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DMA - 08 - 13

Département de mathématiques et applications

CNRS UMR 8553

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September 2008

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Relevance of the slip condition for fluid flows near an irregular boundary

David Gérard-Varet *and Nader Masmoudi†

Abstract

We consider the Navier-Stokes equation in a domain with rough boundaries. The small irregularity is modeled by a small amplitude and small wavelength boundary, with typical lengthscale $\varepsilon \ll 1$. For periodic roughness, it is well-known that the best homogenized (that is regular in ε) boundary condition is of Navier type. Such result still holds for random stationary irregularities, as shown recently by the first author [5, 13]. We study here arbitrary irregularity patterns.

Keywords: Wall laws, rough boundaries,homogenization, ergodicity, almost periodic functions

1 Introduction

The interaction between a fluid and a solid boundary is still today a matter of debate. The main reason is that underlying molecular processes are still unclear, see [20] for a review. But even at larger scales, it has been recognized that the small irregularities of the solid surface can alter deeply various aspects of the fluid dynamics. The understanding of such roughness-induced effects has been the topic of many recent papers, with regards to friction phenomena in microfluidics [23], or to stability issues [22].

Fortunately enough, in most situations, one does not need an accurate description of the dynamics near the irregular boundary. One only looks for an averaged effect. Among the practical ways used to describe this averaged effect, many physicists and numerists rely on *wall laws*: the rough boundary is replaced by an artificial smoothed one, and a homogenized boundary condition (a wall law) is prescribed there, that should reflect the mean impact of the small irregularities. The main question is then: what is the good wall law ? The aim of the present paper is to address this question from a mathematical perspective.

There are numerous mathematical studies on this boundary homogenization problem. On wall laws for scalar elliptic equations, we refer to [1]. On wall laws for fluid flows, see [2, 3, 4, 16, 17]. These works go along with more formal computations, *cf* for instance [6, 21]. Broadly, these studies have been carried under two assumptions: *i)* Compact domains, typically bounded channels with periodic or inflow/outflow boundary conditions *ii)* Periodic irregularities, leading to periodic homogenization problems. The first restriction is just a small mathematical convenience, that gives direct compactness properties through Rellich

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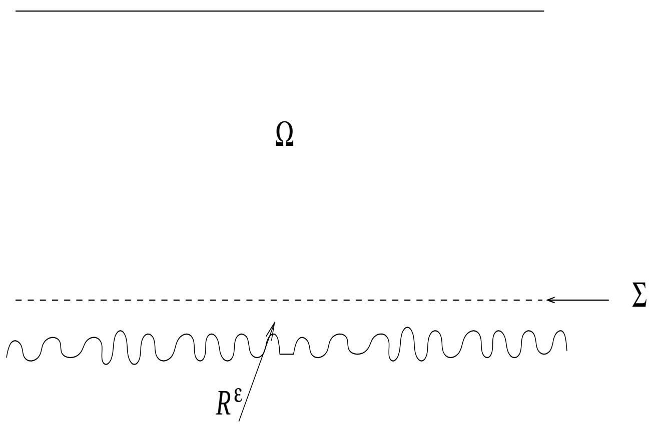

Figure 1: The rough domain Ω^ε.

type theorems. The second assumption is far more stringent, both on mathematics and physics sides. In the recent articles [5] (with A. Basson) and [13], the first author has relaxed substantially these assumptions, considering unbounded channels and random homogenenous irregularity.

Let us briefly describe these previous results. We will restrict ourselves to a simple model, namely a two-dimensional rough channel

$$\Omega^\varepsilon \;=\; \Omega \cup \Sigma \cup R^\varepsilon$$

where $\Omega = \mathbb{R} \times (0,1)$ is the *smooth part*, R^ε is the rough part, and $\Sigma = \mathbb{R} \times \{0\}$ their interface. We assume that the rough part has typical size ε, that is

$$R^\varepsilon \;=\; \left\{x,\; x_2 > \varepsilon\omega\left(\frac{x_1}{\varepsilon}\right)\right\}$$

for a Lipschitz function $\omega : \mathbb{R} \mapsto (-1,0)$. See Figure 1 for an example of such a domain. We consider in this channel a steady flow, governed by stationary Navier-Stokes system with given flux

$$\left\{\begin{aligned} -\Delta u + u\cdot\nabla u + \nabla p = 0,\; x \in \Omega^\varepsilon,\\ \mathrm{div}\; u = 0,\; x \in \Omega^\varepsilon,\\ u|_{\partial\Omega} = 0, \quad \int_{\sigma^\varepsilon} u_1 = \phi. \end{aligned}\right. \tag{NS$^\varepsilon$}$$

The third equation corresponds to a standard no-slip condition at the boundary of the rough channel. The last equation expresses that a flux ϕ is imposed across a vertical cross-section σ^ε of Ω^ε. Note that the flux integral does not depend on the location of the cross-section, thanks to the divergence-free and Dirichlet conditions.

Note also that this problem has singularities in ε, due to the high frequency oscillation of the boundary. The idea of wall laws is to *replace this singular problem in* Ω^ε *by a regular* *problem in* Ω. One keeps the same Navier-Stokes equations

$$\begin{cases} -\Delta u + u\cdot\nabla u + \nabla p = 0, \; x\in\Omega, \\ \qquad\qquad\qquad \mathrm{div}\; u = 0, \; x\in\Omega, \\ u|_{x_2=1} = 0, \quad \displaystyle\int_\sigma u_1 = \phi, \end{cases} \tag{NS}$$

but with a regular boundary condition at the artificial boundary Σ. The point is to find the most accurate (and regular in ε) condition.

In all papers previously mentioned, the starting point is a formal expansion of u^ε:

$$u^\varepsilon(x) \;\sim\; u^0(x) \;+\; \varepsilon\, u^1(x, x/\varepsilon)$$

Formally, the leading term u^0 satisfies (NS) together with the simple no-slip condition

$$u = 0 \;\text{ at }\; \Sigma \tag{Di}$$

System (NS)-(Di) has an explicit solution, the famous Poiseuille flow :

$$u^0(x) \;=\; \left(U^0(x_2), 0\right), \quad U^0(x_2) \;=\; 6\phi x_2(1-x_2)$$

Note that u^0 is defined in all $\mathbb{R}^2$. This zeroth order asymptotics is mathematically justified in [5], for small fluxes ϕ:

Theorem 1 *There exists $\phi_0 > 0$, such that for all $|\phi| < \phi_0$, for all ε, system* (NS$^\varepsilon$) *has a unique solution u^ε in $H^1_{uloc}(\Omega^\varepsilon)$. Moreover, for $\varepsilon < \varepsilon_0$ small enough*

$$\|u^\varepsilon - u^0\|_{H^1_{uloc}(\Omega^\varepsilon)} \le C\sqrt{\varepsilon}, \quad \|u^\varepsilon - u^0\|_{L^2_{uloc}(\Omega)} \le C\varepsilon.$$

Briefly, the Dirichlet wall law provides a $O(\varepsilon)$ approximation of the exact solution u^ε in $L^2_{uloc}(\Omega)$. We emphasize that to get a Dirichlet condition at the limit $\varepsilon = 0$ is true in many settings, even starting from a slip condition at the rough boundaries. See [11, 9]. Nevertheless, the Dirichlet wall law is in some sense crude: Theorem 1 is obtained through energy estimates, that do not ditinguish the behavior of the flow near the boundary. Therefore, a natural question is: can we find a better wall law ?

A widespread idea is that the approximation can be refined, considering a Navier condition:

$$v_1 = \varepsilon\alpha\partial_2 v_1, \quad v_2 = 0 \;\text{ at }\Sigma, \tag{Na}$$

where α is a parameter linked to the roughness profile ω. In our model, a formal explanation is as follows. As the Poiseuille flow u^0 does not vanish at the lower part of $\partial\Omega^\varepsilon$, a boundary layer corrector

$$u^1(x, x/\varepsilon) \;=\; 6\phi\varepsilon v(x/\varepsilon)$$

must be added to describe the dynamics near the irregular boundary. The (normalized) boundary layer $v = v(y)$ is defined on the rescaled infinite domain

$$\Omega^{bl} \;=\; \{y, \; y_2 > \omega(y_1)\}$$

It is made to cancel the trace of u^0 at the rough boundary. Formally, it satisfies the following Stokes problem

$$\left\{\begin{array}{rcl} -\Delta v + \nabla q & = & 0,\ x \in \Omega^{bl}, \\ \nabla \cdot v & = & 0,\ x \in \Omega^{bl}, \\ v(y_1, \omega(y_1)) & = & -(\omega(y_1), 0). \end{array}\right. \tag{BL}$$

The belief which leads to the Navier wall law is that v should converge to a constant field at infinity, which is classical in boundary layer theory. More precisely, one expects

$$v \to v^\infty = (\alpha, 0), \quad \text{as } y_2 \to +\infty$$

for some constant α. Back to the approximation of u^ε, one obtains formally

$$u^\varepsilon \sim u^0 + 6\varepsilon\phi\,(\alpha, 0) + o(\varepsilon) \quad \text{in } L^2$$

and the sum of the first two terms at the r.h.s satisify (Na).

The problem we consider in this paper is the mathematical justification of this formal reasoning. The main point is to understand the properties of the boundary layer system (BL). Although linear, the analysis of this is system is not easy, at least at two levels.

1. Well-posedness is not clear. As the boundary data ω is not decreasing at infinity, one must work with functions of infinite energy. To identify the appropriate functional spaces and to obtain local bounds is not obvious. The Stokes operator being vectorial, one can not work in L^∞ using scalar tools such as the maximum principle or Harnack inequality. Moreover, as Ω^{bl} is unbounded in all directions, the Poincaré inequality does not hold. This is a big difference with the case of the channel: the well-posedness and estimates of Theorem 1 rely in a crucial way on this inequality.

2. Even if a solution v is built, its behavior as $y_2 \to +\infty$, especially its convergence to a constant field, is also a delicate question.

The difficulties raised by system (BL) *explain the periodicity assumption on* ω *in previous studies.* Indeed, under such assumption, the analysis of (BL) gets easy. If ω is say L periodic in y_1, a simple application of Lax-Milgram lemma yields well-posedness in the space

$$\Big\{ v \in H^1_{loc}(\overline{\Omega_{bl}}),\ v\ \ L-\text{periodic in } y_1,\ \int_0^L \int_{\omega(y_1)}^{+\infty} |\nabla v|^2 dy_2 dy_1 < +\infty \Big\}.$$

Moreover, a simple Fourier transform in y_1 shows that

$$\|v(y) - v^\infty\| \le C\, e^{-\delta y_2/L}, \quad v^\infty = (\alpha, 0), \quad \alpha = \frac{1}{L}\int_0^L v_1(s)ds, \quad \delta > 0,$$

that is exponential convergence to a constant field $v^\infty = (\alpha, 0)$ at infinity. As a consequence, *in this periodic framework, the solution* u^N *of* (NS)-(Na) *satisfies*

$$\|u^\varepsilon - u^N\|_{L^2_{uloc}(\Omega)} \ \le\ C\,\varepsilon^{3/2}.$$

We refer to [16] for a similar result in the case of a bounded channel. The error estimate $\varepsilon^{3/2}$ comes from the fact that

$$\|\varepsilon(v(x/\varepsilon) - (\alpha, 0))\|_{L^2_{uloc}} = O(\varepsilon^{3/2}).$$

As discussed in [8], *the Navier wall law is the best homogenized boundary condition*: the boundary layer oscillations are $O(\varepsilon^{3/2})$ and thus prevent any improvement at Σ.

The periodicity hypothesis is a stringent one, and has been considerably relaxed in recent papers by the first author [5, 13]. In these studies, *one makes the much more realistic assumption that the rough profile is given by a random stationary process* $(\omega, y_1) \mapsto \omega(y_1)$ *defined on the probablity space*

$$P \; = \; \{\omega : \mathbb{R} \mapsto (-1,0), \quad \omega \, K - \mathrm{Lip}\}\,, \quad K > 0,$$

of all admissible rough boundaries, together with the cylindrical $\sigma-$ field $\mathcal{C}$ and with a stationary measure π. We refer to [5] for precise statements. Using this probabilistic structure, one can extend the results of the periodic case. Key elements of the analysis are:

1. the well-posedness of (BL), obtained in a functional space encoding stationarity with respect to y_1. Let us emphasize that it is a space of functions depending on both ω (the random parameter) and y. It provides the existence of a solution in the distribution sense *almost surely. However, it yields neither existence nor uniqueness of a solution for a given* ω. We refer to [5] for all details.

2. the use of the ergodic theorem that substitutes to the analysis with Fourier series.

The main result of [5] is

Theorem 2 *There exists* $\alpha = \alpha(\omega) \in L^2(P)$ *such that the solution* u^N *of* (NS), (Na) *satisfies*

$$\|u^\varepsilon - u^N\|_{L^2_{uloc}(P\times\Omega)} = o(\varepsilon),$$

where $\|w\|_{L^2_{uloc}(P\times\Omega)} := sup_x \left(\int_P \int_{B(x,1)\cap\Omega} |w|^2 dx d\pi\right)^{1/2}$. Note that the $o(\varepsilon)$ bound is only a slight improvement of the $O(\varepsilon)$ in Theorem 1. Contrary to the periodic case, the simple use of the ergodic theorem does not yield any speed rate. This poor bound is due to the lack of information on the way v converges at infinity. However, in article [13], we have shown that *under a main assumption of decorrelation at large distances of the roughness distribution*, one has the sharp error estimate:

$$\|u^\varepsilon - u^N\|_{L^2_{uloc}(P\times\Omega)} = O(\varepsilon^{3/2}|\ln\varepsilon|^{1/2}).$$

This bound comes from a central limit theorem for weakly dependent variables. We refer to [13] for detailed statements and proofs. Note that this assumption of independence at large distances is "orthogonal" to a periodicity assumption. For such roughness patterns, one shows in brief that $v \to v^\infty$ as $y_2^{-1/2}$ instead of $e^{-\delta y_2}$ in the periodic case. As the first function is almost but not square integrable in y_2, it is responsible for an extra $|\ln\varepsilon|^{1/2}$ term in the estimate.

Note that Theorem 2 and the following refined estimate are by nature probabilistic. An arbitrary non-periodic boundary ω being given, one can not say if a Navier condition is the correct wall law to homogenize this boundary. In other words, the following deterministic questions remain:

- Is the formal reasoning described above valid for an arbitrary irregularity profile ω ? More precisely, is the system (BL) well-posed for an arbitrary ω ? Then, does v converge to a constant field $(\alpha, 0)$ at infinity ?

- If this homogenization is not valid in general, can we identify a class of functions (other than periodic) for which it is correct ?

Broadly, we address all these questions in the present paper. We first show in section 2 that the boundary layer system (BL) is well-posed for any Lipschitz bounded function ω. Indeed,

Theorem 3 *System* (BL) *has a unique solution* $v \in H^1_{loc}(\overline{\Omega_{bl}})$ *satisfying:*

$$\sup_k \int_{\Omega^{bl}_{k,k+1}} |\nabla v|^2 < +\infty \quad \textit{where for all } k,l, \quad \Omega^{bl}_{k,l} := \Omega_{bl} \cap \{k < y_1 < l\}.$$

We believe that this theorem is interesting in itself, as stationary Navier-Stokes equations in unbounded domains are still a source of interesting open questions, see for instance [7]. The proof of the theorem will rely on two ideas. First, we will establish an equivalent formulation of (BL) in the channel $\Omega^{bl,-} := \Omega^{bl} \cap \{y_2 < 0\}$, with a so-called transparent boundary condition at $y_2 = 0$. Then, we will solve this new formulation, using ideas of [19] in a bounded channel.

The solution v at hand, we will investigate its asymptotic behaviour as y_2 goes to infinity. *We claim that v is very unlikely to converge to a constant field for any rough boundary.* This claim relies on the study of a similar simpler model. Transposed to (BL), it suggests that convergence to some v^∞ may not hold if the roughness profile ω does not have ergodicity properties. All details will be provided in section 3. Note that this is coherent with the random setting studied by the first author, for which an ergodic theorem is used.

To stick to a deterministic setting, whereas preserving ergodicity, it is then natural to consider almost periodic functions. More precisely we introduce the set:

$$PT(\mathbb{R}) := \left\{ \omega : \mathbb{R} \mapsto \mathbb{R},\ y_1 \mapsto \sum_{j \in J} a_j e^{i\xi_j y_1}, \quad a_j \in \mathbb{C},\ \xi_j \in \mathbb{R},\ J \text{ finite} \right\}$$

of real valued trigonometric polynomials, and the set $AP(\mathbb{R}) := \overline{PT(\mathbb{R})}^{W^{2,\infty}}$ of functions that are in the closure of $PT(\mathbb{R})$ for the $W^{2,\infty}$ norm. We shall justify the Navier wall law for all elements of $AP(\mathbb{R})$. Namely,

Theorem 4 *For all* $\omega \in AP(\mathbb{R})$*, there exists* α *such that the solution* u^N *of* (NS)-(Na) *satisfies*

$$\|u^\varepsilon - u^N\|_{L^2_{uloc}(\Omega)} = o(\varepsilon).$$

This theorem will be proved in section 4. We point out the $o(\varepsilon)$ in the error estimate: again, the simple use of ergodicity properties will not provide any rate. Nevertheless, as in the random case, we can identify a subclass for which we can say more. Let us consider a *quasiperiodic function* ω, that is, following [18], $\omega(y_1) = F(\lambda y_1)$, for some smooth periodic $F = F(\theta)$, $\theta \in \mathbb{T}^d$ and some constant vector $\lambda \in \mathbb{R}^d$. We assume the following diophantine condition:

$$\text{(H) for all } \delta > 0, \text{ there exists } c > 0, \quad |\lambda \cdot \xi| \geq c|\xi|^{-d-\delta}, \quad \forall \xi \in \mathbb{Z}^d \setminus \{0\}.$$

It is well-known that this small divisor assumption is satisfied for almost every λ, see [10]. In this framework, we have the following refined result:

Theorem 5 *Assume that* $\omega(y_1) = F(\lambda y_1)$*, where* F *is a smooth periodic function on* $\mathbb{T}^d$ *and* $\lambda \in \mathbb{R}^d$*. Assume that* λ *satisfies (H). Then,*

$$\|u^\varepsilon - u^N\|_{L^2_{uloc}(\Omega)} = O(\varepsilon^{3/2}).$$

This theorem will be proved at the end of section 4.

2 Well-posedness of (BL)

This section is devoted to the proof of Theorem 3. It relies on an equivalent formulation of system (BL), inspired by transparent boundary conditions in numerical analysis. More precisely, we will restrict the Stokes equations to the lower part of Ω^{bl}, namely the channel $\Omega^{bl,-} := \Omega^{bl} \cap \{y_2 = 0\}$. We will of course keep the same inhomogeneous boundary data at the lower boundary $y_2 = \omega(y_1)$. But to get an equivalent problem, we will also need to specify a transparent boundary condition at $\{y_2 = 0\}$, *transparent* meaning that it should be satisfied exactly by the solution v of (BL). This transparent condition will involve a pseudodifferential operator of Dirichlet-to-Neumann type. To introduce this equivalent formulation, we need some preliminary results on the Stokes problem in a half-space.

2.1 Stokes problem in a half-space

We consider the Dirichlet problem for the Stokes operator in the half plane $\mathbb{R}^2_+$:

$$\begin{cases} -\Delta u + \nabla p = 0, \; y_2 > 0, \\ \quad\quad \nabla \cdot u = 0, \; y_2 > 0, \\ \quad\quad u|_{y_2=0} = u_0. \end{cases} \tag{2.1}$$

We have the following well-posedness result:

Proposition 6 *For all $u_0 \in H^{1/2}_{uloc}(\mathbb{R})$ there exists a unique solution $u \in H^1_{loc}(\overline{\mathbb{R}^2_+})$ of (2.1) satisfying*

$$\sup_{k\in\mathbb{Z}} \int_k^{k+1} \int_0^{+\infty} |\nabla u|^2 \, dy_2 dy_1 < +\infty. \tag{2.2}$$

Proof.

Uniqueness

Suppose that $u_0 = 0$ and that u satisfies (2.1)-(2.2). We wish to show that $u = 0$. The key ingredient is the Fourier transform with respect to y_1. To apply this transform to the equation, we must ensure first that the velocity and pressure fields have enough regularity.

By Poincaré's inequality, we infer from (2.2): for all $a \geq 0$

$$\sup_k \int_k^{k+1} \int_0^a |u|^2 \, dy_1 dy_2 \; \leq \; C_a \sup_k \int_k^{k+1} \int_0^{+\infty} |\nabla u|^2 dy_2 dy_1 \; < \; +\infty. \tag{2.3}$$

Moreover, standard elliptic regularity results yield: for all $\beta \in \mathbb{N}^2$,

$$\int_{k+1/4}^{k+3/4} \int_0^1 |\partial_y^\beta u|^2 dy_2 dy_1 \; \leq \; C \int_k^{k+1} \int_0^{5/4} \left(|\nabla u|^2 + |u|^2\right) dy_2 dy_1$$

Combining this inequality with inequality (2.3), $a = 5/4$, we obtain the following estimate near the boundary: for all $\beta \in \mathbb{N}^2$,

$$\int_{k+1/4}^{k+3/4} \int_0^1 |\partial_y^\beta u|^2 dy_2 dy_1 \; < \; +\infty \tag{2.4}$$

As $v_i = \partial_i u$, $i = 1,2$, satisfies a homogeneous Stokes equation in $\mathbb{R}^2_+$, we can apply the Caccioppoli's inequality to the v_i's, see [15]. We deduce the following interior estimate: for all $a \geq 3/4$,

$$\int_k^{k+1} \int_a^{a+1} |\nabla^2 u|^2 dy_2 dy_1 \ \leq \ C \int_{k-1/4}^{k+5/4} \int_{a-1/4}^{a+5/4} |\nabla u|^2 dy_2 dy_1 \tag{2.5}$$

where C does not depend on a. By the same elliptic regularity results as before, but applied inside the domain and to the v^i's, we get: for all $\beta \in \mathbb{N}^2$,

$$\int_{k+1/4}^{k+3/4} \int_{a+1/4}^{a+3/4} |\partial_y^\beta \nabla u|^2 dy_2 dy_1 \ \leq \ C \int_k^{k+1} \int_a^{a+1} \left(|\nabla^2 u|^2 + |\nabla u|^2 \right) dy_2 dy_1$$

Together with (2.5), this yields

$$\int_{k+1/4}^{k+3/4} \int_{a+1/4}^{a+3/4} |\partial_y^\beta \nabla u|^2 dy_2 dy_1 \ \leq \ C \int_{k-1/4}^{k+5/4} \int_{a-1/4}^{a+5/4} |\nabla u|^2 dy_2 dy_1$$

Summing over $a = 3/4, 1+1/4, 2+1/4...$ and adding the boundary estimate (2.4), we obtain

$$\forall \beta \in \mathbb{N}^2, \quad \sup_k \int_k^{k+1} \int_0^{+\infty} |\partial_y^\beta \nabla u|^2 + |\partial_y^\beta \nabla p|^2 \, dy_2 dy_1 \ < \ +\infty. \tag{2.6}$$

As a consequence of these bounds, there exists a unique smooth ψ such that

$$u = \nabla^\perp \psi, \quad \psi|_{\{y_2=0\}} = \partial_2 \psi|_{\{y_2=0\}} = 0, \quad \Delta^2 \psi = 0$$

by the Stokes equation. Note that

$$y_2 \mapsto \psi(y_1, y_2) \in C^\infty(\mathbb{R}^+; \, C^0_b(\mathbb{R})).$$

Moreover,

$$\|\partial_y^\beta \nabla^2 \psi(\cdot, y_2)\|_{C^0_b(\mathbb{R})} \to 0, \ \text{ as } y_2 \to +\infty, \quad \forall \beta \in \mathbb{N}^2. \tag{2.7}$$

As $C^0_b(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R})$, the space of tempered distributions, we can take the Fourier transform with respect to y_1, considering $\hat{\psi}(\xi, y_2) = \mathcal{F}\psi(\cdot, y_2)(\xi)$. One can apply the Fourier transform to the biharmonic equation: it yields

$$\left(\partial_2^2 - |\xi|^2\right)^2 \hat{\psi}(\xi, y_2) \ = \ 0.$$

To avoid any problem with possible singularities at $\xi = 0$, we introduce a smooth function $\chi = \chi(\xi)$ compactly supported in $\mathbb{R}^*_-$. Then, $\varphi := \chi \hat{\psi}$ satisfies the same equation as $\hat{\psi}$, is smooth with respect to y_2, is a temperate distribution in ξ with compact support in $\mathbb{R}^*_-$. By standard integrating factor method, one shows easily that

$$\varphi(\xi, y_2) \ = \ (A_1(\xi)\, y_2 \ + \ A_2(\xi))\, e^{-\xi y_2} \ + \ (B_1(\xi)\, y_2 \ + \ B_2(\xi))\, e^{\xi y_2},$$

for temperate distributions A_i, B_i compactly supported in $\mathbb{R}^*_-$, which makes the product with the exponential terms meaningful. Then, we can use the boundary conditions $\varphi|_{\{y_2=0\}} = \partial_2 \varphi|_{\{y_2=0\}} = 0$ and the conditions at infinity

$$\partial^\beta \nabla^2 \varphi(\cdot, y_2) \to 0, \ \text{as } y_2 \to +\infty, \quad \text{in } \mathcal{S}'(\mathbb{R}), \quad \forall \beta \in \mathbb{N}^2,$$

all inherited from ψ. This implies easily $\varphi = 0$, and from there $\hat{\psi} = 0$ for $\xi \in \mathbb{R}^*_-$. The same result holds for $\xi \in \mathbb{R}^*_+$. Thus, for all y_2, $\hat{\psi}(\cdot, y_2)$ has support in $\{0\}$. It is therefore a combination of a Dirac mass and its derivatives. As $\psi(y_2, \cdot) \in C^0_b(\mathbb{R})$, no derivative can be involved, which means

$$\psi(y_2, \xi) \;=\; p(x_2)\,\delta(\xi).$$

The equation yields $p^{(4)}(x_2) = 0$. Again, by conditions at $y_2 = 0$ and $y_2 = +\infty$, we obtain $p = 0$. Thus, $\psi = 0$, which concludes the proof of uniqueness.

Existence

Let $u_0 \in H^{1/2}_{uloc}(\mathbb{R})$. We build a solution (u, p) of (2.1) in terms of the Poisson kernel for the Stokes operator. We set for all $y_2 > 0$:

$$u(y) \;=\; \int_{\mathbb{R}} G(t, y_2) u_0(y_1 - t)\, dt, \quad p(y) \;=\; \int_{\mathbb{R}} \nabla g(t, y_2) \cdot u_0(y_1 - t)\, dt \tag{2.8}$$

$$\text{where} \quad G(y) \;=\; \frac{2y_2}{\pi(y_1^2 + y_2^2)^2} \begin{pmatrix} y_1^2 & y_1\, y_2 \\ y_1\, y_2 & y_2^2 \end{pmatrix}, \quad g(y) = -\frac{2y_2}{\pi(y_1^2 + y_2^2)}.$$

One can check easily that u, p belong to $C^\infty(\mathbb{R}^2_+)$ and that one can differentiate under the integral sign. Moreover, one can show that for all $a > 0$, for all $\beta \in \mathbb{N}^2$,

$$\sup_k \int_k^{k+1} \int_a^{+\infty} \left(|\partial^\beta_y \nabla u|^2 \;+\; \int_k^{k+1} \int_a^{+\infty} |\partial^\beta_y p|^2 \right) dy_2\, dy_1 \;\le\; C(a, \alpha) < +\infty.$$

Let us just show one of these inequalities, namely: for all k

$$\int_k^{k+1} \int_a^{+\infty} |\nabla u|^2 \le C < +\infty,$$

where C is independent of k. The higher derivatives and pressure term are handled in the exact same way. Considering the form of (2.8), it is enough to prove the bound for $k = 0$. We write

$$\begin{aligned}
\int_0^1 \int_a^{+\infty} |\nabla u|^2 dy_2 dy_1 \;&\le\; C \int_0^1 \int_a^{+\infty} \Big| \int_{\mathbb{R}} \frac{1}{t^2 + y_2^2} |u_0(y_1 - t)| dt \Big|^2 dy_2 dy_1 \\
&\le\; C \int_0^1 \int_a^{+\infty} \left(\int_{\mathbb{R}} \frac{1}{t^2 + y_2^2} \right) \int_{\mathbb{R}} \frac{1}{t^2 + y_2^2} |u_0(y_1 - t)|^2 dt dy_1 dy_2 \\
&\le\; C' \int_a^{+\infty} \frac{1}{y_2} \int_{\mathbb{R}} \frac{1}{t^2 + y_2^2}\, dt\, dy_2\, \|u_0\|_{L^2_{uloc}} \;\le\; C'' \int_a^{+\infty} \frac{dy_2}{y_2^2} < +\infty
\end{aligned}$$

By well-known properties of G and g, the fields u and p satisfy the Stokes equation in $\mathbb{R}^2_+$. It remains to show that for $a > 0$,

$$\int_k^{k+1} \int_0^a |u|^2 + |\nabla u|^2 dy_2 dy_1 \;\le\; C \;<\; +\infty$$

uniformly in k and that $u|_{\{y_2 = 0\}} = u_0$ in the trace sense over $(k, k+1)$. Again, it is clearly enough to show it for $k = 0$. Let $\chi \in C^\infty_c(\mathbb{R})$, $\chi = 1$ on $[-1, 2]$. We decompose

$$u \;=\; U \;+\; V \;:=\; \int_{\mathbb{R}} G(y_1 - t, y_2) \chi(t)\, u_0(t) dt \;+\; \int_{\mathbb{R}} G(y_1 - t, y_2)\, (1 - \chi(t))\; u_0(t) dt$$

To bound U, one uses the fact that $\chi u_0 \in H^{1/2}(\mathbb{R})$. As $t \mapsto G(t, y_2) \in L^1(\mathbb{R})$ for all $y_2 > 0$, one can take the Fourier transform with respect to y_1, $\hat{U}(\xi, y_2) = \hat{G}(\xi, y_2)\, \widehat{\chi u_0}(\xi)$. An explicit calculation yields

$$\hat{U}(\xi, y_2) \;=\; e^{-|\xi| y_2} \chi \hat{u}_0(\xi) \;-\; (\widehat{\chi u_0}_1 + i\ \mathrm{sign}(\xi) \widehat{\chi u_0}_2)\, y_2\, e^{-|\xi| y_2} \begin{pmatrix} |\xi| \\ i\xi \end{pmatrix}. \tag{2.9}$$

Thanks to this expression, one checks that U is in $H^1(\mathbb{R} \times (0, a))$ for all $a > 0$, *a fortiori* in $H^1_{uloc}(\mathbb{R} \times (0, a))$. We still have to bound V. We write directly

$$\int_0^a \int_0^1 |V|^2 dy_1 dy_2 \;\le\; C \int_0^a \int_0^1 \left| \int_{\mathbb{R}} \frac{1}{|y_1 - t|^2 + y_2^2} (1 - \chi(t)) u_0(t) dt \right|^2 dy_1 dy_2$$

Notice that for t in the support of $(1 - \chi) u_0$ and $y_1 \in (0, 1)$, one has $|y_1 - t| \ge |t - 1| \ge 2$. In other words, one does not see the singularity of the kernel G. Hence,

$$\int_0^a \int_0^1 |V|^2 dy_1\, dy_2 \;\le\; C \left| \int_{\mathbb{R}} \frac{1}{(t-1)^2} (1 - \chi(t)) u_0(t) dt \right|^2 \;\le C' \sum_{k \in Z} \frac{1}{k^2} \|u_0\|^2_{L^2_{uloc}} \;<\; +\infty.$$

The same argument works for the gradient and provides the bound in $H^1_{uloc}(\mathbb{R} \times (0, a))$. In the same way, one can decompose

$$p \;=\; P + Q \;:=\; \int_{\mathbb{R}} \nabla g(y_1 - t, y_2) \chi(t)\, u_0(t) dt \;+\; \int_{\mathbb{R}} \nabla g(y_1 - t, y_2)\, (1 - \chi(t))\; u_0(t) dt$$

and show that p belongs to $L^2_{uloc}(\mathbb{R} \times (0, a))$. In particular

$$\widehat{P}(\xi, y_2) \;=\; -2 e^{-|\xi| y_2}\, (\widehat{\chi u_0}_1 \;+\; i\ \mathrm{sign}(\xi)\, \widehat{\chi u_0}_2)\, i\xi \tag{2.10}$$

To show that $u|_{y_2=0}$, we write

$$|u(y) - u_0(y_1)|^2 = \left| \int_{\mathbb{R}} G(t, y_2)\, (u_0(y_1 - t) - u_0(y_1))\ dt \right|^2 \le C \int_{\mathbb{R}} |G(t, x_2)|\, |u_0(x_1 - t) - u_0(t)|^2 dt$$

using Cauchy-Schwartz inequality and homogeneity properties of the kernel. Integrating with respect to y_1, we obtain

$$\int_0^1 |u(y) - u_0(y_1)|^2 dy_1 \;\le\; C \left(\int_{\mathbb{R}} |G(t, y_2)|\, \|\tau_{-t} u_0 - u_0\|^2_{L^2(0,1)}\, dt \right)$$
$$\le\; C \left(\int_{|t| \le \delta} |G(t, y_2)| \|\tau_{-t} u_0 - u_0\|^2_{L^2(0,1)}\, dt \;+\; 2 \int_{|t| \ge \delta} |G(t, y_2)|\, dt\, \|u_0\|_{L^2_{uloc}} \right)$$

The first term at the r.h.s goes to zero as $\delta \to 0$, whereas the second term goes to zero as $y_2 \to 0$. So, on one hand $\displaystyle\int_0^1 |u(y) - u_0(y_1)|^2 dy_1 \to 0$ as $y_2 \to 0$. On the other hand, a direct computation yields

$$\int_0^1 |u(y) - u(y_1, 0)|^2\, dy_1 \;\; \le\; C(a)\, y_2\, \|\partial_2 u\|_{L^2((0,1) \times (0,a))}.$$

Hence, $u|_{\{y_2 = 0\}} = u_0$. It ends the proof of the proposition.

2.2 Dirichlet-to-Neumann operator

Thanks to these preliminary results, we can now introduce a new system in $\Omega^{bl,-}$, equivalent to (BL). The keypoint is to determine a boundary condition at $y_2 = 0$ that the solution v of (BL) should satisfy. Briefly, the idea is that (v, q) is the solution of (2.1) with $u_0 := v|_{\{y_2=0\}}$. Therefore, its expression is given by (2.8). This expression allows to express the stress tensor $\partial_n v - q\, n = -\partial_2 v + q e_2$ at the boundary $\{y_2 = 0\}$ in terms of v at $\{y_2 = 0\}$. Formally,

$$(\partial_n v - q\, n)|_{\{y_2=0\}} \;=\; DN(v|_{\{y_2=0\}})$$

for some Dirichlet-to-Neumann operator DN that we will now properly define.

Usually, such Dirichlet-to-Neumann operators are easier to define in Fourier space, typically over $H^s(\mathbb{R})$. As our boundary data u_0 belongs to $H^{1/2}_{uloc}(\mathbb{R})$, we must extend the definition usually given on $H^{1/2}(\mathbb{R})$. As in the previous pragraph, the idea is to decompose

$$u_0 \;=\; \chi u_0 \;+\; (1 - \chi u_0).$$

The action of DN on χu_0 will be defined in Fourier space, wehereas its action on $(1 - \chi u_0)$ will be defined in the physical space through a singular integral, for which the singularity will not be annoying. Precisely, we define

$$DN : H^{1/2}_{uloc}(\mathbb{R}) \;\mapsto\; \mathcal{D}'(\mathbb{R})$$

in the following way. Let $u_0 \in H^{1/2}_{uloc}(\mathbb{R}),\;\; \varphi \in C^\infty_c(\mathbb{R})$. Let $\chi \in C^\infty_c(\mathbb{R})$, such that $\chi = 1$ on an open set $\mathcal{O}_\chi$ containing the support of φ. We define

$$\begin{aligned} < DN(u_0), \varphi >_{<\mathcal{D}'(\mathbb{R}),\mathcal{D}(\mathbb{R})>} \;:=\; & < \widetilde{DN}(\chi u_0),\; \varphi >_{<H^{-1/2}(\mathbb{R}),H^{1/2}(\mathbb{R})>} \\ & + \int_{\mathbb{R}} K * ((1-\chi)u_0) \cdot \varphi, \end{aligned}$$

where

- $\widetilde{DN} : H^{1/2}(\mathbb{R}) \mapsto H^{-1/2}(\mathbb{R})$ is the “standard” Dirichlet-to-Neumann operator, defined in Fourier space by

$$\mathcal{F}\widetilde{DN}(u)(\xi) \;=\; -|\xi|\mathcal{F}u(\xi) - \begin{pmatrix} |\xi| \\ -i\xi \end{pmatrix} (\mathcal{F}u_1 + i \,\mathrm{sign}(\xi)\mathcal{F}u_2). \tag{2.11}$$

  Remark that
$$\mathcal{F}\widetilde{DN}(\chi u_0) \;=\; -\partial_2 \widehat{U} + \widehat{P} e_2$$
  where $\widehat{U}$,$\widehat{P}$ are defined in (2.9)-(2.10).

- The kernel K is given by

$$\forall t \neq 0, \quad K(t) \;:=\; -\partial_2 G(t,0) + e_2 \otimes \nabla g(t,0) = \begin{pmatrix} -\frac{2}{\pi t^2} & 0 \\ 0 & -\frac{2}{\pi t^2} \end{pmatrix}.$$

  Note that, by the singularity of G and g, this kernel is singular at $t = 0$. However, due to the properties of φ and χ, the integral

$$\int_{\mathbb{R}} K * ((1-\chi)u_0) \cdot \varphi \;=\; \int_{\mathbb{R}} \int_{\mathbb{R}} K(t)((1-\chi)(y_1 - t)\, u_0(y_1 - t)\, \varphi(y_1)\, dt dy_1$$

is well-defined. Indeed, similarly to the previous subsection, for (y_1,t) in the support of $(y_1,t)\mapsto(1-\chi)(y_1-t)\,\varphi(y_1)$, we have $|t|\geq\delta>0$, so that the singularity is not a problem. One shows easily

$$\int_{\mathbb{R}}|K*((1-\chi)u_0)\cdot\varphi|\;\leq\;C\,\|\varphi\|_{L^2}\,\|u_0\|_{L^2_{uloc}}\int_{|t|\geq\delta}\frac{1}{t^2}dt<+\infty \tag{2.12}$$

Remark that

$$\int_{\mathbb{R}}K*((1-\chi)u_0)\cdot\varphi=\int_{\mathbb{R}}(-\partial_2V+Qe_2)|_{\{y_2=0\}}$$

where V,Q were introduced in the previous subsection.

This definition depends *a priori* on the truncation function χ. However, it is intrinsic, as stated in

Lemma 7 *The quantity $<DN(u_0),\,\varphi>$ defined above does not depend on the choice of χ. Moreover, $DN(u_0)$ belongs to $\mathcal{D}'(\mathbb{R})$.*

Proof. Let u_0, φ as above, and χ,χ' two truncation functions as above. One must check that

$$<\widetilde{DN}((\chi-\chi')u_0),\,\varphi>=\int_{\mathbb{R}}K*(\chi-\chi')u_0)\cdot\varphi.$$

Taking the inverse Fourier transform in (2.11), we obtain

$$\widetilde{DN}((\chi-\chi')u_0)\;=\;-\frac{2}{\pi}\partial_1\left(\mathrm{pv}\frac{1}{y_1}*(\chi-\chi')u_0\right)+\partial_1\begin{pmatrix}-(\chi-\chi')u_{0,2}\\(\chi-\chi')u_{0,1}\end{pmatrix}.$$

where pv denotes the principal value. Thus,

$$\begin{aligned}<\widetilde{DN}((\chi-\chi')u_0),\,\varphi>&=<\frac{2}{\pi}\left(\mathrm{pv}\frac{1}{y_1}*(\chi-\chi')u_0\right)+(\chi-\chi')\begin{pmatrix}u_{0,2}\\-u_{0,1}\end{pmatrix},\,\varphi>\\&=<\frac{2}{\pi}\left(\mathrm{pv}\frac{1}{y_1}*(\chi-\chi')u_0\right),\,\varphi>.\end{aligned}$$

The second term cancels because the support of $\chi-\chi'$ is disjoint from the support of φ. By definition of the principal value,

$$<\widetilde{DN}((\chi-\chi')u_0),\,\varphi>=\frac{2}{\pi}\int_{\mathbb{R}}\lim_{\varepsilon\to0}\int_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]}\frac{1}{t}(\chi-\chi'(x_1-t)u_0(x_1-t)dt\,\partial_1\varphi(y_1)dy_1$$

By the assumption on the support of $\chi-\chi'$, one can replace for ε small enough the integral over $\mathbb{R}\setminus[-\varepsilon,\varepsilon]$ by the integral over $\mathbb{R}$:

$$<\widetilde{DN}((\chi-\chi')u_0),\,\varphi>=\frac{2}{\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{1}{t}(\chi-\chi'(x_1-t)u_0(x_1-t)dt\,\partial_1\varphi(y_1)dy_1$$

Then, changing t for x_1-t and integrating by parts, we end up with

$$\begin{aligned}<\widetilde{DN}((\chi-\chi')u_0),\,\varphi>&=-\frac{2}{\pi}\int_{\mathbb{R}}\frac{1}{(x_1-t)^2}(\chi-\chi')(t)\,u_0(t)\,\varphi(y_1)\,dy_1\\&=\int_{\mathbb{R}}K*(\chi-\chi')u_0\cdot\varphi\end{aligned}$$

which shows that DN is well-defined. The fact that $DN(u_0)$ belongs to $\mathcal{D}'(\mathbb{R})$ follows from the fact that $\widetilde{DN}(\chi u_0) \in H^{-1/2}(\mathbb{R})$ and from the estimate (2.12). In fact, keeping the same construction, $DN(u_0)$ can be extended to a continuous linear form (that we still denote DN) over the space $H^{1/2}_c(\mathbb{R})$ of $H^{1/2}$ functions with compact support. That is, for all $\varphi \in H^{1/2}(\mathbb{R})$, with support in K,

$$| < DN(u^0), \varphi > | \;\le\; C(K)\, \|u_0\|_{H^{1/2}_{uloc}(\mathbb{R})}\, \|\varphi\|_{H^{1/2}(\mathbb{R})}.$$

Lemma 8 *Let $u_0 \in H^{1/2}_{uloc}(\mathbb{R})$, and (u,p) the solution of (2.1) provided by Proposition 6. For all $\varphi \in C^\infty_c(\overline{\mathbb{R}^2_+})$ with $\nabla \cdot \varphi = 0$,*

$$\int_{\mathbb{R}^2_+} \nabla u \cdot \nabla \varphi \;=< DN(u_0),\, \varphi|_{\{y_2=0\}} > . \tag{2.13}$$

In particular, if u, p are regular enough, $DN(u_0) \;=\; (\partial_n u - p\, n)|_{\{y_2=0\}}$.

Proof: A look at the proof of Proposition 6 shows that the mapping

$$H^{1/2}_{uloc}(\mathbb{R}) \mapsto \left\{ v,\, \sup_k \int_k^{k+1} \int_0^{+\infty} |v|^2 dy_2 dy_1 < +\infty \right\}, \quad u_0 \mapsto \nabla u, \quad u \text{ solution of (2.1)}$$

is continuous. Moreover, a look at the proof of Lemma 7 shows that the r.h.s. in (2.13) is continuous over $H^{1/2}_{uloc}(\mathbb{R})$ as well. Thus, it is enough to prove (2.13) for $u_0 \in C^\infty_b(\mathbb{R})$. By elliptic regularity u, p are then in $C^\infty(\overline{\mathbb{R}^2_+})$.

We take again a smooth function χ, compactly supported, with $\chi = 1$ in an open set $\mathcal{O}_\chi$ containing the support of φ. Let $U_0 := \chi u_0$, *resp.* $V_0 := (1-\chi) u_0$, and (U,P), *resp.* (V,Q) the corresponding solutions of (2.1). It is enough to show that

$$\int \nabla U \cdot \nabla \varphi \;=< \widetilde{DN}(U_0),\, \varphi >, \qquad \int \nabla V \cdot \nabla \varphi \;=\; \int K * V_0 \cdot \varphi.$$

It is a straightforward computation, that is left to the reader.

We are now ready to provide an equivalent formulation for (BL).

Proposition 9 *Let (v,q) a solution of (BL) in $H^1_{loc}(\overline{\Omega_{bl}})$ with $\sup_k \int_{\Omega^{bl}_{k,k+1}} |\nabla v|^2 < +\infty$. Then it satisfies*

$$\left\{ \begin{aligned} -\Delta v + \nabla q = 0,\; y \in \Omega^{bl,-}, \\ \nabla \cdot v = 0,\; y \in \Omega^{bl,-}, \\ v(y_1, \omega(y_1)) \;=\; -(\omega(y_1), 0), \\ (-\partial_2 v + q e_2)|_{\{y_2=0\}} \;=\; DN(v|_{x_2=0}) \end{aligned} \right. \tag{2.14}$$

Moreover, $v = \displaystyle\int_{\mathbb{R}} G(y_1 - t, y_2)\, v|_{\{y_2=0\}}(t)\, dt, \;\; y_2 > 0.$

Conversely, let v^- in $H^1_{uloc}(\Omega^{bl,-})$ solution of (2.14). Then, the field v defined by

$$v \;:=\; v^- \;\textit{in}\; \Omega^{bl,-}, \quad v \;:=\; \int_{\mathbb{R}} G(y_1 - t, y_2)\, v^-(t,0)\, dt \;\; \textit{for}\; y_2 > 0$$

is a solution of (BL) in $H^1_{loc}(\overline{\Omega^{bl}})$ such that $\sup_k \int_{\Omega^{bl}_{k,k+1}} |\nabla v|^2 < +\infty$.

We emphasize that v^- *solves* (2.14) means: v^- solves (2.14c) in the trace sense, and for all $\varphi \in C_c^\infty(\overline{\Omega^{bl,-}})$ such that div $\varphi = 0$, $\varphi|_{\{y_2=\omega(y_1)\}} = 0$,

$$\int_{\Omega^{bl,-}} \nabla v^- \cdot \nabla \varphi \;=\; - < DN(v^-|_{\{y_2=0\}}), \varphi > .$$

Note that $v^- \in H^1_{uloc}(\Omega^{bl,-})$, so that $v^-|_{\{y_2=0\}} \in H^{1/2}_{uloc}(\mathbb{R})$ and so $DN(v^-|_{y_2=0})$ is well-defined. The proof is a straightforward consequence of Proposition 6 and Lemma 8, and we do not give further details for the sake of brevity.

2.3 Well-posedness of the equivalent problem

By the previous proposition, well-posedness of (BL) is the same as well-posedness of (2.14), in the channel $\Omega^{bl,-}$. In order to get an homogenenous Dirichlet condition at the lower boundary, we introduce the new unknowns $u := v + (y_2, 0)$, $p := q$. They satisfy formally

$$\begin{cases} -\Delta u + \nabla p = 0, \; y \in \Omega^{bl,-}, \\ \nabla \cdot u = 0, \; y \in \Omega^{bl,-}, \\ u(y_1, \omega(y_1)) = 0, \\ (-\partial_2 u + p e_2)|_{\{y_2=0\}} \;=\; DN(u|_{\{y_2=0\}}) + (1,0) \end{cases} \tag{BL$^-$}$$

Theorem 3 will be a consequence of

Proposition 10 *System* (BL$^-$) *has a unique solution* $u \in H^1_{uloc}(\Omega^{bl,-})$.

Proof. In order to lighten notations, we will write Ω instead of $\Omega^{bl,-}$ in the whole proof. As the smooth part $\Omega = \mathbb{R} \times (0,1)$ of the rough channel Ω^ε is not involved in the proof, there is no risk of confusion. We will use notations

$$\Omega_{k,l} \;=\; \Omega \cap \{k < |y_1| < l\}, \quad \Sigma_{k,l} \;=\; \{y_2 = 0, \; k < |y_1| < l\}, \quad \Omega_k \;:=\; \Omega_{-k,k}, \quad \Sigma_k \;:=\; \Sigma_{-k,k}.$$

Existence

We will construct a solution u as the limit of approximations u^n, solving the following problem: $u^n \in V$,

$$\int \nabla u^n \cdot \nabla \varphi \;=\; - < DN(u^n|_{\{y_2=0\}}), \varphi|_{\{y_2=0\}} > - \int_{\{y_2=0\}} \varphi_1, \quad \forall \varphi \in V \tag{2.15}$$

where

$$V \;=\; \left\{ \varphi \in H^1(\Omega), \text{ div } \varphi = 0, \; \varphi = 0 \text{ on } \Omega \setminus \Omega_n, \; \varphi|_{\{y_2=\omega(y_1)\}} = 0 \right\}, \quad \|\varphi\|^2_V = \int_\Omega |\nabla \varphi|^2.$$

As mentioned at the end of the proof of Lemma 7, $< DN(u^n|_{\{y_2=0\}}), \varphi|_{\{y_2=0\}} >$ is well-defined for all $\varphi \in H^1(\Omega)$ with compact support, especially for $\varphi \in V$. The variational formulation (2.15) is well-posed in V by Lax-Milgram Lemma. By little adaptation of Proposition 9, the fonction

$$v^n := u^n - y_2 \text{ in } \Omega, \quad v^n := G(\cdot, y_2) * u^n|(\cdot, 0) \text{ in } \mathbb{R} \times \mathbb{R}_+$$

satisfies

$$\begin{cases} -\Delta v^n + \nabla q^n = 0, & y_1 \in (-n,n),\ y_2 > \omega(y_1) \\ \nabla \cdot v^n = 0, & y_1 \in (-n,n),\ y_2 > \omega(y_1) \\ v^n(x) = -(\omega(y_1), 0), & y_1 \in (-n,n),\ y_2 = \omega(y_1). \end{cases}$$

for some pressure q^n. Standard elliptic regularity arguments show that v^n, q^n are smooth inside their domain. Back to u^n, we get

$$\begin{cases} -\Delta u^n + \nabla p^n = 0, & y \in \Omega_n, \\ \nabla \cdot u^n = 0, & y \in \Omega_n, \\ u^n(x) = 0, & y \in \partial\Omega_n \setminus \Sigma_n, \\ (-\partial_2 u^n + p^n e_2)|_{\Sigma_n} = -\, DN(u^n|_{\{y_2=0\}}) + (1,0). \end{cases} \tag{2.16}$$

for some pressure p^n, with

$$u^n \in C^\infty(\Omega_n \cup \Sigma_n), \quad p^n \in C^\infty(\Omega_n \cup \Sigma_n).$$

It is easy to deduce from the identity (2.13) that

$$< DN(\varphi|_{\{y_2=0\}}),\, \varphi|_{\{y_2=0\}} > \ > \ 0, \quad \forall \varphi \in V. \tag{2.17}$$

Taking $\varphi = u^n$ in (2.15), this gives

$$\int_\Omega |\nabla u^n|^2 \ \le\ C \int_{\{y_2=0\}} |u^n| \ \le\ C\sqrt{n} \left(\int_{\{y_2=0\}} |u^n|^2 \right)^{1/2} \ \le\ C'\sqrt{n} \left(\int_\Omega |\partial_2 u^n|^2 \right)^{1/2}$$

using successively Cauchy-Schwartz inequality over $\{|y_1| < n\}$, and Poincaré inequality over the whole channel Ω, with $u^n|_{\{y_2=\omega(y_1)\}} = 0$ (therefore, constant C' does not depend on n). We get the global estimate

$$\int_\Omega |\nabla u^n|^2 \ =\ \int_{\Omega_n} |\nabla u^n|^2 \ \le\ C_0\, n \tag{2.18}$$

Of course, this bound explodes as $n \to +\infty$. It is reminiscent of the fact that u should be in $H^1_{uloc}(\mathbb{R})$, therefore of infinite energy. The main point is therefore to obtain a local uniform bound on ∇u^n. This question has been adressed by Solonnikov and Ladyzenskaya [19], when the non-local condition (2.16c) is replaced by a homogeneous Dirichlet condition (and some appropriate forcing is added inside the domain). Their analysis is the starting point of the well-posedness result in Theorem 1. In a very crude way, the idea is to introduce the linear quantity

$$E^n_k \ :=\ \int_{\Omega_k} |\nabla u^n|^2, \quad k \le n.$$

By the global estimate (2.18), $E^n_n \ \le\ C\, n$. Then, one shows by induction on $n-k$ that $E^n_k \le Ck$. Using this inequality for $k=1$ yields $\int_{\Omega_1} |\nabla u^n|^2 \ \le C$. Finally, one uses the same reasoning on the translated channel $\Omega + (k,0)$ to get a uniform local bound. Typically, the induction relies on an induction relation between E^n_k and E^n_{k+1}, like

$$E^n_k \ \le\ C \left(E^n_{k+1} - E^n_k + k + 1 \right).$$

Such relation is obtained using a truncation over Ω_k and energy estimates. We stress that the fact that Poincaré's inequality applies in a channel is a crucial ingredient in this reasoning. Indeed, the truncation over Ω_k involves terms containing u^n, whereas the Laplacian gives a control of ∇u^n. This is why we wanted to replace the original system in Ω^{bl} by a system in the channel $\Omega(= \Omega^{bl,-})$. The problem is that the new Dirichlet-to-Neumann operator is not local, so that the induction relation will be harder to derive, and more elaborate. To lighten notations, we shall denote E_k instead of E_k^n. We shall prove that there exists $\eta > 0$ such that, for any $m > 1$

$$E_k \;\le\; C_1 \left(k + 1 + \frac{1}{m^\eta} \sup_{j \ge k+m} (E_{j+1} - E_j) \;+\; m \sup_{k+m \ge j \ge k} (E_{j+1} - E_j) \right). \tag{2.19}$$

Before we establish this inequality, let us indicate how it implies an H^1_{uloc} bound on u^n. More precisely, let us show first by induction on $n-k$ that for m large enough, (2.19) implies

$$E_k \;\le\; C_2 \left(k + 1 + \frac{1}{m^\eta} \sup_{j \ge k+m} (E_{j+1} - E_j) \;+ m^3 \right), \quad \forall k \le n. \tag{2.20}$$

for some C_2 depending on the constants C_0 in (2.18) and C_1 in (2.19). The inequality is of course true when $k = n$, as soon as $C_1 > C_0$. Now, we assume that

$$E_{k'} \;\le\; C_2 \left(k' + 1 + \frac{1}{m^\eta} \sup_{j \ge k'+m} (E_{j+1} - E_j) \;+ m^3 \right)$$

holds for all indices $k' = k+1, ..., n$, and try to show it holds for index k. If not, one has

$$E_k \;\ge\; C_2 \left(k + 1 + \frac{1}{m^\eta} \sup_{j \ge k+m} (E_{j+1} - E_j) + m^3 \right). \tag{2.21}$$

Combining these last two inequalities, we have for all $k + m \ge j \ge k$

$$E_{j+1} - E_j \;\le\; E_{j+1} - E_k \;\le\; C_2(m+1).$$

By (2.19), we deduce

$$E_k \;\le\; C_1 \left(k + 1 + \frac{1}{m^\eta} \sup_{j \le k+m} (E_{j+1} - E_j) \;+\; C_2 m(m+1) \right). \tag{2.22}$$

Comparison between (2.21) and (2.22) yields a contradiction if $C_2 > C_1$ and $C_1\, C_2 m(m+1) \le C_2 m^3$, which is satisfied if $C_2 > C_1$ and m large enough. Thus, inequality (2.20) is valid for all $k \le n$. For $k = 1$, we get

$$E_1 \;\le\; C_1 \left(2 + \frac{1}{m^\eta} \sup_{j \in \mathbb{N}} (E_{j+1} - E_j) \;+ m^3 \right).$$

It will be clear from the proof of (2.19) below that it is invariant by a horizontal translation. Thus, previous inequality generalizes to

$$E_{k+1} - E_k \;\le\; C_1 \left(2 + \frac{1}{m^\eta} \sup_{j \in \mathbb{N}} (E_{j+1} - E_j) \;+ m^3 \right)$$

for all k, so that for m large enough, we end up with

$$\sup_k \,(E_{k+1} - E_k) \;\le\; C_1 \frac{(2+m^3)}{1-m^{-\eta}} = C < +\infty$$

which is a H^1_{uloc} bound on u^n. We can therefore extract a subsequence of u^n that converges weakly to some $u \in H^1_{uloc}(\Omega)$, clearly satisfying (BL$^-$).

It remains to prove (2.19). The case $k = n$ follows again from (2.18). Let $k < n$. We introduce a smooth truncation function $\chi_k = \chi_k(y_1)) \in [0,1]$, compactly supported in $(k-1,\, k+1)$, with $\chi = 1$ on $[-k,k]$, $|\chi_k'| \le 2$. We will note u instead of u^n. One multiplies (2.16a) by $\chi_k u$ and integrate by parts:

$$\int_\Omega \chi_k |\nabla u|^2 \;\le\; \int_\Omega |\nabla \chi_k|\,|u|^2 \;-\; \int DN(u)\cdot \chi_k u|_{\{y_2=0\}} + \int_\Sigma |\chi_k|\,|u_1| + \left| \int_\Omega p \nabla \chi_k \cdot u \right| \;=\; \sum_{i=1}^4 I_i.$$

Note that we use an integral sign instead of a bracket sign for the term containing $DN(u)$. Indeed, u, p are regular over Σ_n, so that $DN(u) = (-\partial_2 u + p e_2)|_{\{y_2=0\}}$ is a real smooth function. We must evaluate the four terms at the r.h.s. Clearly,

$$I_1 \;\le\; 2 \int_{\Omega_{k,k+1}} |u|^2 \;\le\; C \int_{\Omega^{k,k_1}} |\partial_2 u|^2 \;\le\; C(E_{k+1} - E_k),$$

using Poincaré inequality in the channel. Then, by Cauchy-Schwartz inequality

$$I_3 \;\le\; C\,\sqrt{2(k+1)} \left(\int_\Omega |\sqrt{\chi_k} u|^2 \right)^{1/2} \;\le\; C_\nu (k+1) \;+\; \nu \int_\Omega \chi_k |u|^2$$

where ν will be taken small to absorb the second term in the r.h.s.

Let us decompose I_2 as follows, for $m > 1$:

$$\begin{aligned} I_2 \;&=\; -\int DN(\chi_k u)\cdot \chi_k u - \int_\Omega DN((\chi_{k+m} - \chi_k)u)\cdot \chi_k u - \int_\Omega DN((1-\chi_{k+m})u)\cdot \chi_k u \\ &\le \left| \int_\Omega DN((\chi_{k+m} - \chi_k)u)\cdot \chi_k u \right| \;+\; \left| \int_\Omega DN((1-\chi_{k+m})u)\cdot \chi_k u \right| \;=\; J_1 \;+\; J_2 \end{aligned}$$

where we have used (2.17). First term is dominated through

$$\begin{aligned} |J_1| \;&\le\; C\, \|\chi_k u\|_{H^{1/2}(\Sigma)}\, \|(\chi_{k+m} - \chi_k)u\|_{H^{1/2}(\Sigma)} \\ &\le\; C\, \|\chi_k u\|_{H^1(\Omega)}\, \|(\chi_{k+m} - \chi_k)u\|_{H^1(\Omega)} \;\le\; C_\nu\, m \sup_{k+m \ge j \ge k} (E_{j+1} - E_j) \;+\; \nu\, E_k, \end{aligned}$$

where again ν will be taken small. To bound J_2,we use the convolution formula with the kernel K. It yields:

$$
\begin{aligned}
|J_2| \;\le\; & C \int_{\Sigma_{k+1}} \int_{\Sigma \setminus \Sigma_{k+m}} \frac{1}{|t-y_1|^2} |u(t)|\, dt\, |\chi_k(y_1) u(y_1)|\, dy_1, \\
\le\; & C \left(\int_{\Sigma} \chi_k(y_1) |u(y_1)|^2 dy_1 \right)^{1/2} \\
& \qquad \left(\int_{\Sigma_{k+1}} dy_1 \int_{\Sigma \setminus \Sigma_{k+m}} \frac{1}{|t-y_1|^{2+2\eta}} dt \int_{\Sigma \setminus \Sigma_{k+m}} \frac{1}{|t-y_1|^{2-2\eta}} |u(t)|^2 dt \right)^{1/2} \\
\le & C E_k^{1/2}\, m^{-\eta/2} \left(\int_{\Sigma_{k+1}} \int_{\Sigma \setminus \Sigma_{k+m}} \frac{1}{|t-y_1|^{2+\eta}} dt dy_1 \sup_{j \ge k+m} (E_{j+1} - E_j) \sum_{j \ge k+m} (j-k)^{2\eta - 2} \right)^{1/2}
\end{aligned}
$$

for any $0 < \eta < 1/2$. A direct computation leads to

$$
\int_{\Sigma_{k+1}} \int_{\Sigma \setminus \Sigma_{k+m}} \frac{1}{|t-y_1|^{2+\eta}} dt dy_1 \;\le\; C' \int_{\mathbb{R} \setminus [0,1]} \frac{1}{x^{1+\eta}} dx < +\infty.
$$

Thus,

$$
\begin{aligned}
|J_2| \;\le\; & \frac{C E_k^{1/2}}{m^{\eta/2}} \left(\sup_{j \ge k+m} \sup_{j \ge k+m} (E_{j+1} - E_j) \right)^{1/2} \\
\le\; & \frac{C_\nu}{m^\eta} \sup_{j \ge k+m} (E_{j+1} - E_j) \;+\; \nu E_k
\end{aligned}
$$

We end up with

$$
I_2 \;\le\; \nu E_k \;+\; \frac{C_\nu}{m^\eta} \sup_{j \ge k+m} (E_{j+1} - E_j) \;+\; C_\nu\, m \sup_{k+m \ge j \ge k} (E_{j+1} - E_j)
$$

The integral I_4 coming from the pressure term is

$$
I_4 = \int_\Omega p\, \chi_k' u_1 \;=\; \int_\Omega p(y_1, 0)\, (\chi_k' u_1)(y)\, dy \;+\; \int_\Omega \int_0^{y_2} \partial_2 p(y_1, t) dt\, (\chi_k' u_1)(y) dy \;=\; H_1 \;+\; H_2.
$$

One writes, using the boundary condition at $\{y_2 = 0\}$

$$
\begin{aligned}
H_1 \;=\; & \int_\Omega \partial_2 u_2(y_1, 0)\;\; (\chi_k' u_1)(y)\, dy \quad + \int_\Omega DN(u|_{\{y_2=0\}}) \cdot e_2\; (\chi_k' u_1)(y)\, dy \\
=\; & \int_\Omega -\partial_1 u_1(y_1, 0)\;\; (\chi_k' u_1)(y)\, dy \;+\; \int_\Omega DN(u|_{\{y_2=0\}}) \cdot e_2\;\; (\chi_k' u_1)(y)\, dy
\end{aligned}
$$

where the last line comes from the divergence-free condition. After an integration by parts, the first term is easily bounded by $C \, \|u_1\|_{L^2(\Sigma_{k,k+1})} \|\nabla u\|_{L^2(\Omega_{k,k+1})}$, thus by $C\,(E_{k+1} - E_k)$. The second term is treated similarly to I_2, as $\int (\chi_k' u_1)(y_1, y_2) dy_2$ substitutes to $\chi_k u|_{y_2=0}$. We get

$$
\begin{aligned}
| \int_\Omega DN(u|_{\{y_2=0\}}) \cdot e_2\, (\chi_k' u_1)(y)\, dy| \;\le\; & C (E_{k+1} - E_k)^{1/2} \\
& \left(\|\nabla u\|_{L^2(\Omega_k)} + \frac{1}{m^{\eta/2}} \sup_{j \ge k+m} (E_{j+1} - E_j) \;+\; m \sup_{k+m \ge j \ge k} (E_{j+1} - E_j) \right)^{1/2}.
\end{aligned}
$$

This yields

$$|H_1| \;\le\; \nu E_k \;+\; (C_\nu + m) \sup_{k+m \ge j \ge k} (E_{j+1} - E_j) \;+\; \frac{C}{m^\eta} \sup_{j \ge k+m} (E_{j+1} - E_j)$$

Using the Stokes equation, we then get

$$\begin{aligned} H_2 \;&=\; \int_{\Omega_{k,k+1}} \int_0^{y_2} (\partial_1^2 + \partial_2^2) u_2(y_1, t)\, dt \;\; (\chi_k' u_1)(y)\, dy \\ &=\; \int_{\Omega_{k,k+1}} \int_0^{y_2} -\partial_1 u_2(y_1, t)\, dt \;\; \partial_1 (\chi_k' u_1)(y)\, dy \\ &+\; \int_{\Omega_{k,k+1}} \partial_2 u_2(y) \;\; (\chi_k' u_1)(y)\, dy \;-\; \int_{\Omega_{k,k+1}} \partial_2 u_2(y_1, 0) \;\; (\chi_k' u_1)(y)\, dy \end{aligned}$$

The first two terms are easily bounded by $C(E_{k+1} - E_k)$. For the last one, one can again replace $\partial_2 u_2$ by $-\partial_1 u_1$ and integrate by parts with respect to y_1. Finally, $H_2 \le C(E_{k+1} - E_k)$ and

$$|I_4| \;\le\; \nu E_k \;+\; \frac{C_\nu}{m^\eta} \sup_{j \ge k+m} (E_{j+1} - E_j) \;+\; C_\nu\, m \sup_{k+m \ge j \ge k} (E_{j+1} - E_j)$$

Gathering the bounds on the I_j's and taking ν small enough lead to the induction relation (2.19). This ends the existence part.

Uniqueness

Let u be the difference of two solutions of (BL$^-$) in $H^1_{uloc}(\Omega)$. It satisfies

$$\begin{cases} -\Delta u + \nabla p = 0, \; y \in \Omega^{bl,-}, \\ \nabla \cdot u = 0, \; y \in \Omega^{bl,-}, \\ u(y_1, \omega(y_1)) = 0, \\ (-\partial_2 u + p e_2)|_{\{y_2 = 0\}} \;=\; DN(u|_{\{y_2=0\}}) \end{cases}$$

Applying the same estimates as in the "existence part", the induction relation (2.19) is modified into:

$$E_k \;\le\; C \left(\frac{1}{m^\eta} \sup_{j \ge k+m} (E_{j+1} - E_j) \;+\; m \sup_{k+m \le j \le k} (E_{j+1} - E_j) \right).$$

The difference with (2.19) is of course the lack of a $(k+1)$ term, because of the homogeneous condition on the stress tensor at $\{y_2 = 0\}$. Using the H^1_{uloc} bound on u, we get

$$E_k \;\le\; C'$$

uniformly in k, which means that the difference u between the two solutions belongs to $H^1(\mathbb{R})$. We can then multiply the Stokes equation on u by u itself and integrate by parts. By positivity of the DN operator ($= \widetilde{DN}$ in this context), the energy estimate yields $\int_\Omega |\nabla u|^2 = 0$, so that $u = 0$. This ends the proof of Proposition 10.

3 Asymptotic behaviour of (BL)

By theorem 3, the well-posedness of the boundary layer system is ensured for all Lipschitz bounded boundary $\{y_2 = \omega(y_1)\}$. It opens the way to the formal scenario explained in the introduction: if v converges to a constant field $v^\infty = (\alpha, 0)$ as $y_2 \to +\infty$, then one can believe that a slip condition of Navier type is the best choice for a wall law. *Unfortunately, convergence of v far from the boundary is unlikely to be true for all roughness profiles ω.* This claim is suggested by what happens for the following similar (but simpler) problem

$$\begin{cases} \Delta v = 0, \; y \in \mathbb{R}^2_+, \\ \quad v = v_0, \; y_2 = 0, \end{cases} \tag{3.1}$$

where $v_0 \in L^\infty(\mathbb{R})$. One can check by standard scalar arguments that this Dirichlet problem has a unique solution $v \in L^\infty(\mathbb{R})$. System (3.1) is a baby version of (BL), oscillations of the boundary being replaced by oscillations of the boundary data. Nevertheless, it shares common features with (BL). For instance, if v_0 is $L-$periodic and smooth, a simple Fourier analysis shows that

$$v \to v^\infty \; := \; \frac{1}{L}\int_0^L v_0(t)\,dt, \qquad \text{as } y_2 \to +\infty$$

and that the convergence is exponential. It is then natural to ask if this convergence property is true in general. However,

Proposition 11 *There exists $v_0 \in L^\infty(\mathbb{R})$ such that $v(0, y_2)$ does not have a limit when $y_2 \to +\infty$.*

When transposed to the original system (BL), this suggest that there may be some ω for which v does not converge transversally to the boundary. For the corresponding rough channels, the Dirichlet boundary condition can certainly not be improved.

Proof of the proposition. We take

$$v_0 \; = \; (-1)^k \text{ on } (x^k, x^{k+1/2}), \quad k \in \mathbb{N}, \quad x > 1 \text{ to be fixed later}, \quad v_0 = 0 \text{ elsewhere.}$$

The proof of non-convergence of v_0 relies on the expression of the solution v of (3.1) in terms of the Poisson kernel:

$$v(0, y_2) \; = \; \int_{\mathbb{R}} \frac{y_2}{\pi(t^2 + y_2^2)}\, v_0(t)dt. \tag{3.2}$$

With $y_2 = x^n$, this yields

$$\begin{aligned} v(0, x^n) \; &= \; \frac{1}{\pi}\sum_{k=0}^{+\infty}(-1)^k \left(\arctan(x^{k+1/2-n}) - \arctan(x^{k-n})\right) \\ &= \; \frac{(-1)^n}{\pi} \sum_{j=-n}^{+\infty} (-1)^{|j|} \left(\arctan(x^{j+1/2}) - \arctan(x^j)\right) = \frac{(-1)^n}{\pi} \; I_n + \; o(1), \end{aligned}$$

where $I_n := \sum_{j=-n}^{n} (-1)^{|j|} \Big(\arctan(x^{j+1/2}) - \arctan(x^j)\Big)$. We will show that I_n has a non-zero limit for x large enough, from where $v(0, x^n)$ will have no limit as $n \to +\infty$.

$$
\begin{aligned}
I_n \;&=\; \sum_{j=0}^{n}(-1)^j \Big(\arctan(x^{j+1/2}) - \arctan(x^j)\Big) \;+\; \sum_{j=1}^{n}(-1)^j \Big(\arctan(x^{-j+1/2}) - \arctan(x^{-j})\Big) \\
&=\; \sum_{j=0}^{n}(-1)^j \Big(\arctan(x^{-j}) - \arctan(x^{-j-1/2})\Big) \;+\; \sum_{j=1}^{n}(-1)^j \Big(\arctan(x^{-j+1/2}) - \arctan(x^{-j})\Big)
\end{aligned}
$$

using that $\arctan(x) + \arctan(1/x) = \pi/2$ for $x > 0$. Note that the terms $\pm \arctan x^{-j}$, $j = 1, ..., n$ cancel. The change of index $n := n-1$ in the second sum yields

$$
\begin{aligned}
I_n \;&=\; \frac{\pi}{4} \;-\; \sum_{j=0}^{n}(-1)^j \arctan(x^{-j-1/2}) \;-\; \sum_{j=0}^{n-1}(-1)^j \arctan(x^{-j-1/2}) \\
\to I \;:=&\; \frac{\pi}{4} \;-\; 2\sum_{j=0}^{+\infty}(-1)^j \arctan(x^{-j-1/2}), \qquad \text{as } n \to +\infty.
\end{aligned}
$$

The right term in the limit is an alternating series. Therefore,

$$
\frac{\pi}{4} \;-\; 2\,\arctan(x^{-1/2}) \;\; \le I \;\le \frac{\pi}{4} \;-\; 2\arctan(x^{-1/2}) \;+ 2\arctan(x^{-3/2})
$$

which is close to $\pi/4$ for any x large enough. This ends the proof.

The "input" v_0 considered in the above proof, built as a sequence of ± 1's, stresses the *analogy between the problem of the asymptotic behaviour and the problem of coin tossing.* Indeed, in this case, formula (3.2) can be seen as an averaging of sequences of ± 1's, not uniform, but following the distribution $\frac{y_2}{\pi(t^2+y_2^2)}$. As y_2 goes to infinity, this corresponds to long time averaging. With such analogy in mind, we can expect bad data v_0's like in the above proposition to be quite exceptional. For instance, the law of large numbers says that almost surely, a Bernoulli sequence of ± 1's converges to 0.

Indeed, as soon as

$$
\frac{1}{L}\int_0^L v_0(y_1 - t)\, dt \;\to\; v^\infty, \quad L \to \pm\infty \tag{3.3}
$$

the solution v of (3.1) satisfies: $v(y_1, y_2) \to v^\infty$, as $y_2 \to +\infty$. This can be deduced from a simple integration by parts in (3.2), writing

$$
\begin{aligned}
v(y_1, y_2) - v^\infty \;&=\; -\frac{1}{\pi}\int_{\mathbb{R}} \frac{y_2\, t}{t^2 + y_2^2}\,\frac{1}{t}\left(\int_0^t v_0(y_1 - s)ds - v^\infty\right) dt \\
&=\; -\frac{1}{\pi}\int_{\{|t|\ge n\}} \frac{y_2\, t}{t^2 + y_2^2}\,\frac{1}{t}\left(\int_0^t v_0(y_1 - s)ds - v^\infty\right) dt \\
&\quad +\; \frac{1}{\pi}\int_{\{|t|\le n\}} \frac{y_2\, t}{t^2 + y_2^2}\,\frac{1}{t}\left(\int_0^t v_0(y_1 - s)ds - v^\infty\right) dt
\end{aligned}
$$

By (3.3), the first term goes to 0 as $n \to +\infty$, uniformly in y_2. Then, for all n, the second term goes to 0 as $y_2 \to +\infty$.

The convergence given in (3.3), especially when v^∞ is independent of y_1 is connected to the ergodicity properties of v_0. Although the original problem (BL) is much more difficult than (3.1), because its dependence with respect to the "input" ω is non linear, this argument suggests that the derivation of the Navier wall law could be made rigorous in the settings where the roughness profile ω satisfies some ergodic theorem. This is coherent with the analysis led in [5], that justifies a Navier condition in the random stationary setting.

4 Navier wall law for almost periodic roughness

From the considerations of the previous sections, one can not expect the Navier wall law to be good for all boundaries, but it may be good for boundaries satisfying ergodicity properties. Besides the random framework considered in [5], it is therefore natural to consider an almost periodic framework. This section is devoted to the proof of Theorem 4, that is justification of a slip boundary condition for $\omega \in AP(\mathbb{R})$, where $AP(\mathbb{R})$ is the set of almost periodic functions defined in the introduction. Again, the key point will be the analysis of the boundary layer system (BL), and more precisely the convergence properties of the solution transversally to the boundary. The scheme of the proof is as follows:

1. We first study the case of quasiperiodic roughness, which includes the case of trigonometric polynomials. A keypoint is to show that the solution $v(y)$ of (BL) is quasiperiodic in y_1 for all $y_2 \geq 0$, that is the quasiperiodicity of the boundary is conveyed to the solution itself. Therefore, we solve (BL) in a (smaller) quasiperiodic setting. Afterwards, we couple the ergodicity of $y_1 \mapsto v(y_1, 0)$ to an integral formula of type (2.8), with $u = v$, and $u_0 = v(\cdot, 0)$. We show in this way that v converges to a constant field $v^\infty = (\alpha, 0)$.

2. By the density of trigonometric polynomials in $AP(\mathbb{R})$, and the stability estimates of section 2, we can go from the quasiperiodic setting to the almost periodic setting. The justification of Navier condition follows from an expansion of the real solution based on the boundary layer analysis.

4.1 The quasiperiodic case

We consider here the case

$$\omega(y_1) \;=\; F(\lambda y_1), \quad F = F(\theta) \in \mathcal{C}^\infty(\mathbb{T}^d), \quad \lambda \in \mathbb{R}^d \tag{4.1}$$

where $d \geq 1$. We will show that v is quasiperiodic in y_1. Therefore we reformulate (BL). We replace v by $v' \;:=\; v + (\delta(y_2)\, y_2, 0)$ for some smooth truncation function δ with $\delta = 1$ for $y_2 \leq 0$ and $\delta = 0$ for $y_2 \geq 1$. We get

$$\left\{ \begin{aligned} -\Delta v' + \nabla q' = f', &\quad y \in \Omega^{bl}, \\ \nabla \cdot v' = 0, &\quad y \in \Omega^{bl}, \\ v' = 0, &\quad y \in \partial\Omega^{bl}. \end{aligned} \right.$$

for $f'(y) = (f_1'(y_2), 0)$ compactly supported in $\{y_2 \leq 1\}$. We introduce

$$\phi(y_1, y_2) \;=\; (y_1, y_2 - \chi(y_2)\omega(y_1))$$

for some smooth truncation function

$$\chi = 1 \text{ for } y_2 \le -\|\omega\|_{L^\infty}, \quad \chi = 0 \text{ for } y_2 \ge 1, \quad \text{with } |\chi'| \le \left(\frac{1}{2} + \|\omega\|_{L^\infty}\right).$$

This defines a diffeomorphism ϕ from Ω^{bl} to $\mathbb{R}^2_+$, such that $\phi = Id$ for $y_2 \ge 1$. Then, we introduce the new functions

$$w(\phi(y)) = v'(y), \quad r(\phi(y)) = q'(y), \quad g(\phi(y)) = f(y),$$

that satisfy

$$\begin{cases} \nabla \cdot A\nabla w + B\,\nabla r = g, & z \in \mathbb{R}^2_+, \\ \nabla \cdot (B^t\, w) = 0, & z \in \mathbb{R}^2_+, \\ w = 0, & z_2 = 0. \end{cases} \tag{4.2}$$

where

$$A := \frac{(\nabla\phi)^t\,\nabla\phi}{\det|\nabla\phi|} = \mathcal{A}(\lambda z_1, z_2), \quad B = \frac{\nabla\phi}{\det|\nabla\phi|} = \mathcal{B}(\lambda z_1, z_2), \quad g = G(\lambda z_1, z_2),$$

for smooth $\mathcal{A} = \mathcal{A}(\theta, t)$, $\mathcal{B} = \mathcal{B}(\theta, t)$, $G = G(\theta, t)$, $\theta \in \mathbb{T}^d, t > 0$. We quote that $\mathcal{A} = \mathcal{B} = Id$ and that $G = 0$ for t large enough. Quasiperiodicity in y_1 for v is equivalent to quasiperiodicity in z_1 for w. In other words, we look for $w = W(\lambda z_1, z_2)$, $W = W(\theta, t)$, $\theta \in \mathbb{T}^d$. Hence, it is natural to solve directly the enlarged system

$$\begin{cases} -\begin{pmatrix} \lambda\cdot\partial_\theta \\ \partial_t \end{pmatrix} \cdot \mathcal{A} \begin{pmatrix} \lambda\cdot\partial_\theta \\ \partial_t \end{pmatrix} W + \mathcal{B} \begin{pmatrix} \lambda\cdot\partial_\theta \\ \partial_t \end{pmatrix} R = G, & \theta \in \mathbb{T}^d,\ t > 0, \\ \begin{pmatrix} \lambda\cdot\partial_\theta \\ \partial_t \end{pmatrix} \cdot \mathcal{B}^t\, W = 0, & \theta \in \mathbb{T}^d,\ t > 0, \\ W(\theta, t) = 0, & t = 0. \end{cases} \tag{4.3}$$

Proposition 12 *System* (4.3) *has a unique smooth solution W satisfying*

$$\int_0^{+\infty} \int_{\mathbb{T}^d} |\partial_\theta^\gamma\, \partial_t^k (\lambda \cdot \partial_\theta) W|^2 + |\partial_\theta^\gamma\, \partial_t^k \partial_t W|^2 \, d\theta\, dt < +\infty, \quad \forall \gamma, k. \tag{4.4}$$

Here, W is a solution means that for all smooth $\varphi \in C^\infty(\mathbb{T}^d \times \mathbb{R}_+)$ satisfying an estimate of type (4.6), and such that $\begin{pmatrix} \lambda\cdot\partial_\theta \\ \partial_t \end{pmatrix} \cdot \mathcal{B}^t\, \varphi = 0$,

$$\int_{\mathbb{T}^d \times \mathbb{R}_+} \mathcal{A} \begin{pmatrix} \lambda\cdot\partial_\theta \\ \partial_t \end{pmatrix} W \cdot \begin{pmatrix} \lambda\cdot\partial_\theta \\ \partial_t \end{pmatrix} \varphi = \int_{\mathbb{T}^d \times \mathbb{R}_+} G \cdot \varphi. \tag{4.5}$$

Proof. The main difficulty is that this system is a degenerate elliptic system, as the $\lambda \cdot \partial_\theta$ derivative does not allow a control on all tangential derivatives. The study of a similar quasiperiodic system has been carried out in the recent paper [14], that deals with homogenization of elliptic operators in polygonal domains. We follow here the same scheme of proof, accounting for the additional difficulties due to the pressure term. To lighten notations, we will denote $D := \begin{pmatrix} \lambda\cdot\partial_\theta \\ \partial_t \end{pmatrix}$.

A priori estimates

Basic estimate. We just multiply by W and integrate by parts to get

$$\int_0^{+\infty} \int_{\mathbb{T}^d} |DW|^2 \, d\theta \, dt \;=\; \int_0^{+\infty} \int_{\mathbb{T}^d} G \cdot W \, d\theta \, dt$$

As $G = 0$ for $t \geq a$, a large enough, we get by Cauchy-Schwartz and Poincaré inequality

$$\begin{aligned}\int_{\mathbb{T}^d \times \mathbb{R}_+} |DW|^2 \;&\leq\; C(a) \left(\int_{\mathbb{T}^d \times \mathbb{R}_+} |G|^2 \right)^{1/2} \left(\int_{\mathbb{R}^2_+} |\partial_t W|^2 \right)^{1/2} \\ &\leq\; C_\nu \int_{\mathbb{T}^d \times \mathbb{R}_+} |G|^2 \;+\; \nu \int_{\mathbb{T}^d \times \mathbb{R}_+} |\partial_t W|^2\end{aligned}$$

Taking ν small enough, we obtain

$$\int_{\mathbb{R}^2_+} |DW|^2 \;<\; +\infty. \tag{4.6}$$

Higher order tangential estimates. We then need to estimate $\partial_\theta^\gamma DW$, for $\gamma \in \mathbb{N}^d$, $|\gamma| \geq 1$. We focus on the estimates corresponding to $|\gamma| = 1$. The control of higher order tangential derivatives follows from an easy induction on the number of derivatives, relying on these estimates. Let i in $\{1, ..., d\}$. We get

$$\begin{aligned}\int_{\mathbb{T}^d \times \mathbb{R}_+} |\partial_{\theta_i} DW|^2 \;\leq\; &- \int_{\mathbb{T}^d \times \mathbb{R}_+} D \cdot (\partial_{\theta_i} \mathcal{A}) DW \cdot \partial_{\theta_i} W - \int_{\mathbb{T}^d \times \mathbb{R}_+} (\partial_{\theta_i} \mathcal{B}) DR \cdot \partial_{\theta_i} W \\ &- \int_{\mathbb{T}^d \times \mathbb{R}_+} \partial_{\theta_i} R \left(D \cdot \left(\partial_{\theta_i} \mathcal{B}^t \right) W \right) \;+\; \int_{\mathbb{T}^d \times \mathbb{R}_+} \partial_{\theta_i} G \cdot \partial_{\theta_i} W \;:=\; \sum_{j=1}^4 I_j\end{aligned}$$

Clearly,

$$|I_1| \;\leq\; C_\nu \int_{\mathbb{T}^d \times \mathbb{R}_+} |\left(\partial_{\theta_i} \mathcal{A} \right) DW|^2 \;+\; \nu \int_{\mathbb{T}^d \times \mathbb{R}_+} |\partial_{\theta_i} DW|^2,$$

so that by the basic estimate(4.6) $|I_1| \;\leq\; C_\nu \;+\; \nu \int_{\mathbb{T}^d \times \mathbb{R}_+} |\partial_{\theta_i} DW|^2$. Also, we have easily

$$|I_4| \;\leq\; C_\nu \int_{\mathbb{T}^d \times \mathbb{R}_+} |\partial_{\theta_i} G|^2 \;+\; \nu \int_{\mathbb{T}^d \times \mathbb{R}_+} |\partial_{\theta_i} \partial_t W|^2.$$

It remains to handle the pressure terms. For the first term, we replace DR using the equation (4.3), so that

$$\begin{aligned}I_2 \;&=\; - \int_{\mathbb{T}^d \times \mathbb{R}_+} (\partial_{\theta_i} \mathcal{B}) \mathcal{B}^{-1} G \cdot \partial_{\theta_i} W \;+\; \int_{\mathbb{T}^d \times \mathbb{R}_+} \mathcal{B}^{-1} (D \cdot \mathcal{A} DW) \cdot \partial_{\theta_i} W \\ &\leq\; C_\nu \int_{\mathbb{T}^d \times \mathbb{R}_+} C_\nu |G|^2 \;+\; \nu \int_{\mathbb{T}^d \times \mathbb{R}_+} |\partial_\theta^i \partial_t W|^2 \;+\; I_2'\end{aligned}$$

where

$$I_2' \;:=\; \int_{\mathbb{T}^d \times \mathbb{R}_+} \mathcal{B}_i (D \cdot \mathcal{A} DW) \cdot \partial_{\theta_i} W, \quad \mathcal{B}_i \;:=\; (\partial_{\theta_i} \mathcal{B}) \mathcal{B}^{-1}.$$

The important remark to bound I_2' is that $\mathcal{B}_i$ has compact support in t, as $\mathcal{B} = Id$ for $t \geq a$, a large enough. This means we will be able to use Poincaré inequality to control $\partial_{\theta_i} W$ by the better quantity $\partial_{\theta_i} DW$. More precisely, a simple integration by parts yields

$$\begin{aligned}
I_2' \, &\leq \, \int_{\mathbb{T}^d \times \mathbb{R}_+} |\mathcal{A} DW| \left(|D\mathcal{B}_i| \, |\partial_{\theta_i} W| \, + \, |\mathcal{B}_i| |D\partial_{\theta_i} W| \right) \\
&\leq \, C \, ||DW||_{L^2(\mathbb{T}^d \times \mathbb{R}_+)} \left(||\partial_{\theta_i} W||_{L^2(\mathbb{T}^d \times [0,a])} \, + \, ||D\partial_{\theta_i} V||_{L^2(\mathbb{T}^d \times \mathbb{R}_+)} \right) \\
&\leq \, C_\nu \, + \, \nu \int_{\mathbb{T}^d \times \mathbb{R}_+} |\partial_{\theta_i} DW|^2
\end{aligned}$$

Finally, a double integration by parts yields

$$I_3 \, = \, \int_{\mathbb{T}^d \times \mathbb{R}_+} -DR \cdot \partial_{\theta_i} \left(D \cdot (\partial_{\theta_i} B^t) W \right)$$

From there, one may proceed as for I_2, replacing DR and using the fact that any derivative of $\mathcal{B}$ is compactly supported. We end up with

$$I_3 \, \leq \, C_\nu \, + \, \nu \int_{\mathbb{T}^d \times \mathbb{R}_+} |\partial_{\theta_i} DW|^2$$

Combining all these inequalities, with ν small enough, we end up with $\|\partial_{\theta_i} DW\|_{L^2} < +\infty$.

Derivatives in t. Let $\mathcal{R}$ the rotation that maps $\frac{\lambda}{|\lambda|}$ on the vector $e_1 = (1, 0, ..., 0)$. Considering θ as an element of $\mathbb{R}^d$ instead of $\mathbb{T}^d$, the variables $(\theta', t) = (\mathcal{R}\theta, t)$ define another coordinates system of $\mathbb{R}^d \times \mathbb{R}_+$, and $\lambda \cdot \partial_\theta = |\lambda| \partial_{\theta_1'}$. In the neighborhood of any point of $\mathbb{R}^d \times \mathbb{R}^+$, system (4.3) can be seen as a strongly elliptic system in θ_1', t, depending smoothly on parameters $\theta_2', ..., \theta_d'$. Thanks to the previous estimates on the tangential derivatives $\partial_{\theta_2'}, ..., \partial_{\theta_d'}$ and to the elliptic regularity on the derivatives $\partial_{\theta_1'}$, ∂_t, we obtain a local bound on all derivatives. Back to the original variables θ, t,

$$\|W\|_{H^k(\mathbb{T}^d \times (0,a))} \, + \, \|DR\|_{H^k(\mathbb{T}^d \times (0,a))} \, \leq \, C(k,a) < +\infty, \quad \forall k, a \geq 0 \tag{4.7}$$

where $C(k,a)$ depends on the data G, $\mathcal{A}$ and $\mathcal{B}$. To go from these estimates (local in t) to the global ones, we can derive the equation with respect to t, and obtain an equation similar to (4.3) on $W' := \partial_t W$. The only difference is that extra inhomogeneous terms are involved: at the boundary, $W'|_{t=0} = \partial_t W|_{t=0} \neq 0$, and in the equations, there are several commutators. Note that these commutators contain derivatives of $\mathcal{A}$ and $\mathcal{B}$, so that they are compactly supported in t. In particular, these extra terms are not annoying, because they are controlled by (4.7). Thus, we can proceed as before, to get

$$\|\partial_\theta^\gamma \, \partial_t DW\|_{L^2(\mathbb{T}^d \times \mathbb{R}_+)} \, < \, +\infty$$

Recursively, one can get a control of all $t-$derivatives. We leave the details to the reader. Eventually, we obtain the estimate (4.4).

Construction of solutions

To build solutions, we approximate (4.3) by a strongly elliptic system. If $d = 1$, the system is already strongly elliptic. If $d \geq 2$ we introduce new scalar fields $w_1^\eta = w_1^\eta(\theta, t)$, ...,$w_{d-1}^\eta = w_{d-1}^\eta(\theta, t)$, indexed by a small parameter $\eta > 0$, and the approximate problem

$$\begin{cases} -\Delta_{\theta,t} \begin{pmatrix} w_1^\eta \\ \vdots \\ w_{d-1}^\eta \end{pmatrix} + \eta \begin{pmatrix} \partial_{\theta_1} \\ \vdots \\ \partial_{\theta_{d-1}} \end{pmatrix} R^\eta \; = \; 0, \\ -\eta \Delta_{\theta,t} W^\eta \; + \; D \cdot ADW^\eta \; + \; BDR^\eta \; = \; G, \quad \theta \in \mathbb{T}^d, \; t > 0, \\ \eta \left(\partial_{\theta_1} w_1^\eta \; + \; ... \; + \; \partial_{\theta_{d-1}} w_{d-1}^\eta \right) \; + \; D \cdot (B^t W^\eta) \; = \; 0, \quad \theta \in \mathbb{T}^d, \; t > 0, \\ \left(w_1^\eta, ..., w_{d-1}^\eta, W^\eta \right) |_{t=0} \; = \; 0 \end{cases}$$

Existence, uniqueness and smoothness of solutions are standard for such strongly elliptic system, for all $\eta > 0$. All the *a priori* estimates above extend easily to this system.

The velocity field satisfies notably

$$\|\partial_\theta^\gamma \, \partial_t^k \, DW^\eta\|_{L^2(\mathbb{T}^d \times \mathbb{R}_+)} \; \leq \; C(\gamma, k) < +\infty, \quad \forall \gamma, k.$$

which allows to pass to the limit as $\eta \to 0$, and obtain a smooth solution V of the variational formulation (4.5) satisfying (4.4).

Let us stress that one does not go straightforwardly from the variational formulation (4.5) to the strong formulation (4.3). Indeed, uniformly in $\eta > 0$, the pressure R^η associated to W^η satisfies

$$\|\partial_\theta^\gamma \, \partial_k^t \, DR^\eta\|_{L^2(\mathbb{T}^d \times \mathbb{R}_+)} \; \leq \; C(\gamma, k) < +\infty, \quad \forall \gamma, k.$$

Contrary to the velocity estimate, which yields a bound on W^η itself (locally in t, thanks to Poincaré inequality), we only get here that

$$DR^\eta \to \mathcal{D}, \quad \eta \to 0,$$

where $\mathcal{D}$ is such that: for all smooth W' satisfying $D \cdot \left(\mathcal{B}^t W'\right) = 0$ and (4.4),

$$\int_{\mathbb{T}^d \times \mathbb{R}_+} \mathcal{B} \, \mathcal{D} \cdot W' = 0. \tag{4.8}$$

In other words, the strong formulation satisfied by W is

$$\begin{cases} -D \cdot \mathcal{A} DW \; + \; \mathcal{B} \mathcal{D} \; = \; G, \quad \theta \in \mathbb{T}^d, \; t > 0, \\ D \cdot \mathcal{B}^t \, W \; = \; 0, \quad \theta \in \mathbb{T}^d, \; t > 0, \\ W(\theta, t) \; = \; 0, \quad t = 0. \end{cases} \tag{4.9}$$

where $\mathcal{D}$ satisfies (4.8). But it is not clear that $\mathcal{D} = DR$ for some smooth R periodic in θ.

Uniqueness

Uniqueness follows from the basic estimate (4.6) on the difference between two solutions. This concludes the proof of the proposition

We can deduce from the previous analysis quasiperiodicity properties for the system (BL). We introduce $w(z) = W(\lambda z_1, z_2)$ and evaluate (4.9) at $\theta = \lambda z_1, t = z_2$. We obtain

$$\left\{\begin{aligned} \nabla \cdot A\nabla w + B\,\mathcal{D}_\lambda = g, &\quad z \in \mathbb{R}^2_+,\\ \nabla \cdot (B^t\, w) = 0, &\quad z \in \mathbb{R}^2_+,\\ w = 0, &\quad z_2 = 0. \end{aligned}\right.$$

where $\mathcal{D}_\lambda(z) = \mathcal{D}(\lambda z_1, z_2)$. Taking

$$W' = \left(\mathcal{B}^t\right)^{-1} D^\perp \phi := \left(\mathcal{B}^t\right)^{-1} \begin{pmatrix} -\partial_t \\ \lambda\cdot\partial_\theta \end{pmatrix} \phi$$

for an arbitrary smooth scalar function ϕ on $\mathbb{T}^d \times \mathbb{R}_+$, we get from (4.8)

$$\int_{\mathbb{T}^d\times\mathbb{R}^+} \left(D^\perp \cdot \mathcal{D}\right) \phi = 0.$$

As ϕ is arbitrary, we obtain $D^\perp \cdot \mathcal{D} = 0$. If we evaluate this identity at $(\theta, t) = (\lambda z_1, z_2)$, it gives $\nabla_z^\perp \mathcal{D}_\lambda = 0$. Thus, one can write $\mathcal{D}_\lambda = \nabla r$ for some smooth pressure field r. Hence, w satisfies (4.2).

We can then go back to the system (BL), by considering

$$v(y) = w(\phi(y)) - (\delta(y_2)y_2, 0) = W\left(\lambda y_1, y_2 - \chi(y_2)\omega(y_1)\right) - (\delta(y_2)y_2, 0) = V(\lambda y_1, y_2)$$

where $V(\theta, t) := W(\theta,\, t - \chi(t)F(\theta)) - \left(\delta(t)t, 0\right)$ is smooth and periodic in θ. Clearly, v is a solution of (BL). Moreover, by estimates (4.4), v belongs to $H^1_{loc}\left(\overline{\Omega_{bl}}\right)$ and satisfies

$$\sup_k \int_{\Omega^{bl}_{k,k+1}} |\nabla v|^2 < +\infty.$$

Thus, it is the solution built in Theorem 3, and it is quasiperiodic in y_1, for all $y_2 \geq 0$.

Proposition 13 *There exists $\alpha \in \mathbb{R}$ such that the solution v of* (BL) *satisfies*

$$v(y) \to (\alpha, 0), \qquad \text{as } y_2 \to +\infty, \quad \text{uniformly in } y_1.$$

Proof. We start from the convolution formula

$$v(y) = \int_{\mathbb{R}} G(t, y_2)\, v(y_1 - t, 0)\, dt, \quad y_2 > 0$$

where the Poisson kernel G is defined in (2.8). Integrating by parts leads to

$$v(y) = -\int_{\mathbb{R}} \left(t\, \partial_t G(t, y_2)\right) \frac{1}{t} \int_0^t v(y_1 - s, 0)\, ds\, dt$$

Thanks to Proposition 12, we can write

$$v(y_1 - s, 0) = V_0(\lambda(y_1 - s)) = \sum_{k\in\mathbb{Z}^d} \hat{V}_{0,k}\, e^{ik\cdot\lambda(y_1 - s)},$$

for some smooth periodic $V_0 = V_0(\theta)$. We define $v^\infty := \sum_{k,\, k\cdot\lambda = 0} \hat{V}_{0,k}$. Let $\delta > 0$. For n large,

$$\Big| \sum_{|k|\le n,\, k\cdot\lambda=0} \hat{V}_{0,k} \; - \; v^\infty \Big| \; \le \; \delta$$

and for n large, uniformly in y_1, uniformly in $t \neq 0$,

$$\Big| \frac{1}{t} \int_0^t \Big(v(y_1 - s, 0) - \sum_{|k|\le n} \hat{V}_{0,k}\, e^{i\lambda\cdot k(y_1 - s)} \Big) ds \Big| \; \le \; \sum_{|k|>n} |\hat{V}_{0,k}| \; \le \; \delta.$$

Moreover, n being fixed, we have, for $|t|$ large enough, uniformly in y_1

$$\begin{aligned}
\Big| \sum_{|k|\le n,\, k\cdot\lambda=0} \hat{V}_{0,k} - \frac{1}{t}\int_0^t \sum_{|k|\le n} \hat{V}_{0,k}\, e^{ik\cdot\lambda(y_1-s)}\, ds \Big| \; &\le \; \sum_{|k|\le n, k\cdot\lambda\neq 0} \Big| \frac{1}{t} \int_0^t e^{ik\cdot\lambda(y_1-s)}\, ds \Big| \\
&\le \; \sum_{|k|\le n, k\cdot\lambda\neq 0} \frac{2}{|k\cdot\lambda|\,|t|} \; \le \; \delta
\end{aligned}$$

Thus, uniformly in y_1

$$\lim_{t\to\pm\infty} \frac{1}{t} \int_0^t v(y_1 - s, 0)\, ds \; = \; v^\infty$$

Back to the convolution formula, we get

$$\begin{aligned}
|v(y) - v^\infty| \; &\le \; \int_{|t|\ge M} |t\,\partial_t G(t, y_2)| \, \Big| t^{-1} \int_0^t v(y_1 - s, 0)\, ds - v^\infty \Big| dt \\
&\quad + \int_{|t|\le M} |t\partial_t G(t, y_2)| \, \Big| t^{-1} \int_0^t v(y_1 - s, 0)\, ds - v^\infty \Big| dt \\
&\le \; C \left(\delta(M) \; + \; \int_{|t|\le M} |t\partial_t G(t, y_2)| \right) \; \le \; C' \left(\delta(M) \; + \; \frac{1}{y_2} \right),
\end{aligned}$$

where $\delta(M) \to 0$ as $M \to +\infty$. This proves the convergence of v to v^∞, uniformly in y_1 as y_2 goes to infinity.

It remains to show that the second component of v^∞ is zero. Therefore, we consider the function $u \; := \; v \; + \; (y_2, 0)$ which is divergence-free and cancels at $\partial\Omega^{bl}$. We integrate the equation $\nabla \cdot u = 0$ for $y_1 \in (0, t)$, $y_2 \in (\omega(y_1), 0)$, $\; t > 0$. We get by the Stokes formula

$$\int_0^t u_2(s, 0) ds \; = \; \int_{\omega(t)}^0 u_1(t, y_2) dy_2 - \int_{\omega(0)}^0 u_1(0, y_2) dy_2$$

which is exactly

$$\int_0^t v_2(s, 0) ds \; = \; \int_{\omega(t)}^0 v_1(t, y_2) dy_2 - \int_{\omega(0)}^0 v_1(0, y_2) dy_2 \; + \; \frac{\omega(t)^2 - \omega(0)^2}{2}.$$

We divide by t and let $t \to +\infty$. As v is bounded, we obtain $v_2^\infty = 0$, which means $v^\infty = (\alpha, 0)$ for some $\alpha \in \mathbb{R}$. This ends the proof.

Using the homogeneity properties of the Poisson kernel G, a slight modification of the previous argument yields

Corollary 1 *For all $\beta \in \mathbb{N}^2$, uniformly in y_1,*

$$y_2^{|\beta|} \partial_y^\beta (v(y) - v^\infty) \to 0, \qquad \text{as } y_2 \to +\infty.$$

We refer to [5] for the analogue of this corollary in the (more complicated) random case.

4.2 Extension to the almost periodic case

In this subsection, we want to show that the properties valid for for quasiperiodic roughness extend to the class $AP(\mathbb{R})$. Let $\omega \in AP(\mathbb{R})$, and v the corresponding solution of (BL). We want to show that $v(\cdot, y_2) \in AP(\mathbb{R})$ for all $y_2 > 0$ and that it converges to some $(\alpha, 0)$ as y_2 goes to infinity, uniformly with respect to y_1.

Let $\omega^n \in PT(\mathbb{R})$ such that $\omega^n \to \omega$ in $W^{2,\infty}(\mathbb{R})$. We can associate to ω^n a boundary layer solution v^n. By the results of the previous subsection, v^n is quasiperiodic in y_1 for all $y_2 > 0$, and converges to some field $(\alpha^n, 0)$, as y_2 goes to infinity, uniformly in y_1. Clearly, it is enough to show that

$$v^n \xrightarrow[n\to\infty]{} v \ \text{ in } \ W^{2,\infty}(\mathbb{R} \times \{y_2 > a\}), \qquad \text{for all } a > 0 \tag{4.10}$$

to get all the properties we want on v.

To compare directly v and v^n is difficult as these functions are not defined on the same domain. Like in the previous subsection, we introduce the diffeomorphism ϕ and the new fields w, r solutions of the modified Stokes problem (4.2). Similarly, we introduce ϕ^n and w^n, r^n. All these new fields are defined on the same domain $\mathbb{R}^2_+$. The differences $\tilde{w}^n := w - w^n$ and $\tilde{r}^n = r - r^n$ satisfy, with obvious notations:

$$\left\{ \begin{aligned} \nabla \cdot A \nabla \tilde{w}^n \; + \; B \, \nabla \tilde{r}^n \; &= \; \tilde{g}^n \; := \nabla \cdot (A^n - A) \nabla w^n + (B^n - B) \nabla r^n + g - g^n, \quad z \in \mathbb{R}^2_+, \\ \nabla \cdot (B^t \, \tilde{w}^n) \; &= \nabla \cdot ((B^n - B)^t w^n), \quad z \in \mathbb{R}^2_+, \\ \tilde{w}^n \; &= 0, \quad z_2 = 0. \end{aligned} \right. \tag{4.11}$$

If we manage to prove that

$$\tilde{w}^n \xrightarrow[n\to\infty]{} 0 \ \text{ in } \ H^1_{uloc}(\mathbb{R} \times \{y_2 < a\}), \qquad \text{for all } \ a > 0, \tag{4.12}$$

then property (4.10) follows. Indeed, back to the original fields v and v^n, (4.12) implies that

$$v^n(\cdot, 0) \to v(\cdot, 0) \ \text{ in } \ H^{1/2}_{uloc}(\mathbb{R}), \ \text{ as } \ n \to +\infty\,.$$

Then, the difference $v^n - v$ satisfies a Stokes equation in the half-space $\{y_2 > 0\}$, with a boundary data that goes to zero in $H^{1/2}_{uloc}(\mathbb{R})$. A closer look at the the proof of Proposition 6 (existence part) shows that this property implies (4.10).

It remains to obtain (4.12). From now on, $\delta(n)$ will denote a function going to zero as n goes to infinity, possibly changing from line to line. Let $a > 0$ large enough so that

$$A^n \; = \; A \; = \; B^n \; = \; B \; = \; Id, \quad z > a. \tag{4.13}$$

We wish to show that

$$E^n_k \; := \; \int_{\mathbb{R}} \int_0^a |\nabla \tilde{w}^n|^2 \, dz \; \le \; \delta(n), \tag{4.14}$$

uniformly in k. This bound comes from arguments very similar to those of section 2. For the sake of brevity, we only indicate the main steps and changes to take into account. First, using property (4.13), one can show as in section 2 that the system is equivalent to

$$\left\{ \begin{aligned} \nabla \cdot \, A \nabla \tilde{w}^n \, + \, B \, \nabla \tilde{r}^n \, &= \tilde{g}^n, \quad z \in (0, a), \\ \nabla \cdot (B^t \, \tilde{w}^n) \, &= \nabla \cdot ((B^n - B)^t w^n), \quad z \in (0, a), \\ \tilde{w}^n \, &= 0, \quad z_2 = 0, \\ (-\partial_2 \tilde{w}^n + \tilde{r}^n e_2)|_{\{z_2 = a\}} \, &= DN(\tilde{w}^n|_{\{z_2=a\}}) \end{aligned} \right. \tag{4.15}$$

The keypoint is to use (4.15) to derive the estimate

$$E_k^n \, \le \, C \, \left(\delta(n) \, (k+1) \, + \, \frac{1}{m^\delta} \sup_{j \ge k+m} \, (E_{j+1}^n - E_j^n) \, + \, \delta(n) \, m^3 \right) \tag{4.16}$$

This estimate follows from an induction argument for a sequence of approximate solutions, as in section 2. It is based on the induction relation

$$E_k^n \, \le \, C_1 \left(\delta(n) \, (k+1) \, + \frac{1}{m^\eta} \sup_{j \ge k+m} \, (E_{j+1}^n - E_j^n) \, + \, m \sup_{k+m \ge j \ge k} \left(E_{j+1}^n - E_j^n \right) \right). \tag{4.17}$$

This relation comes from an energy estimate of system (4.11), localized in a truncated channel. Precisely, one multiplies the Stokes like equation by $\chi_k \tilde{w}^n$, where χ_k is the same truncation function as in section 2. After integration over $\mathbb{R} \times (0, a)$ and a few integrations by parts, we end up with

$$\begin{aligned} \int_{\mathbb{R} \times (0,a)} \chi_k |\nabla \tilde{w}^n|^2 \, &\le \, \int_{\mathbb{R} \times (0,a)} \tilde{g}^n \, \chi_k \tilde{w}^n \, + \, \int_{\mathbb{R} \times (0,a)} \tilde{r}^n \nabla \cdot ((B^n - B)^t w^n) \, \chi_k \, + \, \mathcal{R}^n & (4.18) \\ &:= \, I_1 + I_2 + \mathcal{R}^n & (4.19) \end{aligned}$$

where

$$\mathcal{R}^n \, := \, \int_{\mathbb{R} \times (0,a)} A \nabla \tilde{w}^n \cdot (\chi_k', 0) \otimes w^n \, + \, \int_{\mathbb{R} \times (0,a)} \tilde{r}^n \chi_k' (B^t \tilde{w}^n)_1 \, - \, \int_{\{y_2 = a\}} DN(\tilde{w}^n) \cdot (\chi_k \tilde{w}^n)$$

gathers the remaining terms that can be treated exactly as in section 2. In particular,

$$\mathcal{R}^n \, \le \, C_\nu \left(\frac{1}{m^\eta} \sup_{j \ge k+m} \, (E_{j+1}^n - E_j^n) \, + \, m \sup_{k+m \ge j \ge k} \left(E_{j+1}^n - E_j^n \right) \right) \, + \, \nu E_k^n$$

The first integral I_1 is treated in the following way:

$$\begin{aligned} |I_1| \, \le \, & \int_{\mathbb{R} \times (0,a)} |A_n - A| \, |\nabla w^n| \, |\nabla (\chi_k \tilde{w}^n)| \, + \, \int_{\mathbb{R} \times (0,a)} |B^n - B| \, |\nabla r^n| \, |\chi_k \tilde{w}^n| \\ & + \, \int_{\mathbb{R} \times (0,a)} |g - g^n| \, |\chi_k \tilde{w}^n| \end{aligned}$$

Note that the integration by parts responsible for the first term of the r.h.s. does not give any boundary term: indeed, the quantity $(A^n - A)$ cancels at $y_2 = a$ by (4.13), and w^n cancels at $y_2 = 0$. We get:

$$|I_1| \, \le \, \delta(n) \int_{-k-1}^{k+1} \int_0^a \left(|\nabla \tilde{w}^n|^2 \, + |\nabla \tilde{r}^n|^2 + 1 \right) \, + \, \delta(n) \, E_{k+1}^n$$

where one can take

$$\delta(n) \;=\; C\left(\|A^n - A\|_{L^\infty} + \|B^n - B\|_{L^\infty} + \|F^n - F\|_{L^\infty}\right).$$

As the boundaries ω^n are uniformly bounded in $W^{2,\infty}$, one has by standard elliptic regularity

$$\|v^n\|_{H^2_{uloc}(\Omega^{bl,-})} \;+\; \|\nabla q^n\|_{L^2_{uloc}(\Omega^{bl,-})} \;\le\; C\|v^n\|_{H^1_{uloc}(\Omega^{bl,-})} \;\le\; C'$$

uniformly with respect to n. Thus, for the new fields:

$$\|w^n\|_{H^2_{uloc}(\mathbb{R}\times(0,a))} \;+\; \|\nabla r^n\|_{L^2_{uloc}(\mathbb{R}\times(0,a))} \;\le\; C''\,,$$

which in turn leads to

$$|I_1| \;\le\; \delta(n)\left((k+1) \;+\; E^n_{k+1}\right)$$

To handle the second integral, we integrate by parts:

$$|I_2| \;=\; -\int_{\mathbb{R}\times(0,a)} \tilde{r}^n\left(\chi'_k (B^n-B)^t w^n\right)_1 - \int_{\mathbb{R}\times(0,a)} (\nabla\tilde{r}^n)\,(B^n-B)^t w^n\,\chi_k \;:=\; J_1 + J_2.$$

Again, there is no boundary term, as $(B^n - B)^t w^n$ cancels at both boundaries $y_2 = 0, a$. Using the equation,

$$J_2 \;=\; \int_{\mathbb{R}\times(0,a)} B^{-1}\left(\nabla\cdot A\nabla\tilde{w}^n \;-\; \tilde{g}^n\right)\left((B^n-B)^t w^n\right)\chi_k.$$

This term can be treated in the same spirit as I_1. We state without further details

$$|J_2| \;\le\; \delta(n)\,\left(k+1 \;+\; E^n_{k+1}\right)$$

The term J_1 can be treated with minor modifications as the term $I_4 = \int p\,\chi'_k u_1$ in the estimates of section 2. See also the pressure term in R^n above. It leads to:

$$|J_1| \;\le\; \delta(n)\left(E^n_k \;+\; 1 \;+\; \frac{1}{m^\eta}\sup_{j\ge k+m}\left(E^n_{j+1} - E^n_j\right) \;+\; m\sup_{k+m\ge j\ge k}\left(E^n_{j+1} - E^n_j\right)\right).$$

Collecting all these bounds gives the inequality (4.17).

As an easy consequence of the previous result, we get the same convergence properties as in the quasiperiodic case. Namely:

Proposition 14 *There exists $\alpha\in\mathbb{R}$, such that for all $\beta\in\mathbb{N}^2$,*

$$|\,y_2^{|\beta|}\,\partial_y^\beta\left(v(y) \;-\; (\alpha,0)\right)| \to 0, \qquad \textit{as } y_2\to+\infty$$

4.3 Justification of the slip condition

Thanks to the previous proposition, we conclude the proof of Theorem 4. It is very close to the proof of Theorem 2 of [5], related to the stationary random case. *The only difference is that in [5], all estimates involved expectations, because the convergence of the boundary layer solution did not hold a priori uniformly with respect to* y_1. *Here, by Proposition 14, we can obtain a "deterministic" bound, that is in* $L^2_{uloc}(\Omega)$.

For the sake of brevity, we only describe the main steps of proof, and refer to [5] for all necessary details. Let the flux ϕ in (NS$^\varepsilon$) be small enough, and u^ε the solution provided by Theorem 1. We introduce an approximation u^ε_{app} of u^ε of the type

$$u^\varepsilon_{app}(x) \;=\; u^0(x) \;+\; 6\phi\,\varepsilon v(x/\varepsilon) \;+\; 6\phi\,\varepsilon\left(u^1(x) \;+\; r^\varepsilon\right)\mathbf{1}_\Omega(x)$$

where u^0 is the Poiseuille flow and v is the solution of (BL). The additional correctors u^1 and r^ε ensure zero Dirichlet condition at the upper boundary of Ω^ε as well as zero flux. For instance, u^1 satisfies

$$\left\{\begin{aligned} -\Delta u^1 + u^0\cdot\nabla u^1 + u^1\cdot\nabla u^0 + \nabla p^1 \;&=\; 0,\; x\in\Omega\\ \nabla\cdot u^1 \;&=\; 0,\; x\in\Omega,\\ u^1|_{\{x_2=0\}} \;=\; 0,\quad u^1|_{\{x_2=1\}} \;=\; -v^\infty \;&=\; -(\alpha,0),\quad \int_\sigma u^1_1 = -\alpha. \end{aligned}\right. \tag{4.20}$$

where as usual, σ (*resp.* σ^ε) denotes a vertical cross section of the channel Ω (*resp.* Ω^ε). Like u^0, u^1 is explicit. It is a combination of Couette and Poiseuille flows:

$$u^1(x) = \left(U^1(x_2), 0\right), \quad U^1(x_2) \;=\; -4\alpha x_2 + 3\alpha x_2^2.$$

The remainder r^ε satisfies

$$\left\{\begin{aligned} \nabla\cdot r^\varepsilon \;&=\; 0,\; x\in\Omega,\\ r^\varepsilon|_{\{x_2=0\}} \;&=\; 0,\quad r^\varepsilon|_{\{x_2=1\}} = (-v(x/\varepsilon)+v^\infty)|_{\{x_2=1\}},\\ \int_\sigma r^\varepsilon_1 \;&=\; -\int_{\sigma^\varepsilon\setminus\sigma} u^0_1 - \int_{\sigma^\varepsilon} v_1(x/\varepsilon) + \alpha. \end{aligned}\right. \tag{4.21}$$

It is provided by the following:

Lemma 15 *The problem* (4.21) *has a (non-unique) solution* r^ε *such that*

$$\|r^\varepsilon\|_{H^2_{uloc}(\Omega)} \;=\; o(1), \qquad \text{as } \varepsilon\to 0.$$

Proof of the Lemma. It is enough to find some $\tilde r^\varepsilon$ that satisfies the first two lines of (4.21) and the estimates of the Lemma, because the field

$$r^\varepsilon \;:=\; \tilde r^\varepsilon \;-\; \left(6\phi^\varepsilon y_2(1-y_2),\, 0\right), \quad \phi^\varepsilon \;:=\; \int_{\sigma^\varepsilon\setminus\sigma} u^0_1 + \int_{\sigma^\varepsilon} v_1(x/\varepsilon) - \alpha + \int_\sigma \tilde r^\varepsilon_1$$

will then fulfill all requirements. Indeed, the flux term ϕ^ε is independent of x_1, because

$$\phi^\varepsilon \;=\; -\phi + \int_{\sigma^\varepsilon} \tilde u^\varepsilon_{app}, \qquad \text{where } \tilde u^\varepsilon_{app}(x) \;:=\; u^0(x) + v(x/\varepsilon) + \varepsilon\,(\tilde r^\varepsilon - \alpha)\,\mathbf{1}_\Omega(x)$$

is divergence-free and zero at $\partial\Omega^\varepsilon$. Moreover,

$$
\begin{aligned}
|\phi^\varepsilon| \;&\le\; \frac{\varepsilon^2\|\omega\|^2_{L^\infty}}{2} + \varepsilon\int_{\varepsilon\omega(0)}^0 |v(0,x_2/\varepsilon)|dx_2 \;+\; \int_0^1 |v(0,x_2/\varepsilon)-\alpha|dx_2 \;+\; \|r^\varepsilon\|_{L^\infty} \\
&\le\; \delta(\varepsilon) \;+\; \int_0^1 |v(0,x_2/\varepsilon)-\alpha|dx_2, \quad \delta(\varepsilon)\to 0, \qquad \text{as } \varepsilon\to 0.
\end{aligned}
$$

By Proposition 14, for all $\delta>0$, we can find some M such that $|v(0,y_2)-\alpha|\le\delta$ for $y_2>M$. Then,

$$
\int_0^1 |v(0,x_2/\varepsilon)-\alpha|dx_2 \;=\; \int_0^{\varepsilon M} |v(0,x_2/\varepsilon)-\alpha|dx_2 \;+\; \int_{\varepsilon M}^1 |v(0,x_2/\varepsilon)-\alpha|\,dx_2 \le C(M)\,\varepsilon \;+\; \delta.
$$

This shows as needed that $|\phi^\varepsilon| = o(1)$ as ε goes to zero.

The construction of $\tilde{r}^\varepsilon$ follows the exact same lines as the construction of v_l in [5, Proposition 5.1,p979]. The only difference is that thanks to the stronger Proposition 14, we can get a deterministic H^2_{uloc} bound.

This approximation at hand, Theorem 4 is deduced from the two following estimates:

$$
\|u^\varepsilon_{app} - u^N\|_{L^2_{uloc}(\Omega)} \;=\; o(\varepsilon), \tag{4.22}
$$

where u^N is the solution of (NS)-(Na), with the parameter α associated to (BL), and

$$
\|u^\varepsilon_{app} - u^\varepsilon\|_{H^1_{uloc}(\Omega^\varepsilon)} \;=\; o(\varepsilon) \tag{4.23}
$$

For inequality (4.22), one takes advantage that u^N is explicit, namely

$$
u^N = (U^N(x_2),0), \quad U^N(x_2) \;=\; \phi\left(-\frac{6(1+\varepsilon\alpha)}{1+4\varepsilon\alpha}x_2^2 \;+\; \frac{6}{1+4\varepsilon\alpha}x_2 \;+\; \frac{6\varepsilon\alpha}{1+4\varepsilon\alpha}\right).
$$

A direct computation shows that

$$
u^N \;=\; u^0 + 6\phi\,\varepsilon(\alpha,0) + 6\phi\,\varepsilon u^1 + O(\varepsilon^2) \quad \text{ in } L^2_{uloc}(\Omega)
$$

so that

$$
u^\varepsilon_{app} - u^N \;=\; 6\phi\,\varepsilon\,(v(x/\varepsilon)-(\alpha,0)) \;+\; 6\phi\,\varepsilon r^\varepsilon \mathbf{1}_\Omega \;+\; O(\varepsilon^2) \;=\; o(\varepsilon) \quad \text{ in } L^2_{uloc}(\Omega).
$$

The inequality (4.23) comes from an energy estimate on $w^\varepsilon = u^\varepsilon - u^\varepsilon_{app}$. It solves:

$$
\begin{cases}
-\Delta w^\varepsilon \;+\; \nabla r^\varepsilon \;=\; \nabla\cdot G^\varepsilon, \quad x\in\Omega^\varepsilon, \\
\nabla\cdot w^\varepsilon = 0, \\
w^\varepsilon|_{\partial\Omega^\varepsilon} = 0, \\
\displaystyle\int_{\sigma^\varepsilon} w_1^\varepsilon = 0, \\
[w^\varepsilon]|_\Sigma = 0, \quad [\partial_2 w^\varepsilon - r^\varepsilon e_2]|_\Sigma = j^\varepsilon,
\end{cases} \tag{4.24}
$$

where

$$G^\varepsilon \;:=\; -u^\varepsilon \otimes u^\varepsilon \;-\; 6\phi\,\varepsilon\nabla r^\varepsilon \mathbf{1}_\Omega \;=\; O(\varepsilon^{3/2}) \;+\; o(\varepsilon) \;=\; o(\varepsilon) \quad \text{in } L^2_{uloc}(\Omega^\varepsilon)$$

and the jump term

$$j^\varepsilon \;:=\; -24\phi\varepsilon\alpha \;+\; \varepsilon\partial_2 r^\varepsilon|_{x_2=0} \;=\; O(\varepsilon) \quad \text{in } L^2_{uloc}(\Sigma)$$

which means that for all $w \in H^1_{uloc}(\Omega^\varepsilon)$ that cancels at the lower boundary of Ω^ε,

$$\sup_k \Big| \int_{\Sigma_{k,k+1}} j^\varepsilon\, w \Big| \;\le\; C\varepsilon \|u^\varepsilon_{L^2_{uloc}(\Sigma)} \;\le\; C\varepsilon^{3/2} \|\nabla w\|_{H^1_{uloc}(\Omega^\varepsilon)}$$

Following the energy estimates of article [5] (or simplifying those of the present paper!), we get from there inequality (4.23). This concludes the proof of Theorem 4.

4.4 Small divisor assumption

The general Theorem 4 shows that a slip condition with appropriate slip parameter yields a $o(\varepsilon)$ approximation of the solution. This error estimate can be refined in the quasiperiodic case:

$$\omega = F(\lambda y_1), \quad F = F(\theta) \in C^\infty(\mathbb{T}^d), \quad \lambda \in \mathbb{R}^d,$$

when the vector of periods λ satisfies the diophantine assumption **(H)**. In such a case, the same Navier wall law gives a $O(\varepsilon^{3./2})$ estimate, as stated in Theorem 5. This theorem will be a consequence of

Proposition 16 *If λ satisfies* **(H)**, *the solution v of* (BL) *satisfies for all $\beta, \gamma \in \mathbb{N}^2$,*

$$y_2^{|\beta|}\, \partial^\gamma (v - v^\infty) \to 0, \qquad \textit{as } y_2 \to 0, \qquad \textit{uniformly in } y_1.$$

In other words, the boundary layer profile is in the Schwartz class with respect to the variable y_2, uniformly with respect to y_1. This fast decay allows to turn each $o(\varepsilon)$ into $O(\varepsilon^{3/2})$ in all the arguments of the previous subsection. This is due to the fact that any power of $v - v^\infty$ or its derivatives is integrable with respect to y_2. This yields for instance:

$$\varepsilon \|v(x/\varepsilon) - v^\infty\|_{L^2_{uloc}(\Omega)} \;\le\; \varepsilon^{3/2} \sup_{y_1} \left(\int_{\omega(y_1)}^{+\infty} |v(y) - \alpha|^2 dy \right)^{1/2} \;=\; O(\varepsilon^{3/2})$$

The same is true for all related quantities, which leads to Theorem 5.

Proof of the Proposition To establish this speed of convergence, we come back to the field $W = W(\theta, t)$ provided by Proposition 12. It is enough to prove that W is in the Schwartz class with respect to t, uniformly with respect to θ. We remind that it satisfies the estimates (4.4), which express that $DW := \begin{pmatrix} \lambda\cdot\partial_\theta \\ \partial_t \end{pmatrix} W$ belongs to $H^s(\mathbb{T}^d \times \mathbb{R}^+)$ for all s. For any smooth function ϕ defined on $\mathbb{T}^d \times \mathbb{R}$, we will decompose

$$\phi(\theta, t) \;=\; \tilde{\phi}(\theta, t) \;+\; \overline{\phi}(t), \qquad \int_{\mathbb{T}^d} \tilde{\phi}(\theta, t)\, d\theta = 0.$$

Assumption **(H)** yields that

$$\int_{\mathbb{T}^d} (\lambda \cdot \partial_\theta)\tilde{\varphi}|^2 \;\geq\; c\,\|\tilde{\varphi}\|_{H^{-l}(\mathbb{T}^d)} \tag{4.25}$$

for smooth enough $\tilde{\varphi} = \tilde{\varphi}(\theta)$ with zero average. From there and (4.4), we get the following estimate:

$$\int_0^{+\infty} \int_{\mathbb{T}}^{d} |\partial^\beta \tilde{W}|^2 \;+\; |\partial^\beta \partial_t W|^2 \, d\theta \, dt \;\leq\; C(\beta, k) \;<\; +\infty, \quad \forall\, \beta \in \mathbb{N}^{d+1}. \tag{4.26}$$

where $\partial^\beta \;:=\; \partial_{\theta_1}^{\beta_1} ... \partial_{\theta_d}^{\beta_d} \, \partial_t^{\beta_{d+1}}$. Moreover, this yields a genuine strong formulation for system (4.3): we remind that in the general quasiperiodic case, we only had the modified strong formulation (4.9), due to the lack of compactness of the sequence R^η. We only had

$$DR^\eta \to \mathcal{D}, \quad \eta \to 0,$$

in $H^s(\mathbb{T}^d \times \mathbb{R}^+)$ for any s, where $\mathcal{D}$ was satisfying (4.8). Here, thanks to (4.25), we get a bound on $\tilde{R}^\eta$ which allows to extract from R^η a subsequence converging in H^s to some smooth

$$R \;:=\; \lim_{\delta \to 0} \tilde{R}^\eta \;+\; \lim_{\eta \to 0} \int_0^t \partial_t \overline{R}^\eta.$$

Moreover,

$$\int_0^{+\infty} \int_{\mathbb{T}}^{d} |\partial^\beta \tilde{R}|^2 \;+\; |\partial^\beta \partial_t R|^2 \, d\theta \, dt \;\leq\; C(\beta, k) \;<\; +\infty, \quad \forall\, \beta \in \mathbb{N}^{d+1}, \tag{4.27}$$

and W, R satisfy (4.3) in a classical sense.

The last step is to determine the behaviour of W and its derivatives as $t \to 0$. Let M large enough so that $\mathcal{A} = \mathcal{B} = Id$, $\tilde{F} = 0$ for $t \geq M$. Thus,

$$D^2 W + DR = 0, \quad D \cdot W = 0, \quad \text{ for } t \geq M,\, \theta \in \mathbb{T}^d. \tag{4.28}$$

Let $T \geq M$. We define

$$E(T) \;:=\; \int_T^{+\infty} \int_{\mathbb{T}^d} |DW|^2 \; d\theta \, dt, \quad \text{ and } W'(t) \;:=\; W - \int_{\mathbb{T}^d} W(\theta, T) d\theta.$$

Multiplying (4.28) by W and integrating for $\theta \in \mathbb{T}^d$, $t \in (T, +\infty)$,

$$\begin{aligned} E(T) \;&=\; -\int_{\mathbb{T}^d} \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) \cdot DW(\cdot, T) \cdot W'(\cdot, T) \;+\; \int_{\mathbb{T}^d} \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) R(\cdot, T) \cdot W'(\cdot, T) \\ &=\; -\int_{\mathbb{T}^d} \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) \cdot D\tilde{W}(\cdot, T) \cdot \tilde{W}(\cdot, T) \;+\; \int_{\mathbb{T}^d} \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) \tilde{R}(\cdot, T) \cdot \tilde{W}(\cdot, T) \end{aligned}$$

By the Cauchy-Schwartz inequality,

$$E(T) \;\leq\; \left(\left(\int_{\mathbb{T}^d} \left| D\tilde{W}(\cdot, T) \right|^2 \right)^{1/2} + \left(\int_{\mathbb{T}^d} \left| \tilde{R}(\cdot, T) \right|^2 \right)^{1/2} \right) \left(\int_{\mathbb{T}^d} \left| \tilde{W}(\cdot, T) \right|^2 \right)^{1/2} \tag{4.29}$$

By interpolation and use of (4.25), we have: for all $1 < p < +\infty$, for all smooth $\tilde{\varphi}$ with zero average,

$$\|\tilde{\varphi}\|^2_{H^s(\mathbb{T}^d)} \;\le\; C \left(\int_{\mathbb{T}^d} |\lambda \cdot \partial_\theta \tilde{\varphi}|^2 \right)^{1/p} \left(\|\tilde{\varphi}\|_{H^{s'}} \right)^{2-2/p} \tag{4.30}$$

where $s' = \frac{l+s}{p-1} + s$. It implies, together with (4.26), for all $1 < p < +\infty$,

$$\int_{\mathbb{T}^d} \left| \tilde{W}(\cdot, T) \right|^2 \;\le\; C \left(\int_{\mathbb{T}^d} |DW|^2 \right)^{1/p}.$$

Moreover, thanks to (4.26)-(4.27), we have crudely

$$\left(\int_{\mathbb{T}^d} \left| D\tilde{W}(\cdot, T) \right|^2 \right)^{1/2} \;+\; \left(\int_{\mathbb{T}^d} \left| \tilde{R}(\cdot, T) \right|^2 \right)^{1/2} \;\le\; C \tag{4.31}$$

Back to inequality (4.29), we get

$$E(T) \;\le\; C \left(\int_{\mathbb{T}^d} |DW|^2 \right)^{1/2p}, \quad \forall\, p > 1$$

which is

$$E(T) \;\le\; C(-E'(T))^{1/2p}, \quad \forall\, p > 1. \tag{4.32}$$

It yields in turn that

$$E(T) \;\le\; C\,T^{-\eta}, \quad \forall \eta < 1. \tag{4.33}$$

Remark that we can differentiate the equation (4.28) and apply the same reasoning to any derivative of W. We obtain for all $\beta \in \mathbb{N}^{d+1}$

$$E_\beta(T) \;:=\; \int_T^{+\infty} \int_{\mathbb{T}^d} |\partial^\beta DW|^2 \, d\theta \, dt \;\le\; C\,T^{-\eta}, \quad \forall \eta < 1.$$

which yields by (4.25)

$$\int_T^{+\infty} \int_{\mathbb{T}^d} \left(|\partial^\beta \tilde{W}|^2 + |\partial^\beta \partial_t W|^2 \right) d\theta \, dt \;\le\; C\,T^{-\eta}, \quad \forall \eta < 1.$$

By Sobolev imbedding, it implies

$$\left(\int_{\mathbb{T}^d} \left| D\tilde{W}(\cdot, T) \right|^2 \right)^{1/2} \;\le\; C\,T^{-\eta/2}, \quad \forall \eta < 1.$$

It also implies, using (4.25) and the Stokes equation (4.3),

$$\left(\int_{\mathbb{T}^d} \left| \tilde{R}(\cdot, T) \right|^2 \right)^{1/2} \;\le\; C\,T^{-\eta/2}, \quad \forall \eta < 1.$$

Hence, (4.31) can be replaced by

$$\left(\int_{\mathbb{T}^d} \left| D\tilde{W}(\cdot, T) \right|^2 \right)^{1/2} \;+\; \left(\int_{\mathbb{T}^d} \left| \tilde{R}(\cdot, T) \right|^2 \right)^{1/2} \;\le\; C\,T^{-\eta/2}, \quad \forall \eta < 1,$$

and the integro-differential inequality (4.32) is replaced by

$$E(T) \le C\,T^{-\eta/2}(-E'(T))^{1/2p}, \quad \forall \eta < 1.$$

Finally, (4.33) is replaced by

$$E(T) \le C\,T^{-\eta'}, \quad \forall\,\eta' < p+1$$

Proceeding recursively, we can gain an arbitrary power of T in the decay rate. We end up with

$$T^\gamma\,\partial^\beta \tilde{W},\, T^\gamma \partial^\beta \partial_t W \to 0, \quad \text{as } T \to +\infty, \quad \text{uniformly in } \theta.$$

It remains to show the rate of convergence of the average $\overline{W}$. We write

$$|\overline{W}(t+h) - \overline{W}(t)| \le \int_t^{t+h} |\frac{d}{dt}\overline{W}| \le C(\gamma) \int_t^{t+h} (1+s)^{-\gamma} ds$$

for all $\gamma > 0$. This shows that $\overline{W}(t)$ is a Cauchy function, hence convergent to a constant vector W^∞ as t goes to infinity. Back to above inequality, the convergence is faster than any power function of t. This concludes the proof of the proposition.

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