omega_0} \subset \Omega$ and in each Ω_i , there exists a smooth function ψ_i such that $\Omega \cap \Omega_i = \{(x = (x_1, x_2, x_3), x_3 > \psi_i(x_1, x_2))\} \cap \Omega_i$ and $\partial\Omega \cap \Omega_i = \{x_3 = \psi_i(x_1, x_2)\} \cap \Omega_i$.

To define Sobolev conormal spaces, we consider $(Z_k)_{1 \leq k \leq N}$ a finite set of generators of vector fields that are tangent to $\partial\Omega$ and

$$H_{co}^m(\Omega) = \{f \in L^2(\Omega), \quad Z^I \in L^2(\Omega), \quad |I| \leq m\}$$

where for $I = (k_1, \dots, k_m)$, We use the notation

$$\|u\|_m^2 = \sum_{i=1}^3 \sum_{|I| \leq m} \|Z^I u_i\|_{L^2}^2$$

and in the same way

$$\|u\|_{k,\infty} = \sum_{|I|\leq m} \|Z^I u\|_{L^\infty},$$

$$\|\nabla Z^m u\|^2 = \sum_{|I|\leq m} \|\nabla Z^I u\|_{L^2}^2.$$

Note that, by using our covering of Ω , we can always assume that each vector field is supported in one of the Ω_i , moreover, in Ω_0 the $\|\cdot\|_m$ norm yields a control of the standard H^m norm, whereas if $\Omega_i \cap \partial\Omega \neq \emptyset$, there is no control of the normal derivatives.

In the proof C_k will denote a number independent of $\varepsilon \in (0, 1]$ which depends only on the \mathcal{C}^k regularity of the boundary, that is to say on the \mathcal{C}^k norm of the functions ψ_i .

By using that $\partial\Omega$ is given locally by $x_3 = \psi(x_1, x_2)$ (we omit the subscript i for notational convenience), it is convenient to use the coordinates:

$$(4.2) \quad \Psi : (y, z) \mapsto (y, \psi(y) + z).$$

A local basis is thus given by the vector fields $(\partial_{y^1}, \partial_{y^2}, \partial_z)$. On the boundary ∂_{y^1} and ∂_{y^2} are tangent to $\partial\Omega$, but ∂_z is not a normal vector field. We shall sometimes use the notation ∂_{y^3} for ∂_z . By using this parametrization, we can take as suitable vector fields compactly supported in Ω_i in the definition of the $\|\cdot\|_m$ norms:

$$Z_i = \partial_{y^i} = \partial_i + \partial_i \psi \partial_z, \quad i = 1, 2, \quad Z_3 = \varphi(z)(\partial_1 \psi \partial_1 + \partial_2 \psi \partial_2 - \partial_z)$$

where φ is smooth, supported in \mathbb{R}_+ , and such that $\varphi(0) = 0$, $\varphi(s) > 0$, $s > 0$.

In this section, we shall still denote by ∂_i , $i = 1, 2, 3$ or ∇ the derivation with respect to the standard coordinates of \mathbb{R}^n . The coordinates of a vector field u in the basis $(\partial_{y^i})_{1 \leq i \leq 3}$ will be denoted by u^i , thus

$$u = u^1 \partial_{y^1} + u^2 \partial_{y^2} + u^3 \partial_{y^3}$$

whereas we shall still denote by u_i the coordinates in the canonical basis of \mathbb{R}^3 , namely $u = u_1 \partial_1 + u_2 \partial_2 + u_3 \partial_3$ (we warn the reader that this convention does not match with the standard Einstein convention for raising and lowering the indices in differential geometry).

We shall also denote by n the unit outward normal which is given locally by

$$n(\Psi(y, z)) = \frac{1}{(1 + |\nabla \psi(y)|^2)^{\frac{1}{2}}} \begin{pmatrix} \partial_1 \psi(y) \\ \partial_2 \psi(y) \\ -1 \end{pmatrix}$$

(note that n is actually naturally defined in the whole Ω_i and does not depend on x_3) and in the same way, by Π the orthogonal projection

$$\Pi(\Psi(y, z))X = X - X \cdot n(\Psi(y, z)) n(\Psi(y, z))$$

which gives the orthogonal projection onto the tangent space of the boundary.

By using these notations, the Navier boundary condition (1.2) reads:

$$(4.3) \quad u \cdot n = 0, \quad \Pi \partial_n u = \theta(u) - 2\alpha \Pi u$$

where θ is the shape operator (second fundamental form) of the boundary i.e given by

$$\theta(u) = \Pi(u \cdot \nabla n).$$

The crucial step in the proof of Theorem 1.1 is again the proof of an a priori estimate. We shall prove that:

Theorem 4.1. For $m > 6$, and Ω a C^{m+2} domain, there exists $C_{m+2} > 0$ independent of $\varepsilon \in (0, 1]$ and α , $|\alpha| \leq 1$ such that for every sufficiently smooth solution defined on $[0, T]$ of (1.1), (1.2), we have the a priori estimate

$$N_m(t) \leq C_{m+2} \left(N_m(0) + (1+t + \varepsilon^3 t^2) \int_0^t (N_m(s) + N_m(s)^2) ds \right), \quad \forall t \in [0, T]$$

where

$$N_m(t) = \|u(t)\|_m^2 + \|\nabla u(t)\|_{m-1}^2 + \|\nabla u\|_{1,\infty}^2.$$

The steps of the proof of Theorem 4.1 are the same as in the proof of Theorem 3.1. Nevertheless some new difficulties will appear mainly due to the fact that n is not a constant vector field any more.

4.2. Conormal energy estimates.

Proposition 4.2. For every m , the solution of (1.1), (1.2) satisfies the estimate

$$\begin{aligned} & \|u(t)\|_m^2 + \varepsilon \int_0^t \|\nabla u\|_m^2 \\ & \leq C_{m+2} \left(\|u_0\|_m^2 + \int_0^t \left(\|\nabla^2 p_1\|_{m-1} \|u\|_m + \varepsilon^{-1} \|\nabla p_2\|_{m-1}^2 + (1 + \|u\|_{W^{1,\infty}}) (\|u\|_m^2 + \|\nabla u\|_{m-1}^2) \right) \right) \end{aligned}$$

where the pressure p is splitted as $p = p_1 + p_2$ where p_1 is the "Euler" part of the pressure which solves

$$\Delta p_1 = -\nabla \cdot (u \cdot \nabla u), \quad x \in \Omega, \quad \partial_n p = -(u \cdot \nabla u) \cdot n, \quad x \in \partial\Omega$$

and p_2 is the "Navier Stokes part" which solves

$$\Delta p_2 = 0, \quad x \in \Omega, \quad \partial_n p_2 = \varepsilon \Delta u \cdot n, \quad x \in \partial\Omega.$$

Note that the estimate involving the pressure is worse than in Proposition 3.2. Indeed, since $Z^\alpha u \cdot n$ does not vanish on the boundary, we cannot gain one derivative in the estimate of the Euler part of the pressure by using an integration by parts.

4.3. Proof of Proposition 4.2. The estimate for $m = 0$ is already given in Proposition 2.1. Assuming that it is proven for $k \leq m - 1$, we shall prove it for $k = m \geq 1$. By applying Z^I for $|I| = m$ to (1.1) as before, we obtain that

$$(4.4) \quad \partial_t Z^I u + u \cdot \nabla Z^I u + Z^I \nabla p = \varepsilon Z^I \Delta u + \mathcal{C}^1$$

where \mathcal{C}^1 is the commutator defined as

$$\mathcal{C}^1 = [Z^I, u \cdot \nabla].$$

By using again Lemma 3.3, we obtain that

$$(4.5) \quad \|\mathcal{C}^1\| \leq C_{m+1} \|u\|_{W^{1,\infty}} (\|u\|_m + \|\partial_z u\|_{m-1}).$$

Indeed, we can perform this estimate in each coordinate patch. In Ω_0 , this is a direct consequence of the standard tame Gagliardo-Nirenberg-Sobolev inequality. Close to the boundary, we first notice that $u \cdot \nabla u = u_1 \partial_1 u + u_2 \partial_2 u + u_3 \partial_3 u$ can be written

$$u \cdot \nabla u = u_1 \partial_{y^1} u + u_2 \partial_{y^2} u + u \cdot \mathbf{N} \partial_z u$$

where u_i , $i = 1, 2$ and 3 are the coordinates of u in the standard canonical basis of \mathbb{R}^n and \mathbf{N} is defined by

$$\mathbf{N} = \begin{pmatrix} -\partial_1 \psi \\ -\partial_2 \psi \\ 1 \end{pmatrix}.$$

Note, that when the boundary is given by $x_3 = \psi(x_1, x_2)$, \mathbf{N} is a normal (non-unitary) vector field. Moreover, we also have that $Z^I = \partial_{y^1}^{\alpha_1} \partial_{y^2}^{\alpha_2} (\varphi(z) \partial_z)^{\alpha_3}$. Since $u \cdot \mathbf{N}$ vanishes on the boundary $z = 0$, we can use the same estimates as in (3.13), (3.14), (3.15), (3.17) with u_3 replaced by $u \cdot \mathbf{N}$. Note that we have the estimate

$$\|u \cdot \mathbf{N}\|_m \leq C_{m+1} \|u\|_m$$

which explains the dependence in C_{m+1} in (4.5) and also that

$$\|\partial_z u\|_{m-1} \leq C_m \|\nabla u\|_{m-1}.$$

Consequently, a standard energy estimate for (4.4) yields

$$(4.6) \quad \frac{d}{dt} \frac{1}{2} \|Z^I u\|^2 \leq \varepsilon \int_{\Omega} Z^I \Delta u \cdot Z^I u - \int_{\Omega} Z^I \nabla p \cdot Z^I u + C_{m+1} \|u\|_{W^{1,\infty}} (\|u\|_m + \|\partial_z u\|_{m-1}) \|u\|_m$$

We shall first estimate the first term above in the right hand side. To evaluate this term through integration by parts, we shall need estimates of the trace of u on the boundary. At first, thanks to the Navier boundary condition under the form (4.3), we have that

$$(4.7) \quad |\Pi \partial_n u|_{H^m(\partial\Omega)} \leq |\theta(u)|_{H^m(\partial\Omega)} + 2\alpha |u|_{H^m(\partial\Omega)} \leq C_{m+2} |u|_{H^m(\partial\Omega)}, \quad \forall m \geq 0.$$

To estimate the normal part of $\partial_n u$, we can use that $\nabla \cdot u = 0$. Indeed, we have

$$(4.8) \quad \nabla \cdot u = \partial_n u \cdot n + (\Pi \partial_{y^1} u)^1 + (\Pi \partial_{y^2} u)^2$$

and hence, we immediately get that

$$(4.9) \quad |\partial_n u \cdot n|_{H^{m-1}(\partial\Omega)} \leq C_m |u|_{H^m(\partial\Omega)}.$$

Note that by combing these two last estimates, we have in particular that

$$(4.10) \quad |\nabla u|_{H^{m-1}(\partial\Omega)} \leq C_{m+1} |u|_{H^m(\partial\Omega)}.$$

Finally, let us notice that since $u \cdot n = 0$ on the boundary, we have that

$$(4.11) \quad |(Z^\alpha u) \cdot n|_{H^1(\partial\Omega)} \leq C_{m+2} |u|_{H^m(\partial\Omega)}, \quad |\alpha| = m.$$

Next, we can write that

$$\begin{aligned} \varepsilon \int_{\Omega} Z^I \Delta u \cdot Z^I u &= 2\varepsilon \int_{\Omega} (\nabla \cdot Z^I S u) \cdot Z^I u + \varepsilon \int_{\Omega} ([Z^I, \nabla \cdot] S u) \cdot Z^I u \\ &= I + II. \end{aligned}$$

By integration by parts, we get for the first term that

$$I = -\varepsilon \int_{\Omega} Z^I S u \cdot \nabla Z^I u + \varepsilon \int_{\partial\Omega} ((Z^I S u) \cdot n) \cdot Z^I u$$

and we note that

$$-\varepsilon \int_{\Omega} Z^I S u \cdot \nabla Z^I u = -\varepsilon \|S(Z^I u)\|^2 + \varepsilon \int_{\Omega} [Z^I, S] u \cdot \nabla Z^I u.$$

Consequently, thanks to the Korn inequality, there exists $c_0 > 0$ (depending only C_1) such that

$$\varepsilon \int_{\Omega} Z^I S u \cdot \nabla Z^I u \leq -c_0 \varepsilon \|\nabla(Z^I u)\|^2 + C_1 \|u\|_m^2 + \varepsilon \int_{\Omega} [Z^I, S] u \cdot \nabla Z^I u.$$

Moreover, the commutator term can be bounded by

$$\left| \varepsilon \int_{\Omega} [Z^I, S] u \cdot \nabla Z^I u \right| \leq C_{m+1} \varepsilon \|\nabla Z^m u\| \|\nabla u\|_{m-1}.$$

It remains to estimate the boundary term in the expression for I . We can first notice that

$$\int_{\partial\Omega} ((Z^I S u) \cdot n) \cdot Z^I u = \int_{\partial\Omega} Z^I (\Pi(S u \cdot n)) \cdot \Pi Z^I u + \int_{\partial\Omega} Z^I (\partial_n u \cdot n) Z^I u \cdot n + C_b$$

where the commutator term \mathcal{C}_b can be bounded by

$$|\mathcal{C}_b| \leq C_{m+1} |\nabla u|_{H^{m-1}(\partial\Omega)} |u|_{H^m(\partial\Omega)} \leq C_{m+1} |u|_{H^m(\partial\Omega)}^2$$

thanks to a new use of (4.10). For the main term, we write that thanks to the Navier boundary condition (1.2) we have

$$\left| \int_{\partial\Omega} Z^I (\Pi(Su \cdot n)) \cdot \Pi Z^I u \right| \leq C_{m+1} |u|_{H^m(\partial\Omega)}^2$$

and that by integrating once along the boundary, we have that

$$\left| \int_{\partial\Omega} Z^I (\partial_n u \cdot n) Z^I u \cdot n \right| \lesssim |\partial_n u \cdot n|_{H^{m-1}(\partial\Omega)} |Z^I u \cdot n|_{H^1(\partial\Omega)} \leq C_{m+2} |u|_{H^m(\partial\Omega)}^2$$

where the last estimate comes from (4.9), (4.11).

We have thus proven that

$$\varepsilon \left| \int_{\partial\Omega} ((Z^I Su) \cdot n) \cdot Z^I u \right| \leq C_{m+2} \varepsilon |u|_{H^m(\partial\Omega)}^2.$$

This yields

$$(4.12) \quad I \leq -\varepsilon c_0 \|\nabla Z^I u\|^2 + C_{m+2} (\varepsilon \|\nabla Z^m u\| (\|u\|_m + \|\nabla u\|_{m-1}) + |u|_{H^m(\partial\Omega)}^2).$$

It remains to estimate II . We can expand $[Z^I, \nabla \cdot]$ as a sum of terms under the form $\beta_k \partial_k Z^{\tilde{I}}$ with $|\tilde{I}| \leq m-1$ and $|\beta_k|_{L^\infty} \leq C_{m+1}$. Consequently, we need to estimate

$$\int_{\Omega} \beta_k \partial_k (Z^{\tilde{I}} Su) \cdot Z^I u.$$

By using an integration by parts, we get that

$$\varepsilon \left| \int_{\Omega} \beta_k \partial_k (Z^{\tilde{I}} Su) \cdot Z^I u \right| \leq C_{m+2} \varepsilon \left(\|\nabla Z^{m-1} u\| \|\nabla Z^m u\| + \|u\|_m^2 + |\nabla u|_{H^{m-1}(\partial\Omega)} |u|_{H^m(\partial\Omega)} \right).$$

Consequently, from a new use of (4.10) we get that

$$(4.13) \quad |II| \leq C_{m+2} \varepsilon \left(\|\nabla Z^{m-1} u\| \|\nabla Z^m u\| + \|u\|_m^2 + |u|_{H^m(\partial\Omega)}^2 \right).$$

To estimate the term involving the pressure in (4.6), we write

$$\left| \int_{\Omega} Z^I \nabla p \cdot Z^I u \right| \leq \|\nabla^2 p_1\|_{m-1} \|u\|_m + \left| \int_{\Omega} Z^I \nabla p_2 \cdot Z^I u \right|.$$

For the last term, we have

$$\left| \int_{\Omega} Z^I \nabla p_2 \cdot Z^I u \right| \leq \left| \int_{\Omega} \nabla Z^I p_2 \cdot Z^I u \right| + C_{m+1} \|\nabla p_2\|_{m-1} \|u\|_m$$

and we can integrate by parts to get

$$\left| \int_{\Omega} \nabla Z^I p_2 \cdot Z^I u \right| \leq \|\nabla p_2\|_{m-1} \|\nabla Z^I u\| + \left| \int_{\partial\Omega} Z^I p_2 Z^I u \cdot n \right|.$$

To control the boundary term, when $m \geq 2$, we integrate by parts once along the boundary to obtain

$$\left| \int_{\partial\Omega} Z^I p_2 Z^I u \cdot n \right| \leq C_2 \|Z^{\tilde{I}} p_2\|_{L^2(\partial\Omega)} \|Z^I u \cdot n\|_{H^1(\partial\Omega)}$$

where $\tilde{I} = m-1$. Next, we use (4.11) and the trace Theorem to get that

$$\left| \int_{\Omega} Z^I \nabla p_2 \cdot Z^I u \right| \leq C_{m+2} \|\nabla p_2\|_{m-1} (\|\nabla Z^I u\| + \|u\|_m).$$

We have thus proven that

$$\left| \int_{\Omega} Z^I \nabla p \cdot Z^I u \right| \leq C_{m+2} \left(\|\nabla^2 p_1\|_{m-1} \|u\|_m + \|\nabla p_2\|_{m-1} (\|\nabla Z^I u\| + \|u\|_m) \right).$$

Consequently, by collecting the previous estimates, we deduce from (4.6) that

$$\begin{aligned} & \frac{d}{dt} \frac{1}{2} \|u\|_m^2 + \varepsilon c_0 \|\nabla Z^m u\|^2 \\ & \leq C_{m+2} \left(\varepsilon \|\nabla Z^m u\| (\|u\|_m + \|\nabla Z^{m-1} u\|) + |u|_{H^m(\partial\Omega)}^2 \right. \\ & \quad \left. + \|\nabla^2 p_1\|_{m-1} \|u\|_m + \|\nabla p_2\|_{m-1} (\|\nabla Z^m u\| + \|u\|_m) + (1 + \|u\|_{W^{1,\infty}}) (\|u\|_m^2 + \|\partial_z u\|_{m-1}^2) \right). \end{aligned}$$

By using the Trace Theorem and the Young inequality, we finally get that

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} \|u\|_m^2 + \frac{c_0}{2} \varepsilon \|\nabla Z^m u\|^2 & \leq C_{m+2} \left(\varepsilon \|\nabla Z^{m-1} u\|_m^2 + \|\nabla^2 p_1\|_{m-1} \|u\|_m + \varepsilon^{-1} \|\nabla p_2\|_{m-1}^2 \right. \\ & \quad \left. + (1 + \|u\|_{W^{1,\infty}}) (\|u\|_m^2 + \|\partial_z u\|_{m-1}^2) \right) \end{aligned}$$

and the result follows by using the induction assumption to control $\varepsilon \|\nabla Z^{m-1} u\|_m^2$. This ends the proof of Proposition 4.2.

4.4. Normal derivative estimates. In view of Proposition 4.2, we shall now provide an estimate for $\|\nabla u\|_{m-1}$. Of course, the only difficulty is to estimate $\|\chi \partial_z u\|_{m-1}$ or $\|\chi \partial_n u\|_{m-1}$ where χ is compactly supported in one of the Ω_i and with value one in a vicinity of the boundary. Indeed, we have by definition of the norm that $\|\chi \partial_{y^i} u\|_{m-1} \leq C_m \|u\|_m$, $i = 1, 2$. We shall thus use the local coordinates (4.2).

At first, thanks to (4.8), we immediately get that

$$(4.14) \quad \|\chi \partial_n u \cdot n\|_{m-1} \leq C_m \|u\|_m.$$

It thus remains to estimate $\|\chi \Pi(\partial_n u)\|_{m-1}$. Let us set

$$(4.15) \quad \eta = \chi \Pi \left((\nabla u + \nabla u^t) n \right) + 2\alpha \chi \Pi u = \chi \Pi \left(S u n \right) + 2\alpha \chi \Pi u.$$

In view of the Navier condition (1.2), we obviously have that η satisfies an homogeneous Dirichlet boundary condition on the boundary:

$$(4.16) \quad \eta|_{\partial\Omega} = 0.$$

Moreover, since an alternative way to write η in the vicinity of the boundary is

$$(4.17) \quad \eta = \chi \Pi \partial_n u + \chi \Pi \left(\nabla(u \cdot n) - Dn \cdot u - u \times (\nabla \times n) + 2\alpha u \right),$$

we immediately get that

$$\|\chi \Pi \partial_n u\|_{m-1} \leq C_{m+1} (\|\eta\|_{m-1} + \|u\|_m + \|\partial_n u \cdot n\|_{m-1}).$$

and hence thanks to (4.14) that

$$(4.18) \quad \|\chi \Pi \partial_n u\|_{m-1} \leq C_{m+1} (\|\eta\|_{m-1} + \|u\|_m).$$

As before, it is thus equivalent to estimate $\|\Pi \partial_n u\|_{m-1}$ or $\|\eta\|_{m-1}$. Note that we have taken a slightly different definition for η in comparison with the half space case. The reason is that it is better to compute the evolution equation for η with the expression (4.15) than with the expression (4.17) or with the expression involving the vorticity. Indeed, these last two forms require a boundary with more regularity. The price to pay will be that since we do not use the vorticity, the pressure will again appear in our estimates.

We shall establish the following conormal estimates for η :

Proposition 4.3. *For every $m \geq 1$, we have that*

$$(4.19) \quad \begin{aligned} & \|\eta(t)\|_{m-1}^2 + \varepsilon \int_0^t \|\nabla \eta\|_{m-1}^2 \leq C_{m+2} (\|u(0)\|_m^2 + \|\nabla u(0)\|_{m-1}^2) \\ & + C_{m+2} \int_0^t \left((\|\nabla^2 p_1\|_{m-1} + \|\nabla p\|_{m-1}) \|\eta\|_m + \varepsilon^{-1} \|\nabla p_2\|_{m-1}^2 \right. \\ & \quad \left. + (1 + \|u\|_{2,\infty} + \|\nabla u\|_{1,\infty}) (\|\eta\|_{m-1}^2 + \|u\|_m^2 + \|\nabla u\|_{m-1}^2) \right) \end{aligned}$$

Note that by combining Proposition 4.2, Proposition 4.3 and (4.14), (4.18), we immediately obtain the global estimate

$$(4.20) \quad \begin{aligned} & \|u(t)\|_m^2 + \|\nabla u(t)\|_{m-1}^2 + \varepsilon \int_0^t \|\nabla \eta\|_{m-1}^2 \leq C_{m+2} (\|u(0)\|_m^2 + \|\nabla u(0)\|_{m-1}^2) \\ & + C_{m+2} \int_0^t \left(\|\nabla^2 p_1\|_{m-1} (\|u\|_m + \|\nabla u\|_{m-1}) + \varepsilon^{-1} \|\nabla p_2\|_{m-1}^2 \right. \\ & \quad \left. + (1 + \|u\|_{2,\infty} + \|\nabla u\|_{1,\infty}) (\|u\|_m^2 + \|\nabla u\|_{m-1}^2) \right) \end{aligned}$$

for $m \geq 2$.

Proof of Proposition 4.3. Note that $M = \nabla u$ solves the equation

$$\partial_t M + u \cdot \nabla M - \varepsilon \Delta M = -M^2 - \nabla^2 p$$

where $\nabla^2 p$ denotes the Hessian matrix of the pressure. Consequently, we get that η solves the equation

$$(4.21) \quad \partial_t \eta + u \cdot \nabla \eta - \varepsilon \Delta \eta = F - \chi \Pi(\nabla^2 p n)$$

where the source term F can be decomposed into

$$(4.22) \quad F = F^b + F^\chi + F^\kappa$$

where :

$$(4.23) \quad \begin{aligned} F^b &= -\chi \Pi((\nabla u)^2 + (\nabla u^t)^2) n - 2\alpha \chi \Pi \nabla p, \\ F^\chi &= -\varepsilon \Delta \chi \left(\Pi S u n + 2\alpha \Pi u \right) - 2\varepsilon \nabla \chi \cdot \nabla \left(\Pi S u n + 2\alpha \Pi u \right) \\ & \quad + (u \cdot \nabla \chi) \Pi \left((S u n + 2\alpha u) \right), \end{aligned}$$

$$(4.24) \quad \begin{aligned} F^\kappa &= \chi (u \cdot \nabla \Pi) \left(S u n + 2\alpha u \right) + \chi \Pi (S u (u \cdot \nabla n)) \\ & \quad - \varepsilon \chi (\Delta \Pi) \left(S u n + 2\alpha u \right) - 2\varepsilon \chi \nabla \Pi \cdot \nabla (S u n + 2\alpha u) \\ & \quad - \varepsilon \chi \Pi \left(S u \Delta n + 2\nabla S u \cdot \nabla n \right). \end{aligned}$$

Let us start with the proof of the L^2 energy estimate i.e. the case $m = 1$ in Proposition 4.3. By multiplying (4.21) by η , we immediately get that

$$(4.25) \quad \frac{d}{dt} \frac{1}{2} \|\eta\|^2 + \varepsilon \|\nabla \eta\|^2 = \int_\Omega F \cdot \eta - \int_\Omega \chi \Pi(\nabla^2 p n) \cdot \eta.$$

To estimate the right handside, we note that

$$(4.26) \quad \|F^b\|_{m-1} \leq C_m \left(\|u\|_{W^{1,\infty}} \|\nabla u\|_{m-1} + \|\nabla p\|_{m-1} \right)$$

and also that

$$(4.27) \quad \|F^\chi\|_{m-1} \leq C_{m+1} \left(\varepsilon \|\nabla u\|_m + (1 + \|u\|_{W^{1,\infty}}) \|u\|_m \right).$$

Note that we have used that since all the terms in F^χ are supported away from the boundary, we can control all the derivatives by the $\|\cdot\|_m$ norms. Finally, we also have that

$$(4.28) \quad \|F^\kappa\|_{m-1} \leq C_{m+2} \left(\varepsilon \|u\|_m + \varepsilon \|\nabla u\|_{m-1} + \varepsilon \|\chi \nabla^2 u\|_{m-1} + \|u\|_{W^{1,\infty}} (\|u\|_{m-1} + \|\nabla u\|_{m-1}) \right).$$

To estimate the last term in the right hand side of (4.21), we split the pressure to get

$$\left| \int_{\Omega} \chi \Pi(\nabla^2 p n) \cdot \eta \right| \leq \|\nabla^2 p_1\| \|\eta\| + \left| \int_{\Omega} \chi \Pi(\nabla^2 p_2 n) \cdot \eta \right|.$$

Since η vanishes on the boundary, we can integrate by parts the last term to obtain

$$\left| \int_{\Omega} \chi \Pi(\nabla^2 p_2 n) \cdot \eta \right| \leq C_2 \|\nabla p_2\| (\|\nabla \eta\| + \|\eta\|).$$

Consequently, by plugging these estimates into (4.25), we immediately get that

$$(4.29) \quad \begin{aligned} & \frac{d}{dt} \frac{1}{2} \|\eta\|^2 + \varepsilon \|\nabla \eta\|^2 \\ & \leq C_3 \left((\varepsilon \|\nabla u\|_1 + \varepsilon \|\chi \nabla^2 u\|) \|\eta\| + \|\nabla p_2\| (\|\nabla \eta\| + \|\eta\|) \right. \\ & \quad \left. + (\|\nabla^2 p_1\| + \|\nabla p_1\|) \|\eta\| + (1 + \|u\|_{W^{1,\infty}}) (\|u\|_1 + \|\nabla u\|) \right). \end{aligned}$$

To conclude, we only need to estimate $\varepsilon \|\chi \nabla^2 u\|$. Note that we have that

$$\varepsilon \|\chi \nabla^2 u\| \lesssim \varepsilon \|\chi \nabla \partial_n u\| + \varepsilon C_2 \|\nabla u\|_1$$

and hence, by using (4.14) and (4.17) that

$$\|\chi \nabla \partial_n u\| \leq C_3 (\|\nabla \eta\| + \|\nabla u\|_1 + \|u\|_1).$$

Consequently, by using (4.29) and the Young inequality, we finally get that

$$\begin{aligned} & \frac{d}{dt} \frac{1}{2} \|\eta\|^2 + \frac{\varepsilon}{2} \|\nabla \eta\|^2 \\ & \leq C_3 \left(\varepsilon \|\nabla u\|_1 \|\eta\| + (\|\nabla p\| + \|\nabla^2 p_1\|) \|\eta\| + \varepsilon^{-1} \|\nabla p_2\|^2 + (1 + \|u\|_{W^{1,\infty}}) (\|u\|_1 + \|\nabla u\|) \right). \end{aligned}$$

Since $\varepsilon \|\nabla u\|_1$ is already estimated in Proposition 4.2, this yields (4.19) for $m = 1$.

To prove the general case, let us assume that (4.19) is proven for $k \leq m - 2$. We get from (4.21) for $|\alpha| = m - 1$ that

$$\partial_t Z^\alpha \eta + u \cdot \nabla Z^\alpha \eta - Z^\alpha \Delta \eta = Z^\alpha F - Z^\alpha (\chi \Pi(\nabla^2 p n)) + \mathcal{C}$$

where

$$\mathcal{C} = -[Z^\alpha, u \cdot \nabla] \eta.$$

A standard energy estimate yields

$$(4.30) \quad \frac{d}{dt} \frac{1}{2} \|Z^\alpha \eta\|^2 \leq \varepsilon \int_{\Omega} Z^\alpha \Delta \eta \cdot Z^\alpha \eta + (\|F\|_{m-1} + \|\mathcal{C}\|) \|\eta\|_{m-1} - \int_{\Omega} Z^\alpha (\chi \Pi(\nabla^2 p n)) \cdot Z^\alpha \eta.$$

To estimate the first term in the right hand side, we need to estimate

$$I_k = \int_{\Omega} Z^\alpha \partial_{kk} \eta \cdot Z^\alpha \eta, \quad k = 1, 2, 3.$$

Towards this, we write

$$\begin{aligned} I_k &= \int_{\Omega} \partial_k Z^\alpha \partial_k \eta \cdot Z^\alpha \eta + \int_{\Omega} [Z^\alpha, \partial_k] \partial_k \eta \cdot Z^\alpha \eta \\ &= - \int_{\Omega} |\partial_k Z^\alpha \eta|^2 - \int_{\Omega} [Z^\alpha, \partial_k] \eta \cdot \partial_k Z^\alpha \eta + \int_{\Omega} [Z^\alpha, \partial_k] \partial_k \eta \cdot Z^\alpha \eta. \end{aligned}$$

Note that there is no boundary term in the integration by parts since $Z^\alpha \eta$ vanishes on the boundary. To estimate the last two terms above, we need to use the structure of the commutator $[Z^\alpha, \partial_k]$. By using the expansion

$$\partial_k = \beta^1 \partial_{y^1} + \beta^2 \partial_{y^2} + \beta^3 \partial_{y^3},$$

in the local basis, we get an expansion under the form

$$[Z^\alpha, \partial_k] f = \sum_{\gamma, |\gamma| \leq |\alpha| - 1} c_\gamma \partial_z Z^\gamma f + \sum_{\beta, |\beta| \leq |\alpha|} c_\beta Z^\beta f$$

where the \mathcal{C}^l norm of the coefficients is bounded by C_{l+m} . This yields the estimates

$$\begin{aligned} & \left| \int_{\Omega} [Z^\alpha, \partial_k] \eta \cdot \partial_k Z^\alpha \eta \right| \\ & \leq C_m \|\nabla \eta\|_{m-2} \|\nabla Z^{m-1} \eta\| \end{aligned}$$

and

$$\begin{aligned} & \left| \int_{\Omega} [Z^\alpha, \partial_k] \partial_k \eta \cdot Z^\alpha \eta \right| \\ & \leq \sum_{|\gamma| \leq m-2} \left| \int_{\Omega} c_\gamma \partial_z Z^\gamma \partial_k \eta \cdot Z^\alpha \eta \right| + C_m \|\nabla \eta\|_{m-1} \|\eta\|_{m-1}. \end{aligned}$$

Since $Z^\alpha \eta$ vanishes on the boundary, this yields thanks to an integration by parts

$$\left| \int_{\Omega} [Z^\alpha, \partial_k] \partial_k \eta \cdot Z^\alpha \eta \right| \leq C_{m+1} \|\nabla \eta\|_{m-1} (\|\nabla \eta\|_{m-2} + \|\eta\|_{m-1}).$$

Consequently, we get from (4.30) by summing over α and a new use of the Young inequality that

$$\begin{aligned} (4.31) \quad & \frac{d}{dt} \frac{1}{2} \|\eta\|_{m-1}^2 + \frac{\varepsilon}{2} \|\nabla Z^{m-1} \eta\|^2 \\ & \leq C_{m+1} \left(\varepsilon \|\nabla \eta\|_{m-2}^2 + \|\eta\|_{m-1}^2 + (\|F\|_{m-1} + \|C\|) \|\eta\|_{m-1} \right) - \int_{\Omega} Z^\alpha (\chi \Pi(\nabla^2 p n)) \cdot Z^\alpha \eta. \end{aligned}$$

To estimate the right hand side, we first notice that to control the term involving F , we can use (4.26), (4.27) and (4.28). This yields

$$\begin{aligned} (4.32) \quad \|F\|_{m-1} & \leq C_{m+2} \left((\varepsilon \|\nabla u\|_m + \varepsilon \|\chi \nabla^2 u\|_{m-1} + \|\nabla p\|_{m-1}) \|\eta\|_m \right. \\ & \quad \left. + (1 + \|u\|_{W^{1,\infty}}) (\|u\|_m + \|\nabla u\|_{m-1}) \right) \end{aligned}$$

It remains to estimate $\varepsilon \|\chi \nabla^2 u\|_{m-1}$. We can first use that

$$\varepsilon \|\chi \nabla^2 u\|_{m-1} \leq \varepsilon \|\chi \nabla \partial_n u\|_{m-1} + \varepsilon C_{m+1} (\|\nabla u\|_m + \|u\|_m).$$

Next, thanks to (4.17) and (4.14), we also get that

$$\varepsilon \|\chi \nabla \partial_n u\|_{m-1} \leq C_{m+2} \left(\varepsilon \|\nabla u\|_m + \|u\|_m + \|\nabla \eta\|_{m-1} \right)$$

and hence we obtain the estimate

$$(4.33) \quad \|F\|_{m-1} \leq C_{m+2} \left((\varepsilon \|\nabla u\|_m + \varepsilon \|\nabla \eta\|_{m-1} + \|\nabla p\|_{m-1}) \|\eta\|_m \right. \\ \left. + (1 + \|u\|_{W^{1,\infty}}) (\|u\|_m + \|\nabla u\|_{m-1}) \right).$$

In view of (4.31), it remains to estimate $\|\mathcal{C}\|$. Note that by using the local coordinates, we can expand:

$$u \cdot \nabla \eta = u_1 \partial_{y^1} \eta + u_2 \partial_{y^2} \eta + u \cdot \mathbf{N} \partial_z \eta.$$

Consequently, the estimate (3.29) also holds for this term, we thus get that

$$(4.34) \quad \|\mathcal{C}\| \leq C_m \left(\|u\|_{2,\infty} + \|u\|_{W^{1,\infty}} + \|Z\eta\|_{L^\infty} \right) (\|\eta\|_{m-1} + \|u\|_m).$$

Finally, it remains to estimate the last term involving the pressure in the right hand side of (4.31). As before, we use the splitting $p = p_1 + p_2$ and we integrate by parts the term involving p_2 . This yields

$$(4.35) \quad \left| \int_{\Omega} Z^\alpha (\chi \Pi(\nabla^2 p n)) \cdot Z^\alpha \eta \right| \leq C_{m+2} (\|\nabla^2 p_1\|_{m-1} \|\eta\|_m + \|\nabla p_2\|_{m-1} (\|\nabla Z^m \eta\| + \|\eta\|_m)).$$

By combining (4.31), (4.33), (4.34), (4.35) and by using the induction assumption and the Young inequality, we get the result.

4.5. Pressure estimates.

Proposition 4.4. *For $m \geq 2$, we have the following estimate for the pressure:*

$$(4.36) \quad \|\nabla p_1\|_{m-1} + \|\nabla^2 p_1\|_{m-1} \leq C_{m+2} (1 + \|u\|_{W^{1,\infty}}) (\|u\|_m + \|\nabla u\|_{m-1}),$$

$$(4.37) \quad \|\nabla p_2\|_{m-1} \leq C_{m+2} \varepsilon (\|\nabla u\|_{m-1} + \|u\|_m).$$

Note that thanks to (4.37), we have that

$$\varepsilon^{-1} \|\nabla p_2\|_{m-1}^2 \leq C_{m+2} (\|u\|_m^2 + \|\nabla u\|_{m-1}^2).$$

Consequently, by combining (4.20) and Proposition 4.4, we get that

$$(4.38) \quad \|u(t)\|_m^2 + \|\nabla u(t)\|_{m-1}^2 + \varepsilon \int_0^t \|\nabla^2 u\|_{m-1}^2 \\ \leq C_{m+2} (\|u_0\|_m^2 + \|\nabla u_0\|_m^2) + C_{m+2} \int_0^t \left((1 + \|u\|_{2,\infty} + \|\nabla u\|_{1,\infty}) (\|u\|_m^2 + \|\nabla u\|_{m-1}^2) \right)$$

Proof. We recall that we have $p = p_1 + p_2$ where

$$(4.39) \quad \Delta p_1 = -\nabla \cdot (u \cdot \nabla u) = -\nabla u \cdot \nabla u, \quad x \in \Omega, \quad \partial_n p_1 = -(u \cdot \nabla u) \cdot n, \quad x \in \partial\Omega$$

and

$$(4.40) \quad \Delta p_2 = 0, \quad x \in \Omega, \quad \partial_n p_2 = \varepsilon \Delta u \cdot n, \quad x \in \partial\Omega.$$

From standard elliptic regularity results with Neumann boundary conditions, we get that

$$\|\nabla p_1\|_{m-1} + \|\nabla^2 p_1\|_{m-1} \leq C_{m+1} \left(\|\nabla u \cdot \nabla u\|_{m-1} + \|u \cdot \nabla u\| + |(u \cdot \nabla u) \cdot n|_{H^{m-\frac{1}{2}}(\partial\Omega)} \right).$$

Since $u \cdot n = 0$ on the boundary, we note that

$$(u \cdot \nabla u) \cdot n = -(u \cdot \nabla n) \cdot u, \quad x \in \partial\Omega$$

and consequently, thanks to the trace Theorem, we obtain that

$$|(u \cdot \nabla u) \cdot n|_{H^{m-\frac{1}{2}}(\partial\Omega)} \leq C_{m+2} (\|\nabla(u \otimes u)\|_{m-1} + \|u \otimes u\|_{m-1}).$$

Thanks to a new use of Lemma 3.3, this yields

$$\|\nabla p_1\|_{m-1} + \|\nabla^2 p_1\|_{m-1} \leq C_{m+2} \left((1 + \|u\|_{W^{1,\infty}}) (\|u\|_m + \|\nabla u\|_{m-1}) \right).$$

It remains to estimate p_2 . By using again the elliptic regularity for the Neumann problem, we get that for $m \geq 2$,

$$(4.41) \quad \|\nabla p_2\|_{m-1} \leq \varepsilon C_m |\Delta u \cdot n|_{H^{m-\frac{3}{2}}(\partial\Omega)}.$$

To estimate the right hand side, we shall again use the Navier boundary condition (1.2). Since

$$2\Delta u \cdot n = \nabla \cdot (Su n) - \sum_j (Su \partial_j n)_j,$$

we first get that

$$|\Delta u \cdot n|_{H^{m-\frac{3}{2}}(\partial\Omega)} \lesssim |\nabla \cdot (Su n)|_{H^{m-\frac{3}{2}}(\partial\Omega)} + C_{m+1} |\nabla u|_{H^{m-\frac{3}{2}}(\partial\Omega)}$$

and hence thanks to (4.8) and (4.3) that

$$|\Delta u \cdot n|_{H^{m-\frac{3}{2}}(\partial\Omega)} \lesssim |\nabla \cdot (Su n)|_{H^{m-\frac{3}{2}}(\partial\Omega)} + C_{m+1} |u|_{H^{m-\frac{1}{2}}(\partial\Omega)}.$$

To estimate the first term, we can use the expression (4.8) to get

$$|\nabla \cdot (Su n)|_{H^{m-\frac{3}{2}}(\partial\Omega)} \lesssim |\partial_n (Su n) \cdot n|_{H^{m-\frac{3}{2}}(\partial\Omega)} + C_{m+1} \left(|\Pi(Su n)|_{H^{m-\frac{1}{2}}(\partial\Omega)} + |\nabla u|_{H^{m-\frac{3}{2}}(\partial\Omega)} \right)$$

and hence by using again (4.8), (4.3) and (1.2), we obtain that

$$|\nabla \cdot (Su n)|_{H^{m-\frac{3}{2}}(\partial\Omega)} \lesssim |\partial_n (Su n) \cdot n|_{H^{m-\frac{3}{2}}(\partial\Omega)} + C_{m+1} |u|_{H^{m-\frac{1}{2}}(\partial\Omega)}.$$

The first term above in the right hand side can be estimated by

$$\begin{aligned} |\partial_n (Su n) \cdot n|_{H^{m-\frac{3}{2}}(\partial\Omega)} &\lesssim |\partial_n (\partial_n u \cdot n)|_{H^{m-\frac{3}{2}}(\partial\Omega)} + C_{m+1} |\nabla u|_{H^{m-\frac{3}{2}}(\partial\Omega)} \\ &\lesssim |\partial_n (\partial_n u \cdot n)|_{H^{m-\frac{3}{2}}(\partial\Omega)} + C_{m+1} |u|_{H^{m-\frac{1}{2}}(\partial\Omega)}. \end{aligned}$$

Finally, taking the normal derivative of (4.8), we get that

$$\begin{aligned} |\partial_n (\partial_n u \cdot n)|_{H^{m-\frac{3}{2}}(\partial\Omega)} &\lesssim |\Pi \partial_n u|_{H^{m-\frac{1}{2}}(\partial\Omega)} + C_{m+1} |\nabla u|_{H^{m-\frac{3}{2}}(\partial\Omega)} \\ &\lesssim C_{m+2} |u|_{H^{m-\frac{1}{2}}(\partial\Omega)} \end{aligned}$$

where the last line comes from a new use of (4.3). Note that this is the estimate of this term which requires the more regularity of the boundary.

Consequently, we have proven that

$$|\Delta u \cdot n|_{H^{m-\frac{3}{2}}(\partial\Omega)} \leq C_{m+2} |u|_{H^{m-\frac{1}{2}}(\partial\Omega)}$$

and hence by using (4.41) and the trace Theorem, we get that

$$\|\nabla p_2\|_{m-1} \leq C_{m+2} \varepsilon (\|u\|_m + \|\nabla u\|_{m-1}).$$

This ends the proof of Proposition 4.4.

4.6. L^∞ estimates. In order to close the estimates, we need an estimate of the L^∞ norms in the right hand side. As before, let us set

$$N_m(t) = \|u(t)\|_m^2 + \|\nabla u(t)\|_{m-1}^2 + \|\nabla u\|_{1,\infty}^2.$$

Proposition 4.5. *For $m_0 > 1$, we have*

$$(4.42) \quad \|u\|_{2,\infty} \leq C_m (\|u\|_m + \|\nabla u\|_{m-1}), \quad m \geq m_0 + 3,$$

$$(4.43) \quad \|u\|_{W^{1,\infty}} \leq C_m (\|u\|_m + \|\nabla u\|_{m-1}), \quad m \geq m_0 + 2.$$

Proof. It suffices to use local coordinates and Proposition 3.7.

In view of Proposition, we still need to estimate $\|\nabla u\|_{1,\infty}$.

Proposition 4.6. *For $m > 6$, we have the estimate*

$$\|\nabla u(t)\|_{1,\infty}^2 \leq C_{m+2} \left(N_m(0) + (1+t+\varepsilon^3 t^2) \int_0^t (N_m(s) + N_m(s)^2) ds \right).$$

Proof. Away from the boundary, we clearly have by the classical isotropic Sobolev embedding that

$$(4.44) \quad \|\chi \nabla u\|_{1,\infty} \lesssim \|u\|_m, \quad m \geq 4.$$

Consequently, by using a partition of unity subordinated to the covering (4.1) we only have to estimate $\|\chi_i \nabla u\|_{L^\infty}$, $i > 0$. For notational convenience, we shall denote χ_i by χ . Towards this, we want to proceed as in the proof of Proposition 3.8. An important step in this proof was to use Lemma 3.10. It is thus crucial to choose a system of coordinates in which the Laplacian has a convenient form. In this section, we shall use a local parametrization in the vicinity of the boundary given by a normal geodesic system:

$$\Psi^n(y, z) = \begin{pmatrix} y \\ \psi(y) \end{pmatrix} - z n(y)$$

where

$$n(y) = \frac{1}{(1 + |\nabla \psi(y)|^2)^{\frac{1}{2}}} \begin{pmatrix} \partial_1 \psi(y) \\ \partial_2 \psi(y) \\ -1 \end{pmatrix}$$

is the unit outward normal. We have not used this coordinate system to estimate the conormal derivatives because it requires more regularity on the boundary. Nevertheless, it does not yield any restriction on the regularity of the boundary here, since we need to estimate a lower number of derivatives. As before, we can extend n and Π in the interior by setting

$$n(\Psi^n(y, z)) = n(y), \quad \Pi(\Psi^n(y, z)) = \Pi(y).$$

Note that $n(y)$ and $\Pi(y)$ have different definitions from the ones used before. The interest of this parametrization is that in the associated local basis of \mathbb{R}^3 $(\partial_{y^1}, \partial_{y^2}, \partial_z)$, we have $\partial_z = \partial_n$ and

$$\left(\partial_{y^i} \right)_{/\Psi^n(y,z)} \cdot \left(\partial_z \right)_{/\Psi^n(y,z)} = 0.$$

The scalar product on \mathbb{R}^3 thus induces in this coordinate system the Riemannian metric g under the form

$$(4.45) \quad g(y, z) = \begin{pmatrix} \tilde{g}(y, z) & 0 \\ 0 & 1 \end{pmatrix}.$$

Consequently, the Laplacian in this coordinate system reads:

$$(4.46) \quad \Delta f = \partial_{zz} f + \frac{1}{2} \partial_z (\ln |g|) \partial_z f + \Delta_{\tilde{g}} f$$

where $|g|$ denotes the determinant of the matrix g and $\Delta_{\tilde{g}}$ which is defined by

$$\Delta_{\tilde{g}}f = \frac{1}{|\tilde{g}|^{\frac{1}{2}}} \sum_{1 \leq i, j \leq 2} \partial_{y^i} (\tilde{g}^{ij} |\tilde{g}|^{\frac{1}{2}} \partial_{y^j} f)$$

involves only tangential derivatives.

Next, we can observe that thanks to (4.8) (in the coordinate system that we have just defined) and Proposition 4.5 we have that

$$(4.47) \quad \|\chi \nabla u\|_{1,\infty} \leq C_3 (\|\chi \Pi \partial_n u\|_{1,\infty} + \|u\|_{2,\infty}) \leq C_3 (\|\chi \Pi \partial_n u\|_{1,\infty} + \|u\|_m + \|\nabla u\|_{m-1}).$$

Consequently, we need to estimate $\|\chi \Pi \partial_n u\|_{1,\infty}$. To estimate this quantity, it is useful to introduce the vorticity

$$\omega = \nabla \times u.$$

Indeed, by definition, we have

$$(4.48) \quad \Pi(\omega \times n) = \frac{1}{2} \Pi(\nabla u - \nabla u^t) n = \frac{1}{2} \Pi(\partial_n u - \nabla(u \cdot n) + u \cdot \nabla n + u \times (\nabla \times n)).$$

Consequently, we find that

$$\|\chi \Pi \partial_n u\|_{1,\infty} \leq C_3 (\|\chi \Pi(\omega \times n)\|_{1,\infty} + \|u\|_{2,\infty})$$

and hence by a new use of Proposition 4.5, we get that

$$(4.49) \quad \|\chi \Pi \partial_n u\|_{1,\infty} \leq C_3 (\|\chi \Pi(\omega \times n)\|_{1,\infty} + \|u\|_m + \|\nabla u\|_{m-1}).$$

In other words, we only need to estimate $\|\chi \Pi(\omega \times n)\|_{1,\infty}$ in order to conclude. Note that ω solves the vorticity equation

$$(4.50) \quad \partial_t \omega + u \cdot \nabla \omega - \varepsilon \Delta \omega = \omega \cdot \nabla u = F^\omega.$$

Consequently, by setting in the support of χ

$$\tilde{\omega}(y, z) = \omega(\Psi^n(y, z)), \quad \tilde{u}(y, z) = u(\Psi^n(y, z)),$$

we get that

$$(4.51) \quad \partial_t \tilde{\omega} + \tilde{u}^1 \partial_{y^1} \tilde{\omega} + \tilde{u}^2 \partial_{y^2} \tilde{\omega} + \tilde{u} \cdot n \partial_z \tilde{\omega} = \varepsilon (\partial_{zz} \tilde{\omega} + \frac{1}{2} \partial_z (\ln |g|) \partial_z \tilde{\omega} + \Delta_{\tilde{g}} \tilde{\omega}) + F^\omega$$

and

$$(4.52) \quad \partial_t \tilde{u} + \tilde{u}^1 \partial_{y^1} \tilde{u} + \tilde{u}^2 \partial_{y^2} \tilde{u} + \tilde{u} \cdot n \partial_z \tilde{u} = \varepsilon (\partial_{zz} \tilde{u} + \frac{1}{2} \partial_z (\ln |g|) \partial_z \tilde{u} + \Delta_{\tilde{g}} \tilde{u}) - (\nabla p) \circ \Psi^n.$$

Note that we use the same convention as before for a vector u , u^i denotes the components of u in the local basis $(\partial_{y^1}, \partial_{y^2}, \partial_z)$ whereas u_i denotes its components in the canonical basis of \mathbb{R}^3 . The vectorial equations (4.51) and (4.52) have to be understood components by components in the standard basis of \mathbb{R}^3 .

By using (4.48) on the boundary and the Navier boundary condition (4.3), we get that for $z = 0$

$$\Pi(\tilde{\omega} \times n) = \Pi(\tilde{u} \cdot \nabla n - \alpha \tilde{u}).$$

Consequently, we set

$$(4.53) \quad \tilde{\eta}(y, z) = \chi \Pi(\tilde{\omega} \times n - \tilde{u} \cdot \nabla n + \alpha \tilde{u}).$$

We thus get that

$$(4.54) \quad \tilde{\eta}(y, 0) = 0$$

and that $\tilde{\eta}$ solves the equation

$$(4.55) \quad \partial_t \tilde{\eta} + \tilde{u}^1 \partial_{y^1} \tilde{\eta} + \tilde{u}^2 \partial_{y^2} \tilde{\eta} + \tilde{u} \cdot n \partial_z \tilde{\eta} = \varepsilon (\partial_{zz} \tilde{\eta} + \frac{1}{2} \partial_z (\ln |g|) \partial_z \tilde{\eta}) + \chi \Pi F^\omega \times n + F^u + F^\chi + F^\kappa$$

where the source terms are given by

$$(4.56) \quad F^u = \chi \Pi \left(\nabla p \cdot \nabla n - \alpha \nabla p \right) \circ \Psi^n,$$

$$(4.57) \quad F^\chi = \left((\tilde{u}^1 \partial_{y^1} + \tilde{u}^2 \partial_{y^2} + u \cdot n \partial_z) \chi \right) \Pi (\tilde{\omega} \times n - \tilde{u} \cdot \nabla n + \alpha \tilde{u}) \\ - \varepsilon \left(\partial_{zz} \chi + 2\varepsilon \partial_z \chi \partial_z + \varepsilon \frac{1}{2} \partial_z (\ln |g|) \partial_z \chi \right) \Pi (\tilde{\omega} \times n - \tilde{u} \cdot \nabla n + \alpha \tilde{u})$$

$$(4.58) \quad F^\kappa = \left((\tilde{u}^1 \partial_{y^1} + \tilde{u}^2 \partial_{y^2}) \Pi \right) \tilde{\omega} \times n - \tilde{u} \cdot \nabla n + \alpha \tilde{u} + \Pi \left((\tilde{u}^1 \partial_{y^1} + \tilde{u}^2 \partial_{y^2}) n \right) \\ - \Pi \left(((\tilde{u}^1 \partial_{y^1} + \tilde{u}^2 \partial_{y^2}) \nabla n) u \right) \\ - \varepsilon \Delta_{\tilde{g}} \left(\chi \Pi (\tilde{\omega} \times n - \tilde{u} \cdot \nabla n + \alpha \tilde{u}) \right).$$

Note that in computing the source terms and in particular F^κ which contains all the commutators coming from the fact that Π and n are not constant, we have used that in the coordinate system that we have chosen, Π and n do not depend on the normal variable. By using that $\Delta_{\tilde{g}}$ only involves tangential derivatives and that the derivatives of χ are compactly supported away from the boundary, we get the estimates

$$\|F^u\|_{1,\infty} \leq C_3 \|\Pi \nabla p\|_{1,\infty}, \\ \|F^\chi\|_{1,\infty} \leq C_3 \left(\|u\|_{1,\infty} \|u\|_{2,\infty} + \varepsilon \|u\|_{3,\infty} \right), \\ \|F^\kappa\|_{1,\infty} \leq C_4 \left(\|u\|_{1,\infty} \|\nabla u\|_{1,\infty} + \varepsilon (\|\nabla u\|_{3,\infty} + \|u\|_{3,\infty}) \right).$$

Note that the fact that the term $(\nabla p \cdot \nabla) n$ in (4.56) contains only tangential derivatives of the pressure comes from the block diagonal structure of the metric (4.45) and the fact that n does not depend on the normal variable z .

Consequently, by using Proposition 4.5, we get that

$$(4.59) \quad \|F\|_{1,\infty} \leq C_4 \left(\|\Pi \nabla p\|_{1,\infty} + Q_m + \varepsilon \|\nabla u\|_{3,\infty} \right), \quad m \geq m_0 + 4$$

where $F = F^u + F^\chi + F^\kappa$.

In order to be able to use Lemma 3.9, we shall perform a last change of unknown in order to eliminate the term $\partial_z (\ln |g|) \partial_z \tilde{\eta}$ in (4.55). We set

$$\tilde{\eta} = \frac{1}{|g|^{\frac{1}{4}}} \eta = \gamma \eta.$$

Note that we have

$$(4.60) \quad \|\tilde{\eta}\|_{1,\infty} \lesssim C_3 \|\eta\|_{1,\infty}, \quad \|\eta\|_{1,\infty} \lesssim C_3 \|\tilde{\eta}\|_{1,\infty}$$

and that moreover, η solves the equation

$$(4.61) \quad \partial_t \eta + \tilde{u}^1 \partial_{y^1} \tilde{\eta} + \tilde{u}^2 \partial_{y^2} \tilde{\eta} + \tilde{u} \cdot n \partial_z \tilde{\eta} - \varepsilon \partial_{zz} \tilde{\eta}$$

$$(4.62) \quad = \frac{1}{\gamma} \left(\chi \Pi F^\omega \times n + F^u + F^\chi + F^\kappa + \varepsilon \partial_{zz} \gamma \eta + \frac{\varepsilon}{2} \partial_z \ln |g| \partial_z \gamma \eta - (\tilde{u} \cdot \nabla \gamma) \eta \right) := \mathcal{S}.$$

Consequently, by using Lemma 3.9, we get that

$$\begin{aligned} \|\eta(t)\|_{1,\infty} &\lesssim \|\eta_0\|_{1,\infty} + \int_0^t \left((\|\tilde{u}\|_{2,\infty} + \|\partial_z \tilde{u}\|_{1,\infty}) (\|\eta\|_{1,\infty} + \|\eta\|_{m_0+3}) + \|\mathcal{S}\|_{1,\infty} \right) \\ &\lesssim \|\eta_0\|_{1,\infty} + C_3 \int_0^t \left((\|u\|_{2,\infty} + \|\nabla u\|_{1,\infty}) (\|\eta\|_{1,\infty} + \|\eta\|_{m_0+3}) + \|\mathcal{S}\|_{1,\infty} \right). \end{aligned}$$

Consequently, we can use (4.49), (4.53), (4.60), (4.59) and Proposition 4.5 to get as in the proof of Proposition 3.8 that

$$\begin{aligned} \|\chi \Pi \partial_n u(t)\|_{1,\infty}^2 &\leq C_{m+1} \left(\|u(t)\|_m^2 + \|\nabla u(t)\|_{m-1}^2 + N_m(0) + \varepsilon \int_0^t \|\nabla^2 u\|_{m-1}^2 \right. \\ &\quad \left. + (1+t+\varepsilon^3 t^2) \int_0^t (N_m(s) + N_m^2(s) + \|\Pi \nabla p\|_{1,\infty}^2) ds \right). \end{aligned}$$

Since $\Pi \nabla p$ involves only tangential derivatives, we get thanks to the anisotropic Sobolev embedding that for $m \geq 4$

$$\|\Pi \nabla p\|_{1,\infty}^2 \leq C_m \|\nabla p\|_{m-1}^2.$$

Consequently, the proof of Proposition 4.6 follows by using (4.38) and Proposition 4.4.

4.7. Proof of Theorem 4.1. It suffices to combine Proposition 4.6 and the estimate (4.38).

5. PROOF OF THEOREM 1.1

To prove that (1.1), (1.2) is locally well-posed in the function space $E^m \cap \text{Lip}$, one can for example smooth the initial data in order to use a standard well-posedness result and then use the priori estimates given in Theorem 4.1 and a compactness argument to prove the local existence of a solution (we shall not give more details since the compactness argument is almost the same as the one needed for the proof of Theorem 1.2). The uniqueness of the solution is clear since we work with functions with Lipschitz regularity. The fact that the life time of the solution is independent of the viscosity ε then follows by using again Theorem 4.1 and a continuous induction argument.

6. PROOF OF THEOREM 1.2

Thanks to Theorem 1.1, the apriori estimate (1.5) holds on $[0, T]$. In particular, for each t , $u^\varepsilon(t)$ is bounded in H_{co}^m and $\nabla u^\varepsilon(t)$ is bounded in H_{co}^{m-1} . This yields that for each t , $u^\varepsilon(t)$ is compact in H_{co}^{m-1} . Next, by using the equation (1.1), we get that

$$\int_0^T \|\partial_t u^\varepsilon(t)\|_{m-1}^2 \leq \int_0^T (\varepsilon^2 \|\nabla^2 u^\varepsilon\|_{m-1}^2 + \|\nabla p^\varepsilon\|_{m-1}^2 + \|u^\varepsilon \cdot \nabla u^\varepsilon\|_{m-1}^2) ds$$

and hence by using Lemma 3.3 and Proposition 4.4, we get that

$$\int_0^T \|\partial_t u^\varepsilon(t)\|_{m-1}^2 \leq \int_0^T (\varepsilon^2 \|\nabla^2 u^\varepsilon\|_{m-1}^2 + (1 + \|u\|_m^2 + \|\nabla u\|_{m-1}^2 + \|\nabla u\|_{L^\infty}^2) (\|u\|_m^2 + \|\nabla u\|_{m-1}^2)) ds.$$

Consequently, thanks to the uniform estimate (1.5), we get that $\partial_t u^\varepsilon$ is uniformly bounded in $L^2(0, T, H_{co}^{m-1})$.

From the Ascoli Theorem, we thus get that u^ε is compact in $\mathcal{C}([0, T], H_{co}^{m-1})$. In particular, there exists a sequence ε_n and $u \in \mathcal{C}([0, T], H_{co}^{m-1})$ such that u^{ε_n} converges towards u in $\mathcal{C}([0, T], H_{co}^{m-1})$. By using again the uniform bounds (1.5), we get that $u \in \text{Lip}$. Thanks to the anisotropic Sobolev embedding (3.45), we also have that for $m_0 > 1$

$$\sup_{[0, T]} \|u^{\varepsilon_n}(t) - u(t)\|_{L^\infty} \leq \sup_{[0, T]} (\|\nabla(u^{\varepsilon_n} - u)\|_{m_0} \|u^{\varepsilon_n} - u\|_{m_0} + \|u^{\varepsilon_n} - u\|_{m_0}^2)$$

and hence again thanks to the uniform bound (1.5), we get that u^{ε^n} converges uniformly towards u on $[0, T] \times \Omega$. Moreover, it is easy to check that u is a weak solution of the Euler equation.

Finally since $u \in L^\infty([0, T], L^2 \cap \text{Lip})$, u is actually unique and hence we get that the whole family u^ε converges towards u . This ends the proof of Theorem 1.2.

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