ENERGY SCATTERING FOR THE 2D CRITICAL WAVE EQUATION

S. IBRAHIM, M. MAJDOUB, N. MASMOUDI, AND K. NAKANISHI

Abstract. We investigate existence and asymptotic completeness of the wave operators for nonlinear Klein-Gordon and Schrödinger equations with a defocusing exponential nonlinearity in two space dimensions. A certain threshold is defined based on the value of the conserved Hamiltonian, below which the exponential potential energy is dominated by the kinetic energy via a Trudinger-Moser type inequality. We prove that if the energy is below or equal to the critical value, then the solution approaches a free Klein-Gordon solution at the time infinity. The interesting feature in the critical case is that the Strichartz estimate together with Sobolev-type inequalities can not control the nonlinear term uniformly on each time interval, but with constants depending on how much the solution is concentrated. Thus we have to trace concentration of the energy along time, in order to set up favorable nonlinear estimates, and then to implement Bourgain's induction argument. We show the same result for the "subcritical" nonlinear Schrödinger equation.

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1. Introduction

We study the scattering theory in the energy space for nonlinear Klein-Gordon equation (NLKG):

$$\begin{cases} \ddot{u} - \Delta u + u + f(u) = 0, \quad u : \mathbb{R}^{1+2} \to \mathbb{R}, \\ u(0,x) = u_0(x) \in H^1(\mathbb{R}^2), \quad \partial_t u(0,x) = u_1(x) \in L^2(\mathbb{R}^2), \end{cases} \tag{1.1}$$

where the nonlinearity $f : \mathbb{C} \to \mathbb{C}$ is defined by

$$f(u) = \left(e^{4\pi|u|^2} - 1 - 4\pi|u|^2\right) u, \tag{1.2}$$

and for nonlinear Schrödinger equation (NLS):

$$\begin{cases} i\dot{u} + \Delta u = f(u), \quad u : \mathbb{R}^{1+2} \to \mathbb{C}, \\ u(0,x) = u_0(x) \in H^1(\mathbb{R}^2). \end{cases} \tag{1.3}$$

Problem (1.1) has the conserved energy

$$\begin{aligned} E(u,t) &= \int_{\mathbb{R}^2} \left(|\dot{u}|^2 + |\nabla u|^2 + |u|^2 + 2F(u)\right) dx, \\ &:= E_0(u,t) + \int_{\mathbb{R}^2} 2F(u)\, dx. \end{aligned} \tag{1.4}$$

where we denote

$$F(u) := \frac{1}{8\pi} \left(e^{4\pi|u|^2} - 1 - 4\pi|u|^2 - 8\pi^2|u|^4\right). \tag{1.5}$$

Solutions of (1.3) satisfy the conservation of mass and Hamiltonian

$$M(u,t) := \int_{\mathbb{R}^2} |u|^2 dx, \tag{1.6}$$

$$H(u,t) := \int_{\mathbb{R}^2} \left(|\nabla u|^2 + 2F(u)\right) dx. \tag{1.7}$$

The exponential type nonlinearities appear in several applications, as for example the self trapped beams in plasma. (See [27]). ¿From the mathematical point of view, Cazenave in [8] considered the Schrödinger equation with decreasing exponential and showed the global well-posedness and scattering. With increasing exponentials, the situation is much more complicated (since there is no a priori L^∞ control of the nonlinear term). The two dimensional case is particularly interesting because of its relation to the critical Sobolev (or Trudinger-Moser) embedding. On the other hand, we have subtracted the cubic part from our nonlinearity f in order to avoid another critical exponent related to the decay property of solutions. To explain these issues, we start with a brief review of the more familiar power case.

1.1. **The energy critical NLKG.** In any space dimension $d \geq 1$, the monomial defocusing nonlinear Klein-Gordon equation reads

$$\ddot{u} - \Delta u + u + |u|^{p-1}u = 0, \quad u : \mathbb{R}^{1+d} \to \mathbb{R}. \tag{1.8}$$

The mass term u is irrelevant for local time behavior, so that we can ignore it in the well-posedness issue, but it has essential impacts on the long time behavior, so we must distinguish it from the massless wave in the scattering theory.

The global solvability in the energy space of (1.8) has a long history. The Sobolev critical power p appears when $d \geq 3$, namely $p^* := \frac{d+2}{d-2} = \frac{2d}{d-2} - 1$, and there are mainly three cases.

In the subcritical case ($p < p^*$), Ginibre and Velo finally proved in [16] the global well-posedness in the energy space, extending several preceding works which had limitations in the range of power and/or the solution space.

The critical case ($p = p^*$) is much more delicate. The global existence of smooth solutions was first proved by Struwe [42] in the radially symmetric case, then by Grillakis [20, 21] without the symmetry assumption. For the energy space, Ginibre, Soffer and Velo [15] proved the global well-posedness in the radial case, and then Shatah-Struwe [39] in the general case. We note that the uniqueness in the energy space is not yet fully settled (see [36] for the case $d \geq 4$ and [43, 30] for partial results in $d \geq 3$.)

The supercritical case ($p > p^*$) is even much harder and the question remains essentially open, except for the existence of global weak solutions [40], and some negative results about non-smoothness of the solution map [28], and loss of regularity [29].

Concerning the scattering theory of (1.8) another critical value of p appears, namely $p_* := 1 + \frac{4}{d}$. It is linked to the space-time property of the linear solutions. More precisely, the optimal space-time integrability of free Klein-Gordon solutions in the energy space is given by the Strichartz estimate

$$\|u\|_{L^q(\mathbb{R}^{1+d})} \leq CE_0(u)^{1/2}, \quad 2 + \frac{4}{d} \leq q \leq 2 + \frac{6}{d-2}, \tag{1.9}$$

and p_* is determined by the relation $p_*q/(q-1) = q$ for the smallest $q = 2 + 4/d$. The scattering between these two powers $p_* < p < p^*$ was solved in [7, 17] for $d \geq 3$, and it was later extended to $d \leq 2$ in [34], by generalizing the Morawetz estimate to lower dimensions. The scattering for the Sobolev critical case $p = p^*$ was solved in [35], but the lower critical case $p = p_*$ still remains open, which is the reason that we removed the cubic term from our nonlinearity f. We remark that the scattering for the massless wave equation is available in the energy space only if the nonlinearity is dominated by the Sobolev critical power p^* (see [19] for the sub-super-critical case and [4, 3] for the critical case).

For $d \leq 2$, we have no upper bound for the power nonlinearity (1.8). But if we consider more general nonlinearity, we still need some growth condition in the two dimensional case, because of the failure of the Sobolev embedding $H^1(\mathbb{R}^2) \subset L^\infty(\mathbb{R}^2)$. Then the exponential nonlinearity (1.1) naturally arises in the connection to the Trudinger-Moser inequality. One should note, however, the size of solution becomes more crucial for the exponential nonlinearity than the power case. Indeed, solutions of smaller size can be regarded heuristically as giving smaller power in the exponential (it is true if the size is measured in L^∞).

Thus the theory for the exponential nonlinearity (1.1) was worked out first by Nakamura-Ozawa in [32, 33] for sufficiently small data in the energy space, based on the Trudinger-Moser and the Strichartz estimates, establishing the global existence as well as the scattering of such small solutions. Later on, the size of the initial data for which one has global existence was quantified, first in [2] for radially symmetric initial data $(0, u_1)$, and then in [22, 24] for general data in the energy space.

Furthermore, [22] and [24] established the following trichotomy in the dynamic. This trichotomy is similar to the power case for $d \geq 3$, but the difference is that the threshold is given by the energy size, instead of the power. More precisely,

Definition 1.1. *The Cauchy problem* (1.1) *is said to be subcritical if* $E(u, 0) < 1$, *critical if* $E(u, 0) = 1$ *and supercitical if* $E(u, 0) > 1$.

Indeed, one can construct a unique local solution if $\|\nabla u_0\|_{L^2} < 1$, and the time of existence depends only on $\eta := 1 - \|\nabla u_0\|_{L^2}^2$. Actually, it can be constructed even without any size restriction [25], but then the existence time depends fully on the initial data, not only in terms of the norm.

In the subcritical case, the conservation of energy gives a priori lower bound on η, hence the maximal local solutions are global in the subcritical case. In the critical case, the situation is much more delicate, and arguments based on the non-concentration of the energy were investigated to extend the local solutions.

Theorem 1.2 (Global well-posedness [22])**.** *Assume that* $E(u, 0) \leq 1$. *Then* (1.1) *has a unique*[1] *global solution* $u \in \mathcal{C}(\mathbb{R}, H^1) \cap \mathcal{C}^1(\mathbb{R}, L^2)$. *Moreover,* $u \in L^4_{loc}(\mathbb{R}, \mathcal{C}^{1/4})$.

The local well-posedness proof of Theorem 1.2 is based on a fixed point argument. Thus, the solution map (data gives solution) is uniformly continuous. However, in the super-critical case, it is not [24].

We should emphasize that the smallness condition in the works by Nakamura-Ozawa is essentially below the threshold (at most 1/2), even for the global existence part. This is because the Trudinger-Moser controls only the L^1 norm of the critical nonlinearity, which is insufficient for the wave equation in the energy space. Hence the smallness was used for them effectively to improve the exponent to L^2. This gap was filled out in [22] by using a logarithmic inequality.

Moreover, for the scattering problem, the smallness condition was used as the only source to ensure contractiveness for the fixed point argument globally in time. It is very unlikely for such an argument to hold in the full range of the subcritical regime, no matter how we improve the estimates. Hence we need more elaborate

[1] Note that the uniqueness is in the energy space. This is different from the Sobolev critical case in three dimensions, where only partial results are available [43, 30].

arguments to use global dispersion of the nonlinear solutions as another source of smallness, just as in the scattering results in the power case without size restriction.

1.2. **The energy critical NLS.** There are almost parallel stories for the nonlinear Schrödinger equations. Recall the monomial defocusing semilinear Schrödinger equation in space dimension $d \geq 1$

$$i\,\dot{u} + \Delta u = |u|^{p-1}u, \quad u : \mathbb{R}^{1+d} \longmapsto \mathbb{C}, \tag{1.10}$$

which has the same critical exponents $p^* = \frac{d+2}{d-2}$ (only for $d \geq 3$) and $p_* = 1 + \frac{4}{d}$.

For the *energy subcritical* case ($p < p^*$), an iteration of the local-in-time well-posedness result using the *a priori* upper bound on $\|u(t)\|_{H^1}$ implied by the conservation laws establishes global well-posedness for (1.10) in H^1. Those solutions scatter when $p > p_*$ [18, 34].

The *energy critical* case ($p = p^*$) was actually harder than the Klein-Gordon (wave), for which the finite propagation property was crucial to exclude possible concentration of energy, whereas there is no upper bound on the propagation speed for the Schrödinger. Nevertheless, based on new ideas such as induction on the energy size and frequency split propagation estimates, Bourgain [6] proved the global well-posedness and the scattering for radially symmetric data, and it was extended to the general case by [13] using a new interaction Morawetz inequality.

For the exponential nonlinearity in two spatial dimensions, small data global well-posedness together with the scattering was worked out by Nakamura-Ozawa in [31]. Later on, the size of the initial data for which one has local existence was quantified for (1.3) in [14], and a notion of criticality was proposed:

Definition 1.3. *The Cauchy problem* (1.3) *is said to be subcritical if* $H(u_0) < 1$*, critical if* $H(u_0) = 1$ *and supercritical if* $H(u_0) > 1$*.*

Indeed, one can construct a unique local solution if $\|\nabla u_0\|_{L^2} < 1$, and the time of existence depends only on $\eta := 1 - \|\nabla u_0\|_{L^2}$ and $\|u_0\|_{L^2}$. Hence the maximal local solutions are indeed global in the subcritical case. The critical case is more delicate due to the possible concentration of the Hamiltonian. The following result is proved in [14].

Theorem 1.4 (Global well-posedness [14])**.** *Assume that* $H(u_0) \leq 1$*, then the problem* (1.3) *has a unique global solution* u *in the class*

$$\mathcal{C}(\mathbb{R}, H^1(\mathbb{R}^2)).$$

Moreover, $u \in L^4_{loc}(\mathbb{R},\ \mathcal{C}^{1/2}(\mathbb{R}^2))$ *and satisfies the conservation laws* (1.6) *and* (1.7)*.*

Our first goal in this paper is to show that every global solution in the sub- and critical cases to (1.1) approaches solutions to the free Klein-Gordon equation

$$\ddot{v} - \Delta v + v = 0, \tag{1.11}$$

in the energy space $(u(t), \dot{u}(t)) \in H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$ as $t \to \pm\infty$. This can be done by showing global uniform Strichartz estimtes. The main result reads.

Theorem 1.5. *For any solution u of* (1.1) *satisfying $E(u,0) \le 1$, we have $u \in L^4(\mathbb{R}, \mathcal{C}^{1/4})$ and there exist unique free Klein-Gordon solutions $u_\pm$ such that*

$$E_0(u - u_\pm, t) \to 0 \qquad (t \to \pm\infty).$$

Moreover, the maps

$$(u(0), \dot{u}(0)) \longmapsto (u_\pm(0), \dot{u_\pm}(0))$$

are homeomorphisms between the unit balls in the nonlinear energy space and the free energy space, namely from $\{(\varphi,\psi) \in H^1 \times L^2 \; ; \; \|\varphi\|_{H^1}^2 + \|\psi\|_{L^2}^2 + 2\|F(\varphi)\|_{L^1} \le 1\}$ onto $\{(\varphi,\psi) \in H^1 \times L^2 \; ; \; \|\varphi\|_{H^1}^2 + \|\psi\|_{L^2}^2 \le 1\}$.

The peculiarity of this equation is that the Strichartz norms give time-local control of the nonlinearity in the case (1.4), but which is not uniform as $\|\nabla u\|_{L^2} \to 1$, and so in the critical case $E(u) = 1$, it is not *a priori* uniform in time globally, even for a fixed solution. In this respect, our critical case $E(u) = 1$ appears harder than the Sobolev critical case in higher dimensions [35], where the Strichartz norms give uniform control of the nonlinearity on time intervals where they are small.

To have the local uniform estimates of $f(u)$, we choose a norm which takes into account the two behaviors at zero and at infinity of the nonlinearity. Those estimates can be proved if the concentration radius of the H^1 norm is not too small. (See section 3 for the precise definition). Having these estimates in hand, the proof proceeds in the subcritical case $E(u) < 1$ almost verbatim as in [35]. However, in the critical case we need to keep track of the energy distribution and its propagation much more carefully, because of the non-uniform nature of those estimates.

The second goal in this paper is to show that every global solution of (1.3) with $H(u) \le 1$ approaches solutions to the associated free equation

$$i\dot{v} + \Delta v = 0, \tag{1.12}$$

in the energy space H^1 as $t \to \pm\infty$. Unfortunately, we have not succeeded to handle the critical case $H(u) = 1$ and we have to restrict ourselves to the subcritical one. The reason is that to trace the concentration radius, the finite propagation of energy is essential in our argument for NLKG, which is not available for NLS. The main ingredient for the subcritical NLS is a new interaction Morawetz estimate, proved independently by Colliander et al. and Planchon-Vega [12, 37]. This estimate give a priori global bound of u in $L_t^4(L_x^8)$. Hence, by complex interpolation we deduce that some of the Strichartz norms used in the nonlinear estimate go to zero for large time and the scattering in the subcritical case follows. More precisely, we have

Theorem 1.6. *For any global solution u of* (1.3) *in H^1 satisfying $H(u) < 1$, we have $u \in L^4(\mathbb{R}, \mathcal{C}^{1/2})$ and there exist unique free solutions $u_\pm$ of* (1.12) *such that*

$$\|(u - u_\pm)(t)\|_{H^1} \to 0 \qquad (t \to \pm\infty).$$

Moreover, the maps

$$u(0) \longmapsto u_\pm(0)$$

are homeomorphisms between the unit balls in the nonlinear energy space and the free energy space, namely from $\{\varphi \in H^1 \; ; \; H(\varphi) < 1\}$ onto $\{\varphi \in H^1 \; ; \; \|\nabla\varphi\|_{L^2} < 1\}$.

Thus the proof in the subcritical case is much simpler for NLS than NLKG, given the a priori estimate due to [12, 37]. Unfortunately, this type of Morawetz estimates is so far specialized to the Schrödinger, and does not seem easily to apply to the Klein-Gordon equation in any space dimension.

This paper is organized as follows. Section two is devoted to fix the necessary notation we use. For the convenience of the reader, we recall without proofs some useful and quite "standard" lemmas. In section three, we establish the uniform local estimates. In section four, we basically follow the arguments in [35], proving a uniform global bound on the Strichartz norms of solutions under some control on the energy concentration. This implies in particular the scattering Theorem 1.5 in the subcritical case. Section six is devoted to the critical case. First we treat the two extreme (and somehow easier) cases when the solution either completely disperse or does not disperse at all at the time infinity. The wider case is then when the solution repeats dispersion and reconcentration. This necessitates a careful study case by case. Section seven is devoted to show the optimality of the condition on the local H^1 norm. The last section treats the case of NLS equation. The proof of the scattering in the sub-critical case is mainly based on a recent *a priori* bound on the solutions established independently by Colliander et al. [12] and Planchon-Vega [37].

2. Notation and useful Lemmas

In this section, we introduce some notation and recall several lemmas we use to prove the main result.

First, let $B^{\sigma}_{p,q}$ be the inhomogeneous Besov space. In particular, recall that the Hölder space $\mathcal{C}^{\sigma} = B^{\sigma}_{\infty,\infty}$, for any non-integer $\sigma > 0$. Also, we introduce the following function spaces for some specific Strichartz norms. Define the following local H^1 norm:

$$\|\varphi\|_{H^1[R]} := \sup_{c\in\mathbb{R}^2} \int_{|x-c|\le R} |\nabla\varphi(x)|^2 + |\varphi(x)|^2 dx, \tag{2.1}$$

and set

$$\begin{aligned} &H := L^\infty_t(H^1), \quad H_{loc} := L^\infty_t(H^1[6]), \quad B := L^\infty_t(B^{-1/4}_{\infty,\infty}), \\ &X := L^8(L^{16}), \quad K := L^4_t(B^{1/4}_{\infty,2} \cap B^{1/2}_{4,2}), \\ &Y_1 := L^{1/\delta}(I; L^{2/\delta}), \quad Y_2 := L^{4/(1-\delta)}(I; C^{1/4-\delta} \cap L^8), \end{aligned} \tag{2.2}$$

and $Y := Y_1 \cap Y_2$.

Second, define the linear and nonlinear energy densities as

$$e_L(u,t) := |\dot u|^2 + |\nabla u|^2 + |u|^2 \quad \text{and,} \quad e_N(u,t) := e_L(u,t) + 2F(u), \tag{2.3}$$

respectively. Recall that F is given by (1.5). In addition, define $\mathcal{G}(u) = uf(u) - 2F(u)$, $r = |x|$, $\omega = \frac{x}{|x|}$, $u_r = \omega\cdot\nabla u$, and

$$\begin{aligned} t^2 Q(u) :=&(t\dot u + ru_r + u)^2 + (r\dot u + tu_r)^2 \\ &+ (t^2+r^2)(|\nabla u|^2 - |u_r|^2 + u^2). \end{aligned} \tag{2.4}$$

Then we have the inversional identity (see [41, 21] or more specifically [35, (7.4)]):

Lemma 2.1. *For any constant $c > 0$ and for any solution u of (1.1) with $E(u) \leq 1$, we have*

$$\left[\int_{r<ct} t^2 Q(u) + 2(t^2+r^2)F(u)dx\right]_S^T = \int_{cS<r<cT} P_c(u)dx + \int_S^T \int_{r<ct} 4t(u^2 - H(u))dxdt, \tag{2.5}$$

where $H(u) = \frac{\mathcal{G}(u)}{2} - 2F(u)$, and $P_c(u)$ satisfies

$$|P_c(u(r/c,x))| \lesssim \left[e_N(u) + \frac{u^2}{r^2}\right](r/c,x).$$

The formal proof does not require the condition $E(u) \leq 1$. We use it only to extend the estimate from smooth solutions to the energy class by the well-posedness. We also need the generalized Morawetz estimate.

Lemma 2.2 (Morawetz estimate [34], Lemma 5.1)**.** *There exists a positive constant C such that for any solution u of (1.1) with energy $E(u) \leq 1$, one has*

$$\int_{\mathbb{R}^{2+1}} \frac{|x\dot{u} + t\nabla u|^2 + |x \times \nabla u|^2 + (1+t^2)\mathcal{G}(u)}{1+|t|^3+|x|^3} dxdt \leq C. \tag{2.6}$$

To establish uniform estimates on the nonlinearity, we should make sure that energy does not concentrate. This can be quantified through the following notion

Definition 2.3. *For any $\varphi \in H$ such that $\partial_t \varphi \in L_t^\infty(L^2)$, and any $A > 0$, we denote the concentration radius of A amount of the nonlinear energy by*

$$R_A[\varphi](t) := \inf\{r > 0 \mid \exists c \in \mathbb{R}^2, \int_{|x-c|<r} e_N(\varphi,t)\,dx > A\}. \tag{2.7}$$

Observe that lower bounds on the concentration radius yield upper bound on the the local H^1 norm. More precisely

$$R_A[\varphi](t) \geq R \quad \Rightarrow \quad \|\varphi\|_{L_t^\infty(H^1[R])} \leq A. \tag{2.8}$$

Finally, recall the sharp Trudinger-Moser inequality on $\mathbb{R}^2$ [38]. It is the limit case of the Sobolev embedding. For any $\mu > 0$ we have

$$\sup_{\|\varphi\|_{H_\mu}\leq 1} \int (e^{4\pi|\varphi|^2} - 1)dx < \infty, \tag{2.9}$$

where H_μ is defined by the norm $\|u\|_{H_\mu}^2 := \|\nabla u\|_{L^2}^2 + \mu^2\|u\|_{L^2}^2$. We can change $\mu > 0$ just by scaling $\varphi(x) \mapsto \varphi(x/\mu)$.

It is known that the $H^1(\mathbb{R}^2)$ functions are not generally in L^∞. The following lemma shows that we can estimate the L^∞ norm by a stronger norm but with a weaker growth (namely logarithmic).

Lemma 2.4 (Logarithmic inequality [23], Theorem 1.3). *Let $0 < \alpha < 1$. For any real number $\lambda > \frac{1}{2\pi\alpha}$, a constant C_λ exists such that for any function $\varphi \in H_0^1 \cap \dot{\mathcal{C}}^\alpha(|x| < 1)$, one has*

$$\|\varphi\|_{L^\infty}^2 \leq \lambda \|\nabla\varphi\|_{L^2}^2 \log\left(C_\lambda + \frac{\|\varphi\|_{\dot{\mathcal{C}}^\alpha}}{\|\nabla\varphi\|_{L^2}}\right). \tag{2.10}$$

We also recall the whole space version of the above inequality.

Lemma 2.5 ([23], Theorem 1.3). *Let $0 < \alpha < 1$. For any $\lambda > \frac{1}{2\pi\alpha}$ and any $0 < \mu \leq 1$, a constant $C_\lambda > 0$ exists such that, for any function $u \in H^1(\mathbb{R}^2) \cap \mathcal{C}^\alpha(\mathbb{R}^2)$*

$$\|u\|_{L^\infty}^2 \leq \lambda \|u\|_{H_\mu}^2 \log\left(C_\lambda + \frac{8^\alpha \mu^{-\alpha} \|u\|_{\mathcal{C}^\alpha}}{\|u\|_{H_\mu}}\right). \tag{2.11}$$

3. Local uniform estimate on the nonlinearity

In this section, we estimate the nonlinearity in terms of the Strichartz norms in the critical case, such that the constant in the inequality does not depend on the interval as long as the concentration radius is not too small.

Lemma 3.1. *For any $A \in (0, 1)$, there exists $\delta \in (0, 1/8)$, and a continuous increasing function $C_A : [0, \infty) \to [0, \infty)$, such that for any $\varphi_0, \varphi_1 \in H^1(\mathbb{R}^2)$ satisfying*

$$\|\varphi_j\|_{H^1[6]} \leq A, \quad (j = 0, 1) \tag{3.1}$$

we have, putting $\varphi = (\varphi_0, \varphi_1)$

$$\|f(\varphi_1) - f(\varphi_0)\|_{L^2} \leq C_A(\|\varphi\|_{H^1}) \|\varphi\|_{C^{1/4-\delta} \cap L^8}^4 \|\varphi_1 - \varphi_0\|_{L^{2/\delta}}. \tag{3.2}$$

Then by integration and the Hölder inequality in t, we obtain

Corollary 3.2. *Let $A \in (0, 1)$, $\delta \in (0, 1/8)$ and C_A be as above. Let $I \subset \mathbb{R}$ be any measurable subset and assume that $u_0, u_1 \in L^\infty(I; H^1(\mathbb{R}^2))$ satisfy*

$$\sup_{t \in I} \|u_j(t)\|_{H^1[6]} \leq A \quad (j = 0, 1). \tag{3.3}$$

Then we have, putting $u = (u_0, u_1)$,

$$\|f(u_1) - f(u_0)\|_{L^1(I;L^2)} \leq C_A(\|u\|_{L^\infty(I;H^1)}) \|u_1 - u_0\|_{Y_1} \|u\|_{Y_2}^4. \tag{3.4}$$

In particular, if

$$\inf_{t \in I} R_A[u_j](t) \geq 6,$$

then (3.4) *holds.*

Remark 3.3. (1) The estimate on $f(u)$ itself follows by putting $u_0 = 0$ in the above.

(2) By the Strichartz estimate for the Klein-Gordon equation, we have

$$\|u\|_{L^4((0,T); B_{\infty,2}^{1/4} \cap B_{4,2}^{1/2})} \lesssim \|\ddot{u} - \Delta u + u\|_{L^1((0,T);L^2)} + \|u(0)\|_{H^1} + \|\dot{u}(0)\|_{L^2}. \tag{3.5}$$

(3) By the complex interpolation and the embedding between the Besov spaces, we have

$$\begin{aligned}
&\|u\|_{Y_1} = \|u\|_{L_t^{1/\delta}(L^{2/\delta})} \le C_\delta \|u\|_{L_t^\infty(H^1)}^{1-8\delta} \|u\|_{L_t^8(L^{16})}^{8\delta} = C_\delta \|u\|_H^{1-8\delta} \|u\|_X^{8\delta},\\
&\|u\|_{Y_2} = \|u\|_{L^{4/(1-\delta)}(C^{1/4-\delta}\cap L^8)} \lesssim \|u\|_{L^\infty(H^1)}^{\delta} \|u\|_{L^4(B^{1/4}_{\infty,2}\cap B^{3/8}_{8,2})}^{1-\delta},\\
&\|u\|_X = \|u\|_{L_t^8(L^{16})} \lesssim \|u\|_{L^\infty(B^{-1/4}_{\infty,\infty})}^{1/2} \|u\|_{L_t^4(B^{3/8}_{8,2})}^{1/2}.
\end{aligned} \tag{3.6}$$

Also note that $B^{3/8}_{8,2} = [B^{1/4}_{\infty,2}, B^{1/2}_{4,2}]_{1/2}$.

The above lemma is reduced to that on a disk of radius 6 by the following cubic decomposition. We define a radial cut-off function $\chi \in C^{1,1}(\mathbb{R}^2)$ by

$$\chi(x) = \begin{cases} 1, & (|x| \le 2)\\ 1 - (|x|-2)^2/8, & (2 \le |x| \le 4)\\ (|x|-6)^2/8, & (4 \le |x| \le 6)\\ 0, & (|x| \ge 6). \end{cases} \tag{3.7}$$

Then we have $0 \le \chi \le 1$, $\operatorname{supp}\chi = \{|x| \le 6\}$, $|\nabla\chi| \le |\chi'| \le 1/2$, $|\Delta\chi| \le 3/8$. For any $\varphi : \mathbb{R}^2 \to \mathbb{R}$ and $(j,k) \in \mathbb{Z}^2$, we define

$$\varphi^{j,k}(x) := \chi(x - 2\sqrt{2}(j,k))\varphi(x). \tag{3.8}$$

Then for any $x \in \mathbb{R}^2$, there exists $(j,k) \in \mathbb{Z}^2$ such that for any function $\varphi : \mathbb{R}^2 \to \mathbb{R}$,

$$\varphi(x) = \varphi^{j,k}(x). \tag{3.9}$$

In particular, we have for any function $g : \mathbb{R} \to \mathbb{R}$,

$$|g(\varphi(x))| \le \sum_{j,k\in\mathbb{Z}} |g(\varphi^{j,k}(x))|. \tag{3.10}$$

By the support property, we have also

$$\sum_{j,k\in\mathbb{Z}} \|\varphi^{j,k}\|_{L^2}^2 \lesssim \|\varphi\|_{L^2}^2, \quad \sum_{j,k\in\mathbb{Z}} \|\nabla\varphi^{j,k}\|_{L^2}^2 \lesssim \|\varphi\|_{H^1}^2. \tag{3.11}$$

Moreover, by the bounds on χ, $\nabla\chi$ and $\Delta\chi$, we have for any $p \in [1,\infty]$ and any $\theta \in (0,1)$,

$$\begin{aligned}
&\|\varphi^{j,k}\|_{L^p} \le \|\varphi\|_{L^p}, \quad \|\varphi^{j,k}\|_{C^\theta} \le \|\varphi\|_{C^\theta},\\
&\|\nabla\varphi^{j,k}\|_{L^2}^2 + \|\varphi^{j,k}\|_{L^2}^2/2 \le \|\varphi\|_{H^1[6]}^2.
\end{aligned} \tag{3.12}$$

To derive the above $\dot{H}^1$ bound, we use in addition the formula

$$\|\nabla(\psi\varphi)\|_{L^2}^2 = \int (\psi^2|\nabla\varphi|^2 - \psi\Delta\psi|\varphi|^2)dx, \tag{3.13}$$

with $\psi := \chi(x - 2\sqrt{2}(j,k))$.

Proof of Lemma 3.1. Without loss of generality, we may assume that $A \in (1/2, 1)$. We can choose $\delta \in (0, 1/8)$ and $\lambda > 2/\pi$, depending only on A, such that

$$2\pi\lambda(1/4 - \delta) > 1 = 2\pi\lambda(1/4 + \delta/2)A^2. \tag{3.14}$$

For instance, we can choose $\delta = \frac{1-A^2}{8}$. By the mean value theorem we have

$$\begin{aligned} f(\varphi_1) - f(\varphi_0) &= \int_0^1 f'(\varphi_\theta)d\theta \times \varphi', \\ \varphi_\theta &:= \theta\varphi_1 + (1-\theta)\varphi_0, \quad \varphi' := \varphi_1 - \varphi_0, \end{aligned} \tag{3.15}$$

so it suffices to bound $\|f'(\varphi_\theta)\varphi'\|_{L^2}$ uniformly for $\theta \in [0, 1]$. We consider the two cases $\|\varphi_\theta\|_{L^\infty} \le A$ and $\|\varphi_\theta\|_{L^\infty} > A$ separately. In the former case, we have

$$\|\varphi_\theta\|_{L^\infty} \le A < 1, \tag{3.16}$$

hence

$$|f'(\varphi_\theta(x))| \lesssim |\varphi_\theta(x)|^4 \tag{3.17}$$

pointwise in x, so that we can bound it by the Hölder and the Sobolev inequalities

$$\|f'(\varphi_\theta)\varphi'\|_{L^2} \lesssim \|\varphi_\theta\|_{L^8}^{4(1-\delta)} \|\varphi'\|_{L^{2/\delta}} \|\varphi_\theta\|_{L^\infty}^{4\delta}. \tag{3.18}$$

In the remaining case $\|\varphi_\theta\|_{L^\infty} > A$, we apply the cubic decomposition to φ_θ:

$$|f'(\varphi_\theta)| \le \sum_{j,k\in\mathbb{Z}^2} |f'(\varphi_\theta^{j,k})|. \tag{3.19}$$

Then (3.12) implies

$$\|\nabla\varphi_\theta^{j,k}\|_{L^2}^2 + \|\varphi_\theta^{j,k}\|_{L^2}^2/2 \le \|\varphi_\theta\|_{H^1[6]}^2 \le A^2 < 1. \tag{3.20}$$

We apply the Hölder inequality together with the general pointwise estimate

$$|f'(a)| \lesssim |a|^2(e^{4\pi|a|^2} - 1), \tag{3.21}$$

to the decomposed functions,

$$\|f'(\varphi_\theta^{j,k})\varphi'\|_{L^2} \le \|e^{4\pi|\varphi_\theta^{j,k}|^2} - 1\|_{L^1}^{1/2-\delta} \|e^{4\pi|\varphi_\theta^{j,k}|^2} - 1\|_{L^\infty}^{1/2+\delta} \|\varphi_\theta^{j,k}\|_{L^{4/\delta}}^2 \|\varphi'\|_{L^{2/\delta}}. \tag{3.22}$$

The first factor on the right is bounded by the sharp Trudinger-Moser inequality (2.9), for the third factor we use the Sobolev inequality

$$\|\varphi_\theta^{j,k}\|_{L^{4/\delta}} \lesssim \|\varphi_\theta^{j,k}\|_{H^1}, \tag{3.23}$$

and the second factor is bounded by using the logarithmic inequality (2.10)

$$\begin{aligned} \le e^{2\pi(1+2\delta)\|\varphi_\theta^{j,k}\|_{L^\infty}^2} &\le (C_\lambda + \|\varphi_\theta^{j,k}\|_{C^{1/4-\delta}}/\|\nabla\varphi_\theta^{j,k}\|_{L^2})^{2\pi(1+2\delta)\lambda\|\nabla\varphi_\theta^{j,k}\|_{L^2}^2} \\ &\le (C_\lambda + \|\varphi_\theta\|_{C^{1/4-\delta}}/A)^{2\pi(1+2\delta)\lambda A^2} \le (4C_\lambda\|\varphi_\theta\|_{C^{1/4-\delta}})^4, \end{aligned} \tag{3.24}$$

since $\|\varphi_\theta\|_{C^{1/4-\delta}} > A > 1/2$ and $C_\lambda \ge 1$. Thus we obtain

$$\|f'(\varphi_\theta^{j,k})\varphi'\|_{L^2} \lesssim \|\varphi_\theta^{j,k}\|_{H^1}^2 \|\varphi\|_{C^{1/4-\delta}}^4 \|\varphi'\|_{L^{2/\delta}}. \tag{3.25}$$

Then we arrive at the desired estimate after the summation over $(j,k) \in \mathbb{Z}^2$ and using (3.11). $\square$

3.1. **Nonlinear estimate for the critical case.** In the critical case $E(u) = 1$, we will need the nonlinear estimate assuming the subcritical local energy only for one of the functions. It is generally impossible, but we can retain it by additional smallness in the critical Besov space for the difference, which is available when the difference corresponds to a highly localized concentration of energy.

Lemma 3.4. *For any $A \in (0,1)$, there exist $\delta(A), \upsilon(A) \in (0,1/8)$, and a continuous increasing function $C_A : [0,\infty) \to [0,\infty)$, such that for any $\varphi_0, \varphi_1 \in H^1(\mathbb{R}^2)$ satisfying*

$$\|\varphi_0\|_{H^1[6]} \le A, \quad \|\varphi_1\|_{H^1[6]} \le 1, \quad \|\varphi_1 - \varphi_0\|_{B^0_{\infty,2}} \le \upsilon(A), \tag{3.26}$$

we have the estimate (3.2).

Corollary 3.5. *Let $A \in (0,1)$, $\delta(A), \upsilon(A) \in (0,1/8)$, and C_A be as above. Let $I \subset \mathbb{R}$ be any measurable subset and assume that $u_0, u_1 \in L^\infty(I; H^1(\mathbb{R}^2))$ satisfy for all $t \in I$,*

$$\|u_0(t)\|_{H^1[6]} \le A, \quad \|u_1(t)\|_{H^1[6]} \le 1, \quad \|u_1(t) - u_0(t)\|_{B^0_{\infty,2}} \le \upsilon(A). \tag{3.27}$$

Then we have the estimate (3.4).

It suffices to prove the lemma, which is almost the same as before.

Proof. The only difference appears in the use of the logarithmic inequality (3.24), where the assumption $A < 1$ was essential. Now we choose $\delta \in (0,1/8)$ and $\lambda > \frac{2}{\pi}$ such that

$$\begin{aligned} &2\pi\lambda(1/4 - \delta) > 1 = 2\pi\lambda(1/4 + \delta/2)(1+A)^2/4, \\ &1 - \varepsilon/4 = 2\pi\lambda(1/4 + \delta/2)A^2, \end{aligned} \tag{3.28}$$

for some $\varepsilon \in (0,1)$.

In order to use the new assumption, we expand the exponential in (3.24). For simplicity, we omit the superscript j,k for a while.

$$e^{2\pi(1+2\delta)\|\varphi_\theta\|_{L^\infty}^2} \le e^{2\pi(1+2\delta)\|\varphi_0\|_{L^\infty}^2} e^{2\pi(1+2\delta)2\theta\|\varphi_{\theta/2}\|_{L^\infty}\|\varphi'\|_{L^\infty}}. \tag{3.29}$$

For the first term we apply the same estimate as in (3.24), thus it is bounded by

$$(4C_\lambda\|\varphi_0\|_{C^{1/4-\delta}})^{4-\varepsilon}. \tag{3.30}$$

Here and after, the constant $C_\lambda \ge 1$ is determined by λ, but may change from line to line. For the second term, we use

$$\begin{aligned} \|\varphi_{\theta/2}\|_{L^\infty}^2 &\le \lambda\|\nabla\varphi_{\theta/2}\|_{L^2}^2 \log(C_\lambda + \|\varphi_{\theta/2}\|_{C^{1/4-\delta}}/\|\nabla\varphi_{\theta/2}\|_{L^2}) \\ &\le \lambda \log(C_\lambda + \|\varphi_{\theta/2}\|_{C^{1/4-\delta}}), \\ \|\varphi'\|_{L^\infty}^2 &\lesssim \|\varphi'\|_{B^0_{\infty,2}}^2 \log(C + \|\varphi'\|_{C^{1/4-\delta}}/\|\varphi'\|_{B^0_{\infty,2}}) \\ &\lesssim \|\varphi'\|_{B^0_{\infty,2}} \log(C + \|\varphi'\|_{C^{1/4-\delta}}), \end{aligned} \tag{3.31}$$

for some $C \ge 1$ and $C_\lambda \ge 1$, where we used the monotonicity of $x\log(C+B/x)$. For the latter inequality we refer to [26, Theorem 2.1]. Using the convexity of log

$$\log(C+A)\log(C+B) \le [\log(C+(A+B)/2)]^2, \tag{3.32}$$

we obtain

$$\|\varphi_{\theta/2}\|_{L^\infty}\|\varphi'\|_{L^\infty} \le C_\lambda \|\varphi'\|_{B^0_{\infty,2}}^{1/2} \log\{C_\lambda + (\|\varphi_{\theta/2}\|_{C^{1/4-\delta}} + \|\varphi'\|_{C^{1/4-\delta}})/2\}. \tag{3.33}$$

Hence by choosing $\upsilon \ll \varepsilon^2$, the second term in (3.29) is bounded by

$$(C_\lambda + \|\varphi\|_{C^{1/4-\delta}})^\varepsilon \lesssim \|\varphi\|_{C^{1/4-\delta}}^\varepsilon, \tag{3.34}$$

where we used that $\|\varphi\|_{C^{1/4-\delta}} \ge \|\varphi_\theta\|_{C^{1/4-\delta}} > A > 1/2$. The remaining argument is just the same as before. □

3.2. **$L^\infty - L^2$ estimate for localized free Klein-Gordon.** The smallness in the critical Besov space in the above nonlinear estimate will be provided by using a sharp decay estimate given below for the free Klein-Gordon equation. The decay factor $(t/R)^{(1-d)/2}$ is easily obtained by using the usual decay estimates; The nontrivial thing is that we do not lose any regularity compared with the Sobolev, nor any decay compared with the $L^\infty - L^1$ decay for the wave equation. Here we denote $\langle\xi\rangle := \sqrt{1+|\xi|^2}$ and $\langle\nabla\rangle := \sqrt{1-\Delta}$.

Lemma 3.6. *Let $d \ge 1$, $\chi \in \mathcal{S}(\mathbb{R}^d)$ and $\alpha \in [0, d/2]$. Then we have for $R \gtrsim 1$ and $t \in \mathbb{R}$,*

$$\|e^{it\langle\nabla\rangle}\langle\nabla\rangle^{-\alpha}\chi(x/R)f\|_{B^0_{\infty,2}} \lesssim \langle t/R\rangle^{(1-d)/2}\|f\|_{H^{d/2-\alpha}}. \tag{3.35}$$

Proof. First note that the case $|t| < R$ is trivial by the Sobolev embedding $H^{d/2-\alpha} \subset B^{-\alpha}_{\infty,2}$. Hence we assume that $t \gg R > 1$.

Next we dispose of the lower frequency. Denote by $P_{\le 1}$ a fixed smooth cut-off in the Fourier space onto $|\xi| \le 1$. We define $P_{\le R}$ for any $R > 0$ by rescaling $P_{\le 1}$, and also $P_{>R} := 1 - P_{\le R}$. Denote $\chi_R(x) = \chi(x/R)$. By using the L^{2d} decay for the free Klein-Gordon, and the Sobolev embedding $B^{1/2}_{2d,2} \subset B^0_{\infty,2}$, we have for some large $N \in \mathbb{N}$,

$$\begin{aligned} \|P_{\le 1}e^{it\langle\nabla\rangle}\langle\nabla\rangle^{-\alpha}\chi_R f\|_{B^0_{\infty,2}} &\lesssim \|e^{it\langle\nabla\rangle}\chi_R f\|_{B^{-N}_{2d,2}} \\ &\lesssim t^{(1-d)/2}\|\chi_R f\|_{L^{2d/(2d-1)}} \lesssim (t/R)^{(1-d)/2}\|f\|_{L^2}. \end{aligned} \tag{3.36}$$

Similarly, we have

$$\begin{aligned} \|e^{it\langle\nabla\rangle}\langle\nabla\rangle^{-\alpha}\chi_R P_{\le 1}f\|_{B^0_{\infty,2}} &\lesssim t^{(1-d)/2}\|\chi_R P_{\le 1}f\|_{B^N_{2d/(2d-1),2}} \\ &\lesssim t^{(1-d)/2}\|f\|_{W^{N+1,2d/(2d-1)}} \lesssim (t/R)^{(1-d)/2}\|f\|_{L^2}. \end{aligned} \tag{3.37}$$

Hence it suffices to show

$$\|P_{>1}e^{it\langle\nabla\rangle}\chi_R P_{>1}f\|_{\dot B^{-\alpha}_{\infty,2}} \lesssim (t/R)^{(1-d)/2}\|f\|_{\dot H^{d/2-\alpha}}, \tag{3.38}$$

for $t \gg R > 1$. Taking advantage of the homogeneity, we can rescale by $(t, x) \mapsto (Rt, Rx)$, then the above is reduced to

$$\|P_{>R} e^{it\langle\nabla\rangle_R} \chi P_{>R} f\|_{\dot{B}^{-\alpha}_{\infty,2}} \lesssim t^{(1-d)/2} \|f\|_{\dot{H}^{d/2-\alpha}}, \tag{3.39}$$

for $t \gg 1$ and $R > 1$, where we denote $\langle\nabla\rangle_R := \sqrt{R^2 - \Delta}$.

We decompose the left hand side by the dyadic decomposition, and consider the dyadic piece $|\xi| \sim J > R > 1$. We need to estimate the L^∞_x norm of

$$I_J := \iint e^{it\sqrt{|\xi|^2+R^2}+ix\xi} \varphi(\xi/J) \widetilde{\chi}(\xi - \eta) \widetilde{f}(\eta) d\eta d\xi, \tag{3.40}$$

where $\varphi(\xi)$ is the cut-off for the Littlewood-Paley decomposition, satisfying

$$\operatorname{supp} \varphi \subset \{1/2 < |\xi| < 2\}, \quad \sum_{J \in 2^{\mathbb{Z}}} \varphi(\xi/J) = 1, \quad (\xi \neq 0). \tag{3.41}$$

Now we argue by the stationary phase estimate. Let $\Omega = t\sqrt{|\xi|^2 + R^2} + \xi x =: t\langle\xi\rangle_R + \xi x$, then $\nabla\Omega = t\xi/\langle\xi\rangle_R + x$, $|\nabla^{1+N}\Omega| \lesssim \langle\xi\rangle^{-N} t$. For the size of $|\nabla\Omega|$, we take account only of its angular component, i.e.

$$\theta := |\xi/\langle\xi\rangle \times x/|x||, \quad |\nabla\Omega| \geq t\theta. \tag{3.42}$$

Let $g \in C_0^\infty(\mathbb{R}^d)$ and $\mathbb{R}^d \ni p \neq 0$, and assume that $\partial_p \Omega \neq 0$ on $\operatorname{supp} g$. Then we have by the partial integration,

$$\int e^{i\Omega} g(\xi) d\xi = \int e^{i\Omega} (i\partial_p (\partial_p \Omega)^{-1})^N g(\xi) d\xi, \tag{3.43}$$

for any $N \in \mathbb{N}$, where both ∂_p and $(\partial_p\Omega)^{-1}$ in the parenthesis should be regraded as operators. Thus the N-th power of the operator is expanded into a linear combination of terms like

$$e^{i\Omega} \frac{\partial_p^{\beta_1+1}\Omega \cdots \partial_p^{\beta_K+1}\Omega}{(\partial_p\Omega)^\alpha} \partial_p^\gamma, \quad \alpha = N + K, \quad \beta_1 + \cdots + \beta_K + \gamma = N, \tag{3.44}$$

and its integral is bounded by

$$\int (t\theta)^{-N} (J\theta)^{-K} J^{K+\gamma-N} |\partial_p^\gamma g| d\xi \lesssim \int (t\theta)^{-N} (J\theta)^{\gamma-N} |\partial_p^\gamma g| d\xi. \tag{3.45}$$

For the part $J \gtrsim t$, we can decompose $\varphi(\xi/J)\widetilde{\chi}(\xi - \eta)$ into the regions $t\theta \lesssim 1$ and $t\theta \gtrsim 1$, by a smooth cut-off of the form $h(t\theta)$. Then $|\partial_p^\gamma h(t\theta)| \lesssim (t/J)^\gamma \lesssim 1$, hence by choosing $N = 0$ for $t\theta \lesssim 1$ and sufficiently large N for $t\theta \gtrsim 1$, we get

$$\left| \int e^{i\Omega} \varphi(\xi/J) \widetilde{\chi}(\xi - \eta) d\xi \right| \lesssim \int_{|\xi| \sim J} \langle t\theta \rangle^{-N} |\widetilde{\chi}(\xi - \eta)| d\xi, \tag{3.46}$$

for some $\widehat{\chi} \in \mathcal{S}$ given by derivatives of $\widetilde{\chi}$. Applying the Young on the right hand side (L^2 for the first term and L^1 for the second), we get

$$\begin{aligned}|I_J| &\lesssim \sum_{l\in\mathbb{N}} \Big\| \int_{|\xi|\sim 2^j} \langle t\theta\rangle^{-N} |\widehat{\chi}(\xi-\eta)| d\xi \Big\|_{L^2_\eta(|\eta|\sim 2^l)} \|\widetilde{f}\|_{L^2_\eta(|\eta|\sim 2^l)} \\ &\lesssim (2^j/t)^{(d-1)/2} 2^{j/2} \sum_{l\in\mathbb{N}} 2^{-N|j-l|} \|\widetilde{f}\|_{L^2_\eta(|\eta|\sim 2^l)},\end{aligned} \tag{3.47}$$

where we put $J = 2^j$, and the two N's are unrelated arbitrary large numbers. Hence by applying the Young for the convolution on $\mathbb{Z}$, we get

$$\|J^{-\alpha} I_J\|_{\ell^2_{J\gtrsim t} L^\infty_x} \lesssim t^{(1-d)/2} \|f\|_{\dot{H}^{d/2-\alpha}}. \tag{3.48}$$

If $J \ll t$, then we apply the cubic decomposition of size $J\theta_0$ to the ξ integral with $\theta_0 = (tJ)^{-1/2}$, and we use the above partial integration in those cubes with $\theta > \theta_0$. Since the ∂^γ derivative on the cubic cut-off is bounded by $(J\theta_0)^{-\gamma}$, we thus get

$$|\int e^{i\Omega} \varphi(\xi/J) \widetilde{\chi}(\xi-\eta) d\xi| \lesssim \int_{|\xi|\sim J} \langle \theta/\theta_0\rangle^{-N} |\widehat{\chi}(\xi-\eta)| d\xi, \tag{3.49}$$

where $\widehat{\chi} \in \mathcal{S}$ is given by derivatives of $\widetilde{\chi}$. Now we apply the Young to the right hand side, but with different exponents for different variables. In the $d-1$ dimensions perpendicular to x, we apply L^1 and L^2 to the first and the second funcitons, respectively. In the remaining x direction, we apply L^2 and L^1 to them. Then we get with $J = 2^j \ll t$,

$$|I_J| \lesssim \sum_{l\in\mathbb{Z}} (J\theta_0)^{d-1} J^{1/2} 2^{-N|j-l|} \|\widetilde{f}\|_{L^2(||\eta|\sim 2^l)}, \tag{3.50}$$

for arbitrary $N \in \mathbb{N}$. By $\theta_0 = (tJ)^{-1/2}$, the coefficient equals $t^{(1-d)/2} J^{d/2} 2^{-N|j-l|}$. Hence we get the desired bound as the previous case, by the Young inequality on $\mathbb{Z}$. ☐

4. Strichartz estimates in the scattering

In the following lemma, the global energy bound $E(u) \le 1$ is irrelevant. In fact, we have the same conclusion for arbitrary finite $E(u)$, where the estimates depend also on $E(u)$.

Lemma 4.1. *Let u be a solution of* (1.1) *on an interval I satisfying $E(u) \le 1$, $\|u\|_{H_{loc}(I)} \le A < 1$ and $\|u\|_{X(I)} = \eta < \infty$. There exists a constant $\eta_0(A) \in (0,1)$ such that if $\eta \le \eta_0(A)$ then we have*

$$\|u\|_{K(I)} \lesssim 1. \tag{4.1}$$

Moreover, there exist $J \subset I$, $R = R(\eta) \sim \eta^{-3}$ and $c \in \mathbb{R}^2$ such that $|J| \gtrsim \eta^8$ and

$$J \ni \forall t, \quad \int_{|x-c|<R} |u(t,x)|^p dx \ge C_p \eta^{16} \tag{4.2}$$

for any $p \in [1,8]$.

Proof. Let $I = (S, T)$ and let v be a free solution with the same initial data as u at $t = S$. By the Strichartz estimate (3.5) and the interpolation inequalities (3.6), we have

$$\|u - v\|_{K \cap X(I)} \le C\|f(u)\|_{L^1(I;L^2)} \le C(A)\|u\|_{K(I)}^{4(1-\delta)}\|u\|_{X(I)}^{8\delta}, \tag{4.3}$$

where $\delta \in (0, 1/8)$ is given in Lemma 3.1 depending on A. Hence we have

$$\|u\|_{K(I)} \lesssim 1 + C(A)\eta^{4\delta}\|u\|_{K(I)}^{4(1-\delta)}. \tag{4.4}$$

Since $4(1 - \delta) > 1$ and $\|u\|_{K(I)} \to 0$ as $T \to S + 0$, we obtain the desired *a priori* bound if $\eta \le \eta_0(A)$ is sufficiently small. Then (3.6) implies

$$\|u\|_{B(I)} \gtrsim \|u\|_{X(I)}^2\|u\|_{K(I)}^{-1} \gtrsim \eta^2. \tag{4.5}$$

Hence there exists $T \in I$, $c \in \mathbb{R}^2$, and $N \ge 1$ such that

$$\eta^2 \lesssim N^{-1/4}|u_N(T, c)|, \tag{4.6}$$

where u_N stands for the dyadic piece of u. The Sobolev embedding bounds the RHS by $N^{-1/4}$, and also $\eta = \|u\|_{X(I)} \lesssim |I|^{1/8}$. Hence we have $N + 1/|I| \lesssim \eta^{-8}$. Moreover, by the L^2 bound on $\dot{u}$, we have

$$\|u_N(T) - u_N(t)\|_{L^\infty} \lesssim N\|u(T) - u(t)\|_{L^2} \lesssim N|T - t|, \tag{4.7}$$

and so there exists $J \subset I$ such that $|J| \gtrsim \eta^8$ and

$$\eta^2 \lesssim N^{-1/4} \inf_{t \in J} |u_N(t, c)|. \tag{4.8}$$

Writing the frequency localization by using some Schwarz function φ, we have

$$\begin{aligned} N^{1/4}\eta^2 \lesssim |u_N(t, c)| &= |(N^2\varphi(Nx) * u)(t, c)| \\ &\lesssim N^{2/p}\|u(t)\|_{L^p(|x-c|<R)} + N^{1/4}R^{-3/4}\|u(t)\|_{L^2}, \end{aligned} \tag{4.9}$$

where we used the Hölder inequality. Since $N \lesssim \eta^{-8}$ and $\|u(t)\|_{L^2} \le 1$, if $R \gg \eta^{-8/3}$ and $p \le 8$ then we have

$$\|u(t)\|_{L^p(|x-c|<R)} \gtrsim N^{1/4-2/p}\eta^2 \gtrsim \eta^{16/p}. \tag{4.10}$$

□

Lemma 4.2. *Let u be a solution of* (1.1) *on an interval $I = (S, T)$ with $E(u) \le 1$ and $\|u\|_{H_{loc}(I)} \le A < 1$. Let $S = \tau_0 < \tau_1 < \cdots < \tau_N = T$ such that $\|u\|_{X(\tau_j, \tau_{j+1})} \ge \eta > 0$ for all j. Then we have*

$$\sum_{j=1}^{N} \frac{\min(\eta, \eta_0(A))^{40}}{\tau_j - \tau_0 + \eta^{-3}} \lesssim 1. \tag{4.11}$$

Proof. By subdividing each interval (τ_j, τ_{j+1}) if necessary, we may assume $\eta \le \eta_0(A)$ without loss of generality. By Lemma 4.1, there exists $R \sim \eta^{-3}$, $t_j \in (\tau_{j-1}, \tau_j)$ and $c_j \in \mathbb{R}^2$ such that

$$\int_{|x-c_j|<R} |u(t_j, x)|^2 dx \gtrsim \eta^{16}. \tag{4.12}$$

Define the following relation on the set of all indices

$$j \preceq k \iff [j \le k \text{ and } |c_k - c_j| \le |t_k - t_j| + 2R.] \tag{4.13}$$

Then we can inductively find a unique $M \subset \{1, \dots, N\}$ such that

(1) Every $k \in M$ is minimal, i.e. there is no $j \preceq k$.
(2) Each j has a $k \in M$ satisfying $k \preceq j$.

By the first property and the finite propagation of energy, we have

$$1 \ge E(u) \ge \sum_{k\in M} \int_{|x-c_k|<R} e(u(t_k, x))dx \gtrsim \eta^{16}(\#M), \tag{4.14}$$

where $\#M$ denotes the number of the minimal elements, and by the generalized Morawetz inequality we have for each $k \in M$,

$$1 \ge E(u) \gtrsim \int \frac{|u|^6}{|t-t_k| + |x-c_k| + 1} dxdt \gtrsim \sum_{j\succeq k} \frac{\eta^{16}\eta^8}{|t_j - t_k| + \eta^{-3}}. \tag{4.15}$$

Summing this for $k \in M$, and using the second property of M, we obtain

$$\#M \gtrsim \sum_{k\in M} \sum_{j\succeq k} \frac{\eta^{24}}{|t_j - t_k| + \eta^{-3}} \gtrsim \sum_{j=1}^{N} \frac{\eta^{24}}{\tau_j - \tau_0 + \eta^{-3}}, \tag{4.16}$$

which is bounded by η^{-16} due to (4.14). $\square$

Lemma 4.3 (Space time localized energy)**.** *There are functions $L : (0,1)^2 \to (1,\infty)$ and $\nu_0 : (0,1) \to (0,1/2)$ with the following property. Let $T_1 < T_2$ and let u be a solution of (1.1) on (T_1, T_2) satisfying $E(u) \le 1$,*

$$\begin{aligned} &\|u\|_{H_{loc}(T_1,T_2)} \le A, \quad \|u\|_{X(T_1,T_2)} \le 2\eta, \\ &\int_{|x-c|<R} e_N(u, T_1)dx \ge \nu^2, \quad |T_1 - T_2| \ge L(\nu, \kappa)R, \end{aligned} \tag{4.17}$$

for some $A \in (0,1)$, $\kappa \in (0,1)$, $\eta \in (0, \eta_0(A)]$, $\nu \in (0, \nu_0(A))$, $c \in \mathbb{R}^2$ and $R \ge 1$. Then there exists a free solution v and $S \in (T_1, T_2)$ such that

$$\begin{aligned} &E(u - v; S) \le E(u) - \nu^2/2, \quad E_0(v) \le 2\nu^2, \\ &\|v\|_{L^\infty B^0_{\infty,2}(S,\infty)} + \|v\|_{X(S,\infty)} \le \kappa. \end{aligned} \tag{4.18}$$

Proof. We may set $T_1 = 0$ and $c = 0$ by space-time translation. First we seek a bump of energy with very thin Strichartz norm, to be the initial data of v at a time much before S. By Lemma 4.1, we have $\|u\|_{K(T_1,T_2)} \lesssim 1$, and by the finite propagation,

$$\int_{|x|<t+R} e_N(u; t)dx \ge \nu^2.$$

at any $t \ge 0$. Let M, ν' be large and small numbers, which will be determined later depending on ν and κ. Let $t_j := (3 + 3M)^j R$ for $j \in \mathbb{N}$. For any $j \in \mathbb{N}$ and $t_j \le t \le 2t_j$, we have

$$t_{j+1} - t \ge t_{j+1} - 3t_j = 3Mt_j \ge (t + R)M. \tag{4.19}$$

We claim that if $|T_1 - T_2|/R$ is large enough (depending on M and ν'), then for some j with $t_{j+1} < T_2$, we have

$$\|u\|_{(X\cap K)(t_j,t_{j+1})} + \inf_{t_j<t<2t_j} \|\langle x\rangle^{-1}u(t)\|_{L^2} \leq \nu'. \tag{4.20}$$

Denote by N_1 (resp. N_2) the number of j for which $t_{j+1} \leq T_2$ and the first (resp. the second) term is bigger than $\nu'/2$. Since X and K consist of L^8_t and L^4_t, we have

$$\nu' N_1^{1/8} \lesssim \|u\|_{(X\cap K)(T_1,T_2)} \lesssim 1. \tag{4.21}$$

On the other hand, by the generalized Morawetz, we have [34, Lemma 5.3]

$$\int_{\mathbb{R}} \left\|\frac{u(t)}{\langle x\rangle}\right\|_{L^2}^6 \frac{dt}{1+|t|} \lesssim 1, \tag{4.22}$$

so we have $(\nu')^6(3+3M)^{-1}N_2 \lesssim 1$. Therefore we have (4.20) for some j if $|T_1 - T_2|/R \geq (3+3M)^J$ for some $J \gg \kappa^{-8} + M(\nu')^{-6}$. Let $S_0 := t_j$, $S := t_{j+1}$ and $R' := S_0 + R$.

Now we define v. Let $\chi \in C_0^\infty(\mathbb{R}^2)$ satisfy $0 \leq \chi \leq 1$, $\chi(x) = 1$ for $|x| \leq 1$ and $\chi(x) = 0$ for $|x| \geq 2$. Let $\chi_a(x) := a\chi(x/R')$. Then by the continuity in a, there exists $a \in (0,1]$ such that

$$\int_{\mathbb{R}^2} \chi_a^2 e(u;S_0)dx = \nu^2. \tag{4.23}$$

Let v be the free solution satisfying

$$(v(S_0,x), \dot v(S_0,x)) = \chi_a(x)(u(S_0,x), \dot u(S_0,x)).$$

Then by using (3.13), we have

$$\|\nabla v(S_0)\|_{L^2}^2 = \int \chi_a^2|\nabla u(S_0)|^2 - \chi_a\Delta\chi_a|u(S_0)|^2dx, \tag{4.24}$$

and the second term is bounded by

$$\|\langle x\rangle^{-1}u(S_0)\|_{L^2}^2 \leq (\nu')^2, \tag{4.25}$$

so we have

$$E_0(v) = \nu^2 + O((\nu')^2) \leq 2\nu^2, \tag{4.26}$$

by choosing $\nu' \ll \nu$.

On the other hand, by the sharp decay estimate (3.6), we have

$$\|v\|_{L^\infty B^0_{\infty,2}(S,\infty)} \lesssim (R'/|S-S_0|)E_0(v)^{1/2} \lesssim M^{-1}\nu, \tag{4.27}$$

and by interpolation (cf. (3.6)),

$$\begin{aligned}\|v\|_{X(S,\infty)} &\lesssim \|v\|^{1/2}_{L^\infty((S,\infty),B^{-1/4}_{\infty,\infty})}\|v\|^{1/4}_{L^4((S,\infty),B^{1/4}_{\infty,2})}\|v\|^{1/4}_{L^4((S,\infty),B^{1/2}_{4,2})} \\ &\lesssim (R'/|S-S_0|)^{1/2}E_0(v) \lesssim M^{-1/2}\nu.\end{aligned} \tag{4.28}$$

Hence we get $\|v\|_{L^\infty B^0_{\infty,2}(S,\infty)+X(S,\infty)} \leq \kappa$ by choosing $M \gg \kappa^{-2}$.

Now, letting $w = u - v$ and arguing as before we have

$$
\begin{aligned}
E(w, S_0) &\le \int (1-\chi_a^2) e_N(u, S_0) + (1-\chi_a)\Delta\chi_a |u(S_0)|^2 \, dx \\
&\le E(u) - \nu^2 + C\kappa^2.
\end{aligned} \tag{4.29}
$$

Let $2\nu_0(A) < (1-A)/2$, then we can apply our nonlinear estimate to w because

$$
\|w\|_{H_{loc}(S_0,S_1)} \le \|u\|_{H_{loc}(S_0,S_1)} + \|v\|_{H^1} \le (A+1)/2 < 1. \tag{4.30}
$$

By the energy identity together with (3.2), we have

$$
\begin{aligned}
E(w, S) &= E(w, S_0) + \int_{S_0}^{S} 2\langle f(w) - f(u) \mid \dot{w}\rangle_x dt \\
&\le E(w, S_0) + 2\|f(w) - f(u)\|_{L^1L^2(S_0,S)} \|\dot{w}\|_{L^\infty L^2(S_0,S)} \\
&\le E(w, S_0) + C(A)\Big(\|u\|_{Y_2(S_0,S)}^4 + \|v\|_{Y_2}^4\Big)(\|u\|_{Y_1(S_0,S)} + \|v\|_{Y_1}).
\end{aligned} \tag{4.31}
$$

By (4.20) and the Strichartz we have

$$
\|u\|_{(X\cap K)(S_0,S)} \lesssim \nu', \quad \|v\|_{X\cap K} \lesssim E_0(v)^{1/2} \lesssim \nu. \tag{4.32}
$$

Then by the interpolation (3.6), we have, using $\nu' \ll \nu$,

$$
\|u\|_{Y_1(S_0,S)} + \|v\|_{Y_1} \lesssim \nu^{8\delta}, \quad \|u\|_{Y_2(S_0,S)} + \|v\|_{Y_2} \lesssim \nu^{1-\delta}. \tag{4.33}
$$

Plugging them into (4.31), we obtain

$$
\begin{aligned}
E(w, S) &\le E(w, S_0) + C(A)\nu^{4+\delta} \le E(u) - \nu^2 + C(\nu')^2 + C(A)\nu^4 \\
&\le E(u) - \nu^2/2,
\end{aligned} \tag{4.34}
$$

by choosing $\nu_0(A)$ sufficiently small. □

4.1. **Perturbation theory.**

Lemma 4.4. *Let u and w be global solutions to* (1.1) *with* $E(u) \le 1$ *and* $E(w) \le 1$*, and let v be the free solution with the same initial data as $u - w$ at some $t = S$. For any $M \in (0, \infty)$ and $A \in (0, 1)$, there exists $\kappa(A, M) \in (0, 1)$ and $C(A, M) \in (0, \infty)$ such that if*

$$
\begin{aligned}
&\|w\|_{H_{loc}(S,\infty)} \le A, \quad \|v\|_{L^\infty B^0_{\infty,2}(S,\infty)} \le \upsilon(A)/2, \\
&\|w\|_{X(S,\infty)} \le M, \quad \|v\|_{X(S,\infty)} \le \kappa(A, M),
\end{aligned} \tag{4.35}
$$

then we have $\|u\|_{X(S,\infty)} \le C(A, M)$. The above υ is as given in Lemma 3.4.

In the subcritical case, we may replace the smallness of v in $B^0_{\infty,2}$ by something like $\|v\|_{H^1_{loc}} \le (1-A)/4$.

Proof. Using the interpolation (3.6) and $\|v\|_{X(S,\infty)} \le \kappa$, we have $\|v\|_{Y(S,\infty)} \le C_\delta(\kappa^{8\delta} + \kappa^{1-\delta}) := \kappa'$. Let $\eta \in (0, \eta_0(A))$ and $S := T_0 < T_1 < \cdots < T_N < T_{N+1} = \infty$ such that

$$
\eta/2 \le \|w\|_{X(T_j,T_{j+1})} \le \eta \tag{4.36}
$$

for all $j = 0, \dots, N$. Then we have $N^{1/8}\eta \lesssim M$. The actual size of $\eta = \eta(A)$ and $\kappa = \kappa(A, M)$ will be determined later. Again, using (3.6) we have

$$\|w\|_{Y(T_j,T_{j+1})} \le C_\delta(\eta^{8\delta} + \eta^{1-\delta}) := \eta'. \tag{4.37}$$

Let $\Gamma = u - v - w$. Then Γ satisfies

$$\Gamma(t) = \Gamma_j(t) + \int_{T_j}^t U(t-s)\Big(f(w) - f(u)\Big)(s)ds, \tag{4.38}$$

where Γ_j is the free solution with the same initial data as Γ at $t = T_j$.

Let $I = (T_j, T)$ with $T > T_j$. The Strichartz estimates together with (3.6) and Corollary 3.5 give

$$\begin{aligned} \|\Gamma - \Gamma_j\|_{(Y\cap H)(I)} &\le C_0\|f(w) - f(u)\|_{L^1(I,L^2)} \\ &\le C_1\Big(\|w\|_{Y(I)} + \|\Gamma + v\|_{Y(I)}\Big)^4\|\Gamma + v\|_{Y(I)}, \end{aligned} \tag{4.39}$$

where $1 \le C_0 \le C_1$ depend on A, provided that

$$\|\Gamma\|_{L^\infty B^0_{\infty,2}(I)} \le \upsilon(A)/2. \tag{4.40}$$

Denote

$$p_j := \|\Gamma_j\|_{(Y\cap H\cap L^\infty B^0_{\infty,2})(T_j,T)}, \quad q_j(T) := \|\Gamma\|_{(Y\cap H\cap L^\infty B^0_{\infty,2})(T_j,T)}. \tag{4.41}$$

Then q_j is continuous, and from (4.39) we have the following

$$\begin{aligned} &p_0 = q_j(T_j) = 0, \\ &q_j(T) \le p_j + C_1\Big(q_j(T) + \kappa' + \eta'\Big)^4\Big(q_j(T) + \kappa'\Big), \\ &p_{j+1} \le p_j + C_1\Big(q_j(T_{j+1}) + \kappa' + \eta'\Big)^4\Big(q_j(T_{j+1}) + \kappa'\Big), \end{aligned} \tag{4.42}$$

as long as we keep (4.40).

Now we fix $\eta = \eta(A)$ so small that we have

$$C_1(3\eta')^4 \le 1/8, \quad \eta' \le \upsilon(A)/2, \tag{4.43}$$

and then fix $\kappa = \kappa(A, M)$ small enough to have

$$2^{N+1}\kappa' < \eta'. \tag{4.44}$$

If $p_j \le 2^j\kappa'$ and $q(T) \le 2^{j+1}\kappa'$, then we have

$$q_j(T) \le 2^j\kappa' + C_1(3\eta')^4 2^{2+j}\kappa' \le \frac{3}{2}2^j\kappa', \tag{4.45}$$

hence by continuity we have $q(T) \le 2^{j+1}\kappa'$, and in the same way for the limit $T \to T_{j+1} - 0$, we get $p_{j+1} \le 2^{j+1}\kappa'$.

Hence by induction, we obtain $p_j \le 2^j\kappa' < \eta'$ and $q_j(T_{j+1}) < \eta'$ for any $j \le N$. Using the interpolation (3.6), we conclude that

$$\|u\|_{X(S,\infty)} \lesssim \|u\|_{Y(S,\infty)}^{1/2} \lesssim N^{1/8}\eta' \le C(A, M) \tag{4.46}$$

as desired. □

5. Uniform global bound in the sub-critical case

In the sequel, we want to show the following result which is the crucial step in proving Theorem 1.5 in the sub-critical case.

Proposition 5.1. *There exists an increasing function $C : [0,1) \to [0,\infty)$ such that for any $0 \le E < 1$, any global solution u of the (1.1) with $E(u) \le E$ satisfies*

$$\|u\|_{X(\mathbb{R})} \le C(E). \tag{5.1}$$

Proof. First, note that arguing as in Lemma 4.1, one can show that estimate (5.1) is satisfied if the energy is small. Denote by

$$E^* := \sup\{0 \le E < 1 : \sup_{E(u)\le E} \|u\|_{X(\mathbb{R})} < \infty\}. \tag{5.2}$$

¿From the small data scattering, it is clear that $E^* > 0$. The goal is to show that $E^* = 1$. Assume that $E^* < 1$. Then for any $E \in (E^*, 1)$ and any $n \in (0,\infty)$, there exists a global solution u such that $E(u) = E$ and $\|u\|_{X(\mathbb{R})} > n$. We are going to show that if E is close enough to E^*, then n cannot be arbitrarily large.

By time translation, we may assume that $\|u\|_{X(0,\infty)} > n/2$. Since $\|u\|_{H_{loc}(\mathbb{R})} \le \|u\|_{H(\mathbb{R})} \le E < 1$, we can apply all the lemmas in the previous sections with $A := E$.

First we fix $\eta = \eta_0(A)/2$, $R = R(\eta)$ and $\nu = \min(\nu_0(A), \sqrt{C_2}\eta^8)/2$, where $\eta_0(A)$, $R(\eta)$, $\nu_0(A)$ and C_2 are given in Lemmas 4.1 and 4.3.

Next we choose $E \in (E^*, E^* + \nu^2/4)$ such that $E - \nu^2/2 < E^* - \nu^2/4 =: E'$. Then by definition of E^*, there exists $M < \infty$ such that $\|w\|_X \le M$ for any global solution w with $E(w) \le E'$. We fix $\kappa = \min(\upsilon(A), \kappa(A, M))/4$, where υ and κ are given in Corollary 3.5 and Lemma 4.4, respectively. Then we fix $L = L(\nu, \kappa)$ which was given in Lemma 4.3.

Now let $0 = \tau_0 < \tau_1 < \cdots$ such that $\|u\|_{X(\tau_{j-1},\tau_j)} = \eta/2$, and let $J \in \mathbb{N}$ be the smallest satisfying $|\tau_J - \tau_{J+1}| \ge LR$. Then Lemma 4.2 gives an upper bound on J:

$$1 \gtrsim \sum_{j=1}^{J} \frac{\eta^{40}}{jLR + \eta^{-3}} \sim \frac{\eta^{43}}{L} \log J. \tag{5.3}$$

Applying Lemma 4.1 on (τ_{J-1}, τ_J), we get some $T_1 \in (\tau_{J-1}, \tau_J)$, $c \in \mathbb{R}^2$ such that with $T_2 = \tau_{J+1}$, all the assumptions for Lemma 4.3 are fulfilled. By using its conclusion, let w be the global solution with the same initial data as $u - v$ at some $t = S \in (T_1, T_2)$. Since $E(w) \le E - \nu^2/2 < E'$, we have $\|w\|_X \le M$. Also $\|w\|_{H_{loc}} \le E(w) \le E = A$. Thus all the assumptions in Lemma 4.4 hold, so we obtain

$$\|u\|_{X(S,\infty)} \le C(A, M), \tag{5.4}$$

Since $S \in (T_1, T_2)$, we get a bound on n by using the estimate on J as well:

$$n/2 \le \|u\|_{X(0,\infty)} \le (J+1)^{1/8}\eta + C(A, L), \tag{5.5}$$

which is a contradiction. $\square$

6. Scattering in the critical case

Let u be a solution of (1.1) with $E(u) = 1$, and fix it. We denote the concentration radius of energy $1 - \varepsilon$ at time t by

$$r_\varepsilon(t) = R_{1-\varepsilon}[u](t). \tag{6.1}$$

It is easy to see by finite propagation that

$$|r_\varepsilon(t) - r_\varepsilon(s)| \le |t - s|, \tag{6.2}$$

for any ε, s, t. If we have the scattering, then $\lim_{t\to\infty} r_\varepsilon(t) = \infty$ for any $\varepsilon > 0$. We will reduce the problem to the subcritical result with careful investigation on the evolution of $r_\varepsilon(t)$ case by case. The most problematic case is that where $r_\varepsilon(t)$ is neither bounded from above nor away from zero, then we will perform the separation of concentration energy in a similar way as above, but using the oscillatory behavior of the concentration radius. The use of the critical Besov space in the nonlinear estimate is essential only in that case.

6.1. **Dispersive and non-dispersive cases.** First we deal with the two extreme cases. Let $\varepsilon \in (0, 1)$. By Lemma 3.1 together with (2.8), we can control the nonlinearity on any interval I satisfying

$$\inf_{t\in I} r_\varepsilon(t) \ge 6, \tag{6.3}$$

with constants depending only on $A = 1 - \varepsilon$. Therefore, if we have

$$\liminf_{t\to\infty} r_\varepsilon(t) > 6, \tag{6.4}$$

for some $\varepsilon \in (0, 1)$, then the scattering follows in the same way as in the subcritical case. More precisely, here we need two different A's, namely $A_0 = 1 - \varepsilon$ before the perturbation lemma, and $A_1 = \sqrt{1 - \nu(A_0)^2/4}$ for the nonlinear solution w with reduced energy. Indeed, we would fall into a vicious circle if we would insist a single A, since $1 - A_1 \ll 1 - A_0$. Specifically, those parameters are defined by

$$\begin{aligned} &\eta = \eta_0(A_0)/2, \quad \nu = \min(\nu_0(A_0), \sqrt{C_2}\eta^8)/2, \\ &A_1 = \sqrt{1 - \nu(A_0)^2/4}, \quad \kappa = \min(\upsilon(A_1), \kappa(A_1, M))/4, \quad L = L(\nu, \kappa), \end{aligned} \tag{6.5}$$

where $M < \infty$ is given by the subcritical result that we have $\|w\|_X \le M$ for any nonlinear solution w satisfying $E(w) \le A_1^2$. Thus using Lemmas 4.1 and 4.3 with $A = A_0$ and $R = R(\eta)$, and Lemma 4.4 with $A = A_1$, we deduce that $\|u\|_{X(0,\infty)} < \infty$, which implies the scattering for u.

Next consider the case where we have for all $\varepsilon \in (0, 1)$

$$\widetilde{r_\varepsilon} := \limsup_{t\to\infty} r_\varepsilon(t) + 1 < \infty, \tag{6.6}$$

namely the solution does not disperse at all as $t \to \infty$. By the definition of $r_\varepsilon(t)$, there exists a function $c : [0, \infty) \to \mathbb{R}^2$ such that

$$\int_{|x-c(t)|>r_\varepsilon(t)} e_N(u, t)dx \le \varepsilon, \tag{6.7}$$

and the above assumption (6.6) implies that

$$\limsup_{t\to\infty} \int_{|x-c(t)|>\widetilde{r}_\varepsilon} e_N(u,t)dx \le \varepsilon. \tag{6.8}$$

The finite propagation property implies that we can choose $c(t)$ such that

$$\limsup_{t\to\infty} |c(t)|/t+1 \le M \lesssim 1, \tag{6.9}$$

with M independent of ε. Then we obtain using the Hölder inequality and $e_N(u) \ge |u|^2$, that for t large enough

$$\begin{aligned} \|u(t)\|_{L^2} &\lesssim \varepsilon^{1/2} + \|u(t)\|_{L^2(|x|<M|t|)} \\ &\lesssim \varepsilon^{1/2} + (\widetilde{r}_\varepsilon)^{1-2/p} \|u(t)\|_{L^p_x(|x|<M|t|)}, \end{aligned} \tag{6.10}$$

for any $p>2$. On the other hand, the generalized Morawetz inequality (2.6) implies that

$$\lim_{T\to\infty} \frac{1}{T}\int_T^{CT}\int_{|x|\le Mt} \mathcal{G}(u)dxdt = 0 \tag{6.11}$$

for any $C\in(1,\infty)$. Combining this with (6.10), we obtain

$$\lim_{T\to\infty} \frac{1}{T}\int_T^{CT}\int_{\mathbb{R}^2} |u|^2 dxdt = 0. \tag{6.12}$$

By using this and the inversional identity as in Lemma 2.1 , we get

$$\lim_{t\to\infty} \int_{|x|\le Mt} Q(u)dx = 0, \tag{6.13}$$

where $Q(u)$ is as defined in (2.4).

Now suppose that

$$\lim_{n\to\infty} \|\nabla u(t_n)\|_{L^2} = 1 \tag{6.14}$$

for some sequence $t_n\to\infty$. Then $E(u)=1$ implies that

$$\lim_{n\to\infty} \|\dot u(t_n)\|_2^2 + \|u(t_n)\|_2^2 \to 0, \tag{6.15}$$

and hence

$$\lim_{n\to\infty} \int_{|x|\le Ct_n} Q(u(t_n))dx \ge 1, \tag{6.16}$$

contradiction. Therefore we have

$$\limsup_{t\to\infty} \|\nabla u(t)\|_{L^2} < 1. \tag{6.17}$$

On the other hand, (6.13) together with (6.9) and (6.8) implies

$$\limsup_{t\to\infty} \|u(t)\|_{L^2} = 0. \tag{6.18}$$

Hence we have

$$\limsup_{t\to\infty} \|u(t)\|_{H^1} < 1, \tag{6.19}$$

so that we can treat this case in the same way as the above dispersive case, with A_0 between 1 and the left hand side, and the other parameters given by (6.5). However, we get a contradiction in the end between the scattering and the assumption (6.6).

6.2. **Waving concentration case.** This is the wildest case. We have

$$\liminf_{t\to\infty} r_\varepsilon(t) \le 6 \tag{6.20}$$

for all $\varepsilon \in (0,1)$, and

$$\limsup_{t\to\infty} r_\varepsilon(t) = \infty, \tag{6.21}$$

for some $\varepsilon \in (0,1)$; namely the solution repeats dispersing and regathering infinitely many times. Now we fix $\varepsilon \in (0,1/4)$ for which we have (6.21).

6.3. **Extracting very long dispersive era.** By (6.20), there exists $\mathfrak{t}_0 > 0$ such that

$$r_{1/4}(\mathfrak{t}_0) \le 6. \tag{6.22}$$

Let $B : (0,1/4] \to [18,\infty)$ be a continuous function of ε. (6.21) supplies $\mathfrak{t} > \mathfrak{t}_0$ satisfying

$$r_\varepsilon(\mathfrak{t}) \ge B(\varepsilon). \tag{6.23}$$

The actual form of B will be determined later.

By (6.23), the Lipschitz continuity (6.2), preceding concentration (6.22) and the assumption $B(\varepsilon)/2 \ge 9 > 6$, there exist $I = (\mathfrak{t}_1, \mathfrak{t}_2)$ such that

$$\mathfrak{t}_0 < \mathfrak{t}_1 < \mathfrak{t} < \mathfrak{t}_2, \quad |\mathfrak{t}_2 - \mathfrak{t}_1| \ge B(\varepsilon), \quad r_\varepsilon(\mathfrak{t}_1) = 6, \tag{6.24}$$

and

$$[\mathfrak{t}_1, \mathfrak{t}_2] \ni \forall t, \quad r_\varepsilon(t) \ge 6, \tag{6.25}$$

which allows us to use the Strichartz estimate in I depending only on $A := 1-\varepsilon$.

Now the idea is to argue as in the subcritical case inside I. After separating the concentrated energy into a free solution v and another nonlinear solution w with reduced energy, we can apply the perturbative argument beyond $\mathfrak{t}_2$, thanks to the decay of the free solution and Corollary 3.5. Indeed, this is the only place we essentially need that version of nonlinear estimate, otherwise the version with H^1_{loc} would suffice.

Note that v does not decay uniformly in H^1_{loc}. Indeed, if we consider a free wave v whose frequency is supported around $|\xi - Nc| \ll 1$ with $N \gg 1$ and $|c| = 1$, then $v(t,x)$ remains essentially unchanged around $x = tc$ for $|t| \ll R$, and so H^1_{loc} does not decrease during that period. For such high frequency free waves, we can say that $\|v\|^2_{H^1}$ is reduced almost to $E_0(v)/2$ after some time (independent of N) due to the equipartition of energy, but it does not imply any decrease on $\|u\|_{H^1_{loc}}$; without any further information on $w = u - v$, it can still approach 1 arbitrarily closely at later times. On the other hand, when measured in the Besov space $B^0_{\infty,2}$, such a high-frequency wave packet is small from the beginning because of its concentration in high frequency.

Now we set up the assumptions for Lemma 4.3. As before we fix $A_0 = 1 - \varepsilon$, and the other parameters are given by (6.5).

6.4. **Temporary scattering case.** First we consider the case $\|u\|_{X(I)} \leq 2\eta$. The Strichartz norm in I may be too small to apply Lemma 4.1. Hence we just choose $T_1 = \mathfrak{t}_1$, $T_2 = \mathfrak{t}_2$. By the definition of r_ε, there is $c \in \mathbb{R}^2$ such that

$$\int_{|x-c|<R} e_N(u, T_1)dx \geq \nu^2, \tag{6.26}$$

with $R = 6$. Setting $B(\varepsilon) \geq 6L$, we get all the assumptions of Lemma 4.3 with $A = A_0$.

6.5. **Multilayer case.** Next we consider the remaining case $\|u\|_{X(I)} > 2\eta$. Then we split I into subintervals by $\mathfrak{t}_1 = \tau_0 < \tau_1 < \cdots < \tau_N = \mathfrak{t}_2$, with some $N \geq 3$ such that

$$\|u\|_{X(\tau_{j-1},\tau_j)} = \eta, \quad j = 1, 2, \dots N - 1. \tag{6.27}$$

Then Lemma 4.2 implies that

$$\sum_{j=1}^{N} \frac{\eta^{40}}{\tau_j - \tau_0 + \eta^{-3}} \lesssim 1. \tag{6.28}$$

If we have $|\tau_j - \tau_{j-1}| \leq LR(\eta)$ for all $j = 1, \dots N$, then

$$1 \gtrsim \sum_{j=1}^{N} \frac{\eta^{43}}{jL + 1} \gtrsim \eta^{43} \log N, \tag{6.29}$$

and so $|\mathfrak{t}_2 - \mathfrak{t}_1| \leq LRN \lesssim LRe^{C\eta^{-43}}$ with some constant $C \geq 1$.

Hence if we choose $B(\varepsilon) \gg LR(\eta)e^{C\eta^{-43}}$, then there exists $j \in \{0, \dots, N - 1\}$ such that $|\tau_j - \tau_{j+1}| \geq LR(\eta)$. If $j = 0$, then we choose $T_1 = \mathfrak{t}_1 = \tau_0$ and $T_2 = \tau_1$. As in the previous case, there exists $c \in \mathbb{R}^2$ with (6.26) with $R = 6$. Since $R(\eta) \gg 6$, we get all the assumptions of Lemma 4.3 with $A = A_0$.

If $j \geq 1$, then we choose $T_2 = \tau_{j+1}$, and Lemma 4.1 implies that there exists $T_1 \in (\tau_{j-1}, \tau_j)$ and $c \in \mathbb{R}^2$ such that

$$\int_{|x-c|<R} e_N(u, T_1)dx \geq \nu^2, \tag{6.30}$$

with $R = R(\eta)$, hence all the assumptions of Lemma 4.3 are fulfilled with $A = A_0$.

Thus we can use Lemma 4.3 with $A = A_0$ in all cases, hence we get the scattering of u as before, even though it contradict the assumption (6.20).

6.6. **Global Strichartz estimate.** We end this section with a remark about global Strichartz bounds.

Corollary 6.1. *The global solution u of theorem 1.5 satisfies the following global bounds for any $\varepsilon \in (0, 1)$.*

$$\|u\|_{X(\mathbb{R})} + \|f(u)\|_{L^1(\mathbb{R};L^2(\mathbb{R}^2))} \leq C_\varepsilon(\inf_{t\in\mathbb{R}} r_\varepsilon(t)). \tag{6.31}$$

The proof of this corollary is left to the reader. One has just to be more quantitative in the estimates in the previous subsections. Moreover, one has also to use the following lemma, which yields that Lemma 3.1 still holds if (3.1) is replaced by (6.32).

Lemma 6.2. *For any $\theta \in (0,1)$ and $r > 0$, take λ close to 1 such that*

$$(\sqrt{\theta\lambda} + 4\sqrt{2(\lambda-1)}/r)^2 + 2(\lambda-1) < 1 < \lambda.$$

There exists C depending only on θ, r and λ such that if

$$\|\nabla u\|_{L^2}^2 + \|u\|_{L^2}^2 \le 1 \quad \text{and} \quad \|u\|_{H^1[r]} \le \theta, \tag{6.32}$$

then

$$\int_{\mathbb{R}^2} (e^{4\pi\lambda|u|^2} - 1)dx \le C\|u\|_{L^2}^2. \tag{6.33}$$

Proof. By contradiction. Assume (6.33) does not hold, then there exists a sequence u_n such that $\|u_n\|_{H^1} \le 1$, $\|u_n\|_{H^1[r]} \le \theta$ and

$$\int (e^{4\pi\lambda|u_n|^2} - 1) \ge n\|u_n\|_{L^2}^2.$$

Hence, necessarily

$$\liminf_{n\to\infty} \|\nabla u_n\|_{L^2}^2 + \frac{1}{2}\|u_n\|_{L^2}^2 =: \alpha \ge \frac{1}{\lambda}, \tag{6.34}$$

otherwise, we get a contradiction by applying the Trudinger-Moser inequality (see for instance [22, Proposition 1]).

Hence, we deduce that

$$\limsup_{n\to\infty} \frac{1}{2}\|u_n\|_{L^2}^2 \le 1 - \frac{1}{\lambda}. \tag{6.35}$$

Take ϕ a cut-off function such that $\phi = 1$ for $|x| \le r/2$ and $\phi = 0$ for $|x| \ge r$, $0 \le \phi \le 1$ and $\|\nabla\phi\|_{L^\infty} \le 4/r$. Hence,

$$\begin{aligned} &\limsup_{n\to\infty} \|\nabla(\phi u_n)\|_{L^2}^2 + \|\phi u_n\|_{L^2}^2 \\ &\le \limsup_{n\to\infty} (\|\phi\nabla u_n\|_{L^2} + \|u_n\nabla\phi\|_{L^2})^2 + \|\phi u_n\|_{L^2}^2 \\ &\le (\sqrt{\theta} + \sqrt{2(1-1/\lambda)}4/r)^2 + 2(1-1/\lambda) < 1/\lambda. \end{aligned} \tag{6.36}$$

Hence, by Trudinger-Moser, we deduce that for n big enough

$$\int (e^{4\pi\lambda|\phi u_n|^2} - 1) \le C\|\phi u_n\|_{L^2}^2. \tag{6.37}$$

We can cover $\mathbb{R}^2$ by balls of radius $r/2$ in such a way that each point is in at most 10 balls of the same centers and radius r. Adding the estimates (6.37) together, we deduce that

$$\int (e^{4\pi\lambda|u_n|^2} - 1) \le 10C\|u_n\|_{L^2}^2. \tag{6.38}$$

which gives a contradiction. Hence (6.33) holds.

□

7. Criticality of the nonlinear estimate by the Strichartz norms

We see that without the assumption on the local H^1 norm, our nonlinear estimate does not hold generally in the critical case.

Proposition 7.1. *For any $\delta > 0$, there exists a sequence of radial free Klein-Gordon solutions v_N ($N \to \infty$) such that*

$$\begin{aligned}&\int_{\mathbb{R}^2} |\dot{v}_N(t)|^2 + |\nabla v_N(t)|^2 + |v_N(t)| dx < 1, \quad E(v_N, 0) \le 1 + \delta, \\ &\|f(v_N)\|_{L^p_t L^q_x(|t| \ll N^{-1}, |x| \ll N^{-1})} \ge C_\delta (\log N)^{1/2},\end{aligned} \tag{7.1}$$

for any $p, q \in [1, \infty]$ satisfying $1/p + 2/q = 2$.

To apply the Strichartz estimate to the nonlinear term, we have to put it in some $L^p H^{\sigma,q}$ satisfying at least

$$\frac{1}{p} + \frac{1}{q} \ge \frac{3}{2}, \quad \sigma \ge \frac{1}{p} + \frac{2}{q} - 2, \tag{7.2}$$

which is embedded into $L^p L^\rho$ with $1/p + 2/\rho = 2$. Thus the above example implies that in the critical case we cannot control the nonlinearity just by the Strichartz estimate even for the free solutions, which is usually the first step to construct solutions by the iteration argument. However, it does not imply any sort of weak ill-posedness (such as singularity of the solution map) in the critical case, because we can not make the nonlinear energy critical.

Proof. Let $a > 1/2$ and define v_N by the initial data using the Fourier transform

$$\dot{v}_N(0) = 0, \quad v_N(0) = \sqrt{\frac{2\pi}{\log N}} \frac{1}{(2\pi)^2} \int_{1 < |\xi| < e^{-a} N} |\xi|^{-2} e^{i\xi x} d\xi. \tag{7.3}$$

By Plancherel

$$\begin{aligned}\|\nabla v_N(0)\|_{L^2}^2 &= \frac{2\pi}{\log N} \frac{1}{(2\pi)^2} \int_{1 < |\xi| < e^{-a} N} |\xi|^{-2} d\xi = \frac{\log N - a}{\log N}, \\ \|v_N(0)\|_{L^2}^2 &= \frac{2\pi}{\log N} \frac{1}{(2\pi)^2} \int_{1 < |\xi| < e^{-a} N} |\xi|^{-4} d\xi < \frac{1}{2 \log N}.\end{aligned} \tag{7.4}$$

By the sharp Trudinger-Moser inequality (2.9), there exists $M > 0$ such that for any $\mu > 0$

$$\sup_{\|\nabla\psi\|_{L^2}^2 + \mu\|\psi\|_{L^2}^2 \le 1} \int F(\psi) dx \le M/\mu, \tag{7.5}$$

where μ can be removed or inserted by rescaling. Then (7.4) implies that

$$\int 2F(v_N(0)) dx \le \frac{M}{a}, \quad E_0(v_N) < 1, \quad E(v_N, 0) < 1 + \frac{M}{a}, \tag{7.6}$$

so that we get the desired nonlinear energy bound on v_N by choosing $a \ge M/\delta$.

Next, the free solution is given by Fourier transform

$$v_N(t,x) = \sqrt{\frac{2\pi}{\log N}}\frac{1}{(2\pi)^2}\int_{1<|\xi|<e^{-a}N} |\xi|^{-2}\cos(-t\langle\xi\rangle+\xi x)d\xi, \tag{7.7}$$

and hence in the region where $t \sim \varepsilon N^{-1}$ and $|x| \sim \varepsilon N^{-1}$ for some $0 < \varepsilon \ll 1$,

$$\sqrt{\frac{\log N}{2\pi}}(2\pi)^2 v_N(t,x) \geq \sum_{k=1}^{K}\int_{e^{k-1}<|\xi|<e^k} |\xi|^{-2}\cos(-t\langle\xi\rangle+\xi x)d\xi, \tag{7.8}$$

where K is the maximal integer satisfying

$$K \leq \log N - a. \tag{7.9}$$

The cosine is bounded in the region by

$$\cos(-t\langle\xi\rangle+\xi x) \geq 1 - \frac{\varepsilon^2}{2}e^{2(k-K)}, \tag{7.10}$$

so the above integral is bounded from below

$$\geq 2\pi\left[\log N - a - \frac{\varepsilon^2}{2}\sum_{k=1}^{K} e^{2(k-K)}\right], \tag{7.11}$$

and the negative part is bounded by

$$-a - 2\pi\frac{\varepsilon^2}{2}\frac{1}{1-e^{-2}} \geq -2\pi(a+\varepsilon^2). \tag{7.12}$$

Thus we obtain

$$\begin{gathered} v_N(t,x) \geq \sqrt{\frac{\log N}{2\pi}} - \frac{a+\varepsilon^2}{\sqrt{2\pi\log N}}, \\ e^{4\pi|v_N|^2}|v_N|^2 \gtrsim N^2\log N, \end{gathered} \tag{7.13}$$

when $(a+\varepsilon^2)^2 < \log N$.

Thus we conclude

$$\begin{gathered} \inf_{t\sim\varepsilon N^{-1}} \|e^{4\pi|v|^2}v\|_{L^q_x(|x|\sim\varepsilon N^{-1})} \gtrsim N^2\sqrt{\log N}(\varepsilon N^{-1})^{2/q}, \\ \|e^{4\pi|v|^2}v\|_{L^p_tL^q_x(t\sim\varepsilon N^{-1},|x|\sim\varepsilon N^{-1})} \gtrsim \varepsilon^{1/p+2/q}N^{2-2/q-1/p}\sqrt{\log N} \sim \varepsilon^2\sqrt{\log N}. \end{gathered} \tag{7.14}$$

□

8. The case of NLS

For the sake of clarity, we recall the 2D nonlinear Schrödinger equation (NLS):

$$\begin{cases} i\,\dot{u} + \Delta u = \left(e^{4\pi|u|^2} - 1 - 4\pi|u|^2\right)u = f(u), \quad u : \mathbb{R}^{1+2} \to \mathbb{C}, \\ u(0,x) = u_0(x) \in H^1(\mathbb{R}^2), \end{cases}$$

and the conserved quantities

$$M(u,t) = \int_{\mathbb{R}^2} |u|^2 dx,$$

$$H(u,t) = \int_{\mathbb{R}^2} \Big(|\nabla u|^2 + 2F(u)\Big)\, dx,$$

where

$$F(u) = \frac{1}{8\pi}\left(e^{4\pi|u|^2} - 1 - 4\pi|u|^2 - 8\pi^2|u|^4\right).$$

Also we define

$$E(u,t) := H(u,t) + M(u,t). \tag{8.1}$$

For a time slab $I \subset \mathbb{R}$, we define $S^1(I)$ via

$$\|u\|_{S^1(I)} = \|u\|_{L^\infty(I,H^1_x)} + \|u\|_{L^4(I,H^{1,4}_x)}.$$

By the Strichartz estimates we have

$$\|u\|_{S^1} \lesssim \|u(0)\|_{H^1} + \|\langle\nabla\rangle(i\,\dot{u} + \Delta u)\|_{L^{\frac{2}{1+2\eta}}(L_x^{\frac{1}{1-\eta}})}, \tag{8.2}$$

for any $0 < \eta \le 1/2$.

The scattering result Theorem 1.6 is easily proved by the following two lemmas: First we have the Strichartz-type estimate on the nonlinearity

Lemma 8.1. *For any $H \in (0,1)$, there exists $\delta \in (0,1)$, such that for any time slab I, any $T \in I$ and any H^1 solution u of* (1.3) *with $H(u) \le H$, we have*

$$\|u\|_{S^1(I)} \lesssim \|u(T)\|_{H^1} + \|u\|^{4\delta}_{L^4(I,L^8)}\|u\|^{5-4\delta}_{S^1(I)},$$

Next we have a global *a priori* bound. It was proved independently by Planchon-Vega [37] and Colliander et al. [12]

Lemma 8.2. *Let u be a global solution of* (1.3) *in H^1. Then*

$$\|u\|_{L^4(\mathbb{R},L^8)} \lesssim \|u\|^{3/4}_{L^\infty(\mathbb{R};L^2)}\|\nabla u\|^{1/4}_{L^\infty(\mathbb{R};L^2)} \lesssim M(u)^{3/8}H(u)^{1/8}.$$

Actually both of them gave a priori bound on some Sobolev norm on $|u|^2$. The above is a consequence of it via the Sobolev embedding.

By the above global bound, we can decompose $\mathbb{R}$ into a finite number of intervals on which the $\|u\|_{L^4L^8}$ norm is sufficiently small. Then the first lemma gives a uniform bound on $\|u\|_{S^1}$ on each interval, and hence by summing it up for all intervals, we obtain a priori bound

$$\|u\|_{S^1(\mathbb{R}\times\mathbb{R}^2)} \le C(E(u)) < \infty, \tag{8.3}$$

and thereby the scattering for u.

Proof of Lemma 8.1. It suffices to estimate the nonlinear term in some dual Strichartz norm as in (8.2). Choose $0 < \delta < 1$ and $\lambda > 0$ such that

$$K := \frac{H+1}{2} < 1, \quad 2\pi(1+2\delta)\lambda K^2 = 2, \quad \lambda > \frac{1}{\pi(1-\delta)}. \tag{8.4}$$

We estimate only $\nabla f(u)$, since the same estimate on $f(u)$ is easier. Note that

$$|\nabla f(u)| \lesssim |\nabla u||u|^2(e^{4\pi|u|^2} - 1).$$

In the case $\|u\|_{L^\infty_x} \geq K$, we have by the Hölder inequality,

$$\|\nabla f(u)\|_{L_x^{\frac{1}{1-\delta}}} \lesssim \|\nabla u\|_{L^{\frac{2}{1-\delta}}} \|u\|^2_{L^{\frac{4}{\delta}}} \|e^{4\pi|u|^2}-1\|_{L^1}^{1/2-\delta} \|e^{4\pi|u|^2}-1\|_{L^\infty}^{1/2+\delta}. \tag{8.5}$$

The third term on the right is bounded by the Trudinger-Moser (2.9). For the last term we use the H_μ version of the logarithmic inequality (2.11) with $\mu := \min(1, \sqrt{(1-H)/M}) > 0$. Since

$$\|u\|^2_{H_\mu} = \|\nabla u\|^2_{L^2} + \mu^2\|u\|^2_{L^2} \leq H + \mu^2 M \leq \frac{H+1}{2} < 1,$$

that term is bounded by

$$e^{4\pi(1/2+\delta)\|u\|^2_{L^\infty}} \lesssim (1 + \|u\|_{C^{1/2-\delta/2}}/\|u\|_{H_\mu})^{2\pi(1+2\delta)\|u\|^2_{H_\mu}} \lesssim \|u\|^2_{C^{1/2-\delta/2}}, \tag{8.6}$$

where we used (8.4) as well as $\|u\|_{C^{1/2-\delta/2}} \geq \|u\|_{L^\infty} \geq K$. The case $\|u\|_{L^\infty} \leq K$ is easy, since then $|\nabla f(u)| \lesssim |\nabla u||u|^4$.

Now we integrate in time using the Hölder to obtain

$$\|\nabla f(u)\|_{L^{\frac{2}{1+2\delta}}(L^{\frac{1}{1-\delta}})} \lesssim \|\nabla u\|_{L^{\frac{2}{\delta}}(L^{\frac{2}{1-\delta}})} \|u\|^2_{L^{\frac{2}{\delta}}(L^{\frac{4}{\delta}})} \|u\|^2_{L^{\frac{4}{1-\delta}}(C^{1/2-\delta/2})}.$$

Finally, the complex interpolation and the Sobolev embedding imply that

$$\begin{aligned}
&\|\nabla u\|_{L^{\frac{2}{\delta}}(L^{\frac{2}{1-\delta}})} \lesssim \|\nabla u\|^{1-2\delta}_{L^\infty L^2} \|\nabla u\|^{2\delta}_{L^4 L^4} \lesssim \|u\|_{S^1},\\
&\|u\|_{L^{\frac{2}{\delta}}(L^{\frac{4}{\delta}})} \lesssim \|u\|^{1-2\delta}_{L^\infty L^2} \|u\|^{2\delta}_{L^4 L^8},\\
&\|u\|_{L^{\frac{4}{1-\delta}}(C^{1/2-\delta/2})} \lesssim \|u\|_{L^{\frac{4}{1-\delta}}(H^{1,\frac{4}{1+\delta}})} \lesssim \|u\|^{\delta}_{L^\infty H^1} \|u\|^{1-\delta}_{L^4 H^{1,4}} \lesssim \|u\|_{S^1}.
\end{aligned} \tag{8.7}$$

Plugging them into the above, we deduce the result as desired. □

Finally, we observe that the same example as in Proposition 7.1 implies that the linear energy and the Strichartz estimate are not sufficient to control the nonlinearity in the critical case.

Proposition 8.3. *For any $\delta > 0$, there exists a sequence of radial free Schrödinger solutions v_N $(N \to \infty)$ such that*

$$\begin{aligned}
&\int_{\mathbb{R}^2} |\nabla v_N|^2 + |v_N|^2 dx < 1, \quad H(v_N, 0) \leq 1 + \delta,\\
&\|\nabla f(v_N)\|_{L^p_t L^q_x(|t| \ll N^{-2}, |x| \ll N^{-1})} \geq C_\delta (\log N)^{1/2},
\end{aligned} \tag{8.8}$$

for any $(p, q) \in [1, \infty]$ satisfying $1/p + 1/q = 3/2$.

The above norm on $f(v_N)$ is the dual Strichartz norm in H^1_x for the linear Schrödinger equation.

Proof. We take the same initial data as in Proposition 7.1:

$$v_N(0) = \sqrt{\frac{2\pi}{\log N}} \frac{1}{(2\pi)^2} \int_{1<|\xi|<e^{-a}N} |\xi|^{-2} e^{i\xi x} d\xi. \tag{8.9}$$

Then the proof for the previous Proposition gives the desired bounds on the initial data. Also, it implies that for the free solution

$$v_N(t,x) = \sqrt{\frac{2\pi}{\log N}} \frac{1}{(2\pi)^2} \int_{1<|\xi|<e^{-a}N} |\xi|^{-2} e^{-it|\xi|^2+i\xi x} d\xi, \tag{8.10}$$

we have in the region where $t \sim \varepsilon^2 N^{-2}$ and $|x| \sim \varepsilon N^{-1}$ for some small fixed $\varepsilon > 0$ and large $N \in \mathbb{N}$,

$$\Re v_N(t,x) \geq \sqrt{\frac{\log N}{2\pi}} - \frac{a+\varepsilon^2}{\sqrt{2\pi \log N}}, \quad e^{4\pi|v|^2}|v|^2 \gtrsim N^2 \log N. \tag{8.11}$$

In this case we have to estimate ∇v also. By the radial symmetry, it suffices to consider the case $x = (x_1, 0)$ and $\nabla v_N = (\partial_1 v_N, 0)$. Then

$$\begin{aligned} \partial_1 v_N = \partial_r v_N &\sim \frac{1}{\sqrt{\log N}} \int_{1<|\xi|<e^{-a}N} \frac{\xi_1}{|\xi|^2} e^{-it|\xi|^2+i\xi x} d\xi \\ &= \frac{i}{\sqrt{\log N}} \int_1^N e^{-it\rho^2} \int_{-\pi}^{\pi} \cos\theta \sin(r\rho\cos\theta) d\theta d\rho, \end{aligned} \tag{8.12}$$

since $0 < t\rho^2 < \varepsilon^2 \ll 1$ and $0 < r\rho\cos\theta < \varepsilon \ll 1$, we get

$$|\partial_r v_N| \sim \frac{1}{\sqrt{\log N}} \int_1^N r\rho d\rho \sim \frac{rN^2}{\sqrt{\log N}} \sim \frac{\varepsilon N}{\sqrt{\log N}}. \tag{8.13}$$

Thus we conclude

$$\begin{aligned} &\inf_{t\sim\varepsilon^2 N^{-2}} \|e^{4\pi|v_N|^2}|v_N|^2 \nabla v_N\|_{L^q_x(|x|\sim\varepsilon N^{-1})} \gtrsim N^2 \log N \frac{\varepsilon N}{\sqrt{\log N}} (\varepsilon N^{-1})^{2/q}, \\ &\|\nabla f(v_N)\|_{L^p_t L^q_x(t\sim\varepsilon^2 N^{-2}, |x|\sim\varepsilon N^{-1})} \gtrsim \varepsilon^{2/p+2/q} N^{3-2/q-2/p} \sqrt{\log N} \sim \varepsilon^3 \sqrt{\log N}. \end{aligned} \tag{8.14}$$

□

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Arizona State University, Department of Mathematics and Statistics, P.O. Box 871804, Tempe, AZ 85287-1804, USA
E-mail address: `slim.ibrahim@asu.edu`
URL: `http://math.asu.edu/~ibrahim`

Faculty of Sciences of Tunis, Department of Mathematics,
E-mail address: `Mohamed.Majdoub@fst.rnu.tn`

New York University, The Courant Institute for Mathematical Sciences,
E-mail address: `masmoudi@courant.nyu.edu`

Department of Mathematics, Kyoto University
E-mail address: `n-kenji@math.kyoto-u.ac.jp`