GLOBAL EXISTENCE OF WEAK SOLUTIONS TO MACROSCOPIC MODELS OF POLYMERIC FLOWS.

NADER MASMOUDI

Key words Navier-Stokes equations, FENE-P model, micro-macro interactions, defect measure, global existence.
AMS subject classification 35Q30, 82C31, 76A05.

Abstract
One of the most classical closures approximation of the FENE model of polymeric flows is the one proposed by Peterlin, namely the FENE-P model. We prove global existence of weak solutions to the FENE-P model. The proof is based on the propagation of some defect measures that control the lack of strong convergence in an approximating sequence. Using a similar argument, we also prove global existence of weak solutions to the Giesekus and the Phan-Thien and Tannes models.

1. INTRODUCTION

The FENE (Finite Extensible Nonlinear Elastic) model is one of the most used models in polymeric fluids. In this model, a polymer is idealized as an "elastic dumbbell" consisting of two "beads" joined by a spring which can be modeled by a vector R (see Bird, Curtis, Amstrong and Hassager [5], Doi and Edwards [11] and Ottinger [44]). We also refer to Owens and Phillips [46] for the computational aspect and to C. Le Bris and T. Lelièvre [27] and Li and Zhang [34] for very nice mathematical overviews. One then writes a Fokker-Planck equation describing the evolution of the polymer density. This Fokker-Planck equation is coupled to the Navier-Stokes equation. The coupling comes from and extra stress term in the Navier-Stokes equation due to the microscopic effect of the polymers. This is the micro-macro interaction. There is also a drift term in the Fokker-Planck equation that depends on the spatial gradient of the velocity.

The system obtained attempt to describe the behavior of this complex mixture of polymers and fluid, and as such, it presents numerous challenges, simultaneously at the level of their derivation, the level of their numerical simulation and that of their mathematical treatment. There are also many macroscopic models called *closure approximations* that attempt to give a good approximation of the FENE model as well as other microscopic models such as the Doi model (see [12, 9]). The advantage of these models is that they are easier to implement numerically. However, the disadvantage is that they are sometimes unable to describe all the physical properties of the original model (see [23, 24]). An approximate closure of the linear Fokker-Planck equation reduces the description to a closed viscoelastic equation for the added stress. This leads to well-known non-Newtonian fluid models such as the Oldroyd B model that has been studied extensively. Actually, the Oldroyd B model is a macroscopic model that is exactly equivalent to the Hooke model in which each polymer is idealized as a linear "elastic dumbbell".

In this paper we concentrate on the mathematical treatment of an other closure models, namely the FENE-P model. This model was proposed by Peterlin [47] to replace the FENE model by a macroscopic one. It comes from replacing the denominator of the FENE force by the mean value of the elongation. The main result of this paper is to prove global existence of weak solutions for the FENE-P dumbbell model (1). These solutions are the generalization of the Leray weak solutions [33, 32] of the incompressible Navier-Stokes system to the FENE-P model. In the last section of the paper and using a similar strategy, we also prove global existence of weak solutions to the Giesekus [17] and the Phan-Thien and Tannes models.

We end this introduction by mentioning some related mathematical results. In Guillopé and Saut [18] and [19], the existence of local strong solutions to the Oldroyd B model was proved. Also, Fernández-Cara, Guillén and Ortega [15], [14] and [16] proved local well posedness in Sobolev spaces. In Chemin and Masmoudi [6] local and global well-posedness in critical Besov spaces was given. For global existence of weak solutions, we refer to Lions and Masmoudi [37]. We also mention Lin, Liu and Zhang [35] where a formulation based on the deformation tensor is used to study the Oldroyd-B model. Global existence for small data was also proved in [31, 29]. Moreover, non-blow up criteria for Oldroyd-B were given in [26, 30]. There are also many works dealing with compressible viscoelastic fluids as well as their incompressible limit or their Newtonian limit [28, 43, 39]. We also refer to [49, 50] for some numerical works about Oldroyd-B models as well as other macroscopic models.

At the micro-macro level, we can mention Renardy [48] who proved the local existence in Sobolev space when the potential of the dumbbell force $\mathcal{U}$ is given by $\mathcal{U}(R) = (1 - |R|^2)^{1-\sigma}$ for some $\sigma > 1$. W. E, Li and Zhang [13] proved local existence when R is taken in the whole space and under some growth condition on the potential. Also, Jourdain, Lelievre and Le Bris [22] proved local existence in the case $b = 2k > 6$ (where k is the constant appearing in the definition of $\mathcal{U}(R)$ below) for a Couette flow by solving a stochastic differential equation (see also [21] for the use of entropy inequality methods to prove exponential convergence to equilibrium). Zhang and Zhang [51] proved local well-posedness for the FENE model when $b > 76$. Local well-posedness was also proved in [40] when $b = 2k > 0$ (see also [25]). Moreover, Lin, Liu and Zhang [36] proved global existence near equilibrium under some restrictions on the potential. Global existence of weak solutions was proved in [38] for the co-rotational model. Besides, Barrett and Suli [4] studied the problem of global existence for a regularized bead-spring chain model (see also [2, 3]). More recently, global existence of weak solutions to the FENE model was proved in [41]. Other microscopic models such as the Doi model where treated in [45, 7, 8, 42].

The rest of the paper is organized as follows. In the next section we state the main result of the paper, namely the existence of global weak solution to the FENE-P model. Section 3 is devoted to the proof of this result and more precisely to the weak compactness of a sequence of solution. In section 4, we prove global existence for the Giesekus and the Phan-Thien and Tannes models. Finally, in the appendix 5, we recall the notions of Young measures and Chacon weak limit.

2. The FENE-P model

2.1. **Derivation.** The FENE-P model is a macroscopic approximation of the FENE model that reads :

$$\begin{cases} \partial_t u + (u\cdot\nabla)u - \nu\Delta u + \nabla p = \mathrm{div}\tau, \quad \mathrm{div} u = 0, \\ \\ \partial_t A + u.\nabla A = \nabla u A + A(\nabla u)^T - \frac{A}{1-\mathrm{tr}(A)/b} + \mathrm{Id} \\ \\ \tau = \tau(A) = \frac{A}{1-\mathrm{tr}(A)/b} - \mathrm{Id}. \end{cases} \tag{1}$$

In (1), u is the velocity of the fluid, τ is the extra stress tensor due to the polymers, A is sometimes called the mean of the structure tensor and $\nu > 0$ is the viscosity of the fluid. Throughout this paper, we adopt the notation $(\nabla u)_{i,j} = \frac{\partial u_i}{\partial x_j}$. Many other authors use the alternative convention. The system is considered in a domain Ω that can be a bounded domain of $\mathbb{R}^D$, the whole space $\mathbb{R}^D$ or the torus $\mathbb{T}^D$. In the case the problem is considered in a bounded domain Ω, we add the Dirichlet boundary condition $u = 0$ on $\partial\Omega$ in which case there is no need to add a boundary condition for A.

The second and third equations replace the Fokker-Planck equation and the expression of the stress tensor coming from the FENE model, namely

$$\text{(2)} \qquad \begin{cases} \partial_t \psi + u.\nabla\psi = \mathrm{div}_R\Big[-\nabla u\, R\psi + \beta\nabla\psi + \nabla\mathcal{U}\psi\Big] \\ \tau_{ij} = \rho\left[\int_B (R_i \otimes \nabla_j \mathcal{U})\psi(t,x,R)dR \; - \beta I\right] \qquad (\nabla\mathcal{U}\psi + \beta\nabla\psi).n = 0 \text{ on } \partial B(0,R_0). \end{cases}$$

where $\psi(t,x,R)$ denotes the distribution function for the internal configuration, $F(R) = \nabla_R \mathcal{U}$ is the spring force which derives from a potential $\mathcal{U}$ and $\mathcal{U}(R) = -k\log(1 - |R|^2/|R_0|^2)$ for some constant $k > 0$ and ρ is the density of polymers. Here, R is in the bounded ball $B = B(0,R_0)$ of radius R_0 which means that the extensibility of the polymers is finite. We have also to add a boundary condition to insure the conservation of the polymer density ψ, namely $(-\nabla u R\psi + \nabla\mathcal{U}\psi + \beta\nabla\psi).n = 0$ on $\partial B(0,R_0)$.

In (2), we introduce the so-called mean of the structure tensor A which is given by $A_{ij} = \int_B R_i \; R_j \psi(t,x,R)dR$. Hence, A is a positive symmetric matrix and multiplying the first equation of (2) by R_iR_j and integrating over B, we easily get that A solves

$$\partial_t A + u.\nabla A = \nabla u A + A(\nabla u)^T - 2\int_B (R\otimes\nabla\mathcal{U})\psi(t,x,R)dR + 2\beta I.$$

If we choose the constant appropriately, this becomes $\partial_t A + u.\nabla A = \nabla u A + A(\nabla u)^T - \tau$. Notice, that for (2), τ depends on the whole distribution function $\psi(t,x,R)$. The FENE-P approximation consists is setting $\tau = \tau(A) = \frac{A}{1-\mathrm{tr}(A)/b} - \mathrm{Id}$ where $b = R_0^2$.

2.2. **Free energy.** The FENE-P model got some gain of interest after Hu and Lelievre [20] proved that it has a free energy that decays with time. We reproduce here their calculations.

Let $H(t) = \int_\Omega h = \int_\Omega \Big(h_1(t,x) + h_2(t,x) + (b+D)\log(\frac{b}{b+D})\Big)dx$ be given by

$$\text{(3)} \qquad h_1(t,x) = -\log(\det A), \qquad h_2(t,x) = -b\log(1 - \mathrm{tr}(A)/b).$$

Using that $\partial_t \det A = (\det A)\,\mathrm{tr}(A^{-1}\partial_t A)$, we get

$$\text{(4)} \qquad (\partial_t + u.\nabla)h_1 = -\mathrm{tr}(A^{-1}) + \frac{D}{1-\mathrm{tr}(A)/b}.$$

Moreover, we have

$$\text{(5)} \qquad (\partial_t + u.\nabla)h_2 = \frac{2\nabla u : A}{1-\mathrm{tr}(A)/b} + \frac{D}{1-\mathrm{tr}(A)/b} - \frac{\mathrm{tr}(A)}{(1-\mathrm{tr}(A)/b)^2}$$

and

$$\text{(6)} \qquad \partial_t \int_\Omega \frac{|u|^2}{2} = -\int_\Omega \nabla u : \tau - \nu\int_\Omega |\nabla u|^2.$$

Adding (4), (5) and (6) yields the following formal decay of the free-energy

$$\text{(7)} \quad \partial_t \int_\Omega [\frac{h(t,x)}{2} + \frac{|u|^2}{2}] + \int_\Omega \nu|\nabla u|^2 + \frac{1}{2}\left[\frac{\mathrm{tr}(A)}{(1-\mathrm{tr}(A)/b)^2} - \frac{2D}{1-\mathrm{tr}(A)/b} + \mathrm{tr}(A^{-1})\right] = 0.$$

We refer to Hu and Lelièvre [20] for this derivation. We recall that we have the following inequalities for positive symmetric matrices [20]:

$$\frac{\mathrm{tr}(A)}{(1-\mathrm{tr}(A)/b)^2} - \frac{2D}{1-\mathrm{tr}(A)/b} + \mathrm{tr}(A^{-1}) \tag{8}$$

$$\geq -\log(\det A) - b\log(1-\mathrm{tr}(A)/b) + (b+D)\log(\frac{b}{b+D}) \geq 0. \tag{9}$$

Notice that both terms vanish when $\tau(A) = 0$, namely $A = \frac{b}{b+D} Id$.

Based on this decay, we can expect to construct global weak solutions such that $u \in L^\infty((0,T);L^2(\Omega)) \cap L^2((0,T);H^1_0(\Omega))$, $\tau \in L^2((0,T);L^2(\Omega))$, $A \in L^\infty((0,T)\times\Omega))$ and $H(t) \in L^\infty(0,T)$. The fact that τ is in L^2 comes from the dissipation of the free-energy that yields an L^2 bound on $\frac{1}{1-\mathrm{tr}(A)/b}$.

2.3. **Statement of the main result.** The main result of the paper is the proof of global existence of solutions to the FENE-P model (1) that satisfy in addition the free-energy bound (7) (with an inequality $\leq$ instead of the equality).

Theorem 2.1. *Let $u_0(x) \in L^2(\Omega)$ be a divergence free field and $A_0(x)$ a positive definite matrix function of x such that*

$$\int_\Omega -\log(\det A_0) - b\log(1-tr(A_0)/b) + (b+D)\log(\frac{b}{b+D}) < \infty. \tag{10}$$

Then, (1) has a global weak solution (u,A) such that $u \in L^\infty(\mathbb{R}_+;L^2) \cap L^2(\mathbb{R}_+;\dot{H}^1)$, $A \in L^\infty((0,T)\times\Omega))$ and $\tau \in L^2((0,T);L^2(\Omega))$ and (7) holds with an inequality $\leq$ instead of the equality.

Remark 2.2. *1) If Ω is a bounded domain, we can prove existence assuming only that $A_0(x)$ is a positive matrix function of x satisfying $\log(1-tr(A_0)/b) \in L^1$, instead of (10). In a sense this allows $detA_0$ to vanish on a set of non-zero measure. Indeed, one can replace h in the free energy bound (7) by $h = h_2$. This yields a quantity that may grow linearly in time and so gives uniform bounds on any fixed time interval.*

2) As it is classical when proving global existence of weak solutions satisfying the physical a priori estimates, the main difficulty is to prove weak compactness. Indeed, one can easily approximate the system by a sequence of more regular ones for which existence can be easily proved by a fixed point argument and such that a uniform free energy bound holds. This is way we will only concentrate on weak compactness in the next section

3. Weak compactness

We consider (u^n, A^n) a sequence of weak solutions to (1) satisfying the free energy bound (7) with an initial data (u^n_0, A^n_0) such that u^n_0 converges strongly to u_0 in L^2 and $A^n_0(x)$ converges strongly to A_0 in L^p for all $p < \infty$ and such that $-\log(1-\mathrm{tr}(A^n_0)/b)$ converges to $-\log(1-\mathrm{tr}(A_0)/b)$ in L^1.

We extract a subsequence such that u^n converges weakly to u in $L^p((0,T);L^2(\Omega)) \cap L^2((0,T);H^1_0(\Omega))$ and A^n converges weakly to A in $L^p((0,T)\times\Omega)$ for each $p < \infty$. We would like to prove that (u,A) is still a solution of (1). The main difficulty is to pass to the limit in the nonlinear terms.

First, we pass to the limit weakly in (1):

$$\left\{\begin{array}{l}\partial_t u + (u\cdot\nabla)u - \nu\Delta u + \nabla p = \mathrm{div}\tau, \quad \mathrm{div} u = 0, \\ \\ \partial_t A + u.\nabla A = \overline{\nabla u^n A^n} + \overline{A^n(\nabla u^n)^T} - \overline{\frac{A^n}{1-\mathrm{tr}(A^n)/b}} + \mathrm{Id} \\ \\ \tau = \overline{\frac{A^n}{1-\mathrm{tr}(A^n)/b}} - \mathrm{Id}\end{array}\right. \tag{11}$$

where here and below, $\overline{F^n}$ denotes the weak limit (modulo a subsequence extraction) of F^n when n goes to infinity.

Moreover, denoting $\overline{h_2^n} = \overline{-b\log(1-\mathrm{tr}(A^n)/b)}$ and passing to the limit in (5), we get

$$(\partial_t + u.\nabla)\overline{h_2^n} = 2\overline{\nabla u^n : \tau(A^n)} + \overline{\frac{D}{1-\mathrm{tr}(A^n)/b}} - \overline{\frac{\mathrm{tr}(A^n)}{(1-\mathrm{tr}(A^n)/b)^2}}. \tag{12}$$

Denoting $h_2 = -b\log(1-\mathrm{tr}(A)/b)$ and taking the log of the second equation in (11), we get

$$(\partial_t + u.\nabla)h_2 = 2\frac{\overline{\nabla u^n : A^n}}{1-\mathrm{tr}(A)/b} + \frac{D}{1-\mathrm{tr}(A)/b} - \overline{\frac{\mathrm{tr}(A^n)}{(1-\mathrm{tr}(A^n)/b)}}\frac{1}{1-\mathrm{tr}(A)/b}. \tag{13}$$

Notice that here and below, terms of the type $u.\nabla h_2$ are well defined due the divergence free condition and should be understood as $\mathrm{div} h_2 u$. Also, due to the fact that τ is in L^2, we deduce that h_2 and $\overline{h_2^n}$ are in all L^p space for $p<\infty$.

We introduce the following defect measures :

$$\begin{aligned}&\eta = \overline{h_2^n} - h_2 = \overline{-b\log(1-\mathrm{tr}(A^n)/b)} + b\log(1-\mathrm{tr}(A)/b),\\ &\beta = \overline{\nabla u^n : A^n} - \nabla u : A,\\ &\gamma = \overline{\nabla u^n : \tau(A^n)} - \nabla u : \overline{\tau(A^n)}.\end{aligned} \tag{14}$$

Hence, we deduce that η solves

$$\begin{aligned}(\partial_t + u.\nabla)\eta = {} & 2\left[\nabla u : \left(\overline{\tau(A^n)} - \tau(A)\right)\right]\\ & + 2[\gamma - \frac{\beta}{1-\mathrm{tr}(A)/b}] - \left[\overline{\frac{\mathrm{tr}(A^n)}{(1-\mathrm{tr}(A^n)/b)^2}} - \overline{\frac{\mathrm{tr}(A^n)}{(1-\mathrm{tr}(A^n)/b)}}\frac{1}{1-\mathrm{tr}(A)/b}\right]\\ & + D\left[\overline{\frac{1}{(1-\mathrm{tr}(A^n)/b)}} - \frac{1}{1-\mathrm{tr}(A)/b}\right].\end{aligned} \tag{15}$$

It turns out that this identity is not very useful since we can not identify γ. The main difficulty compared with [37] is that we can not prove that $|\tau^n|^2$ is equi-integrable in L^1. To overcome this difficulty, we will introduce some cut-off χ_δ^n before taking the weak limit and then send δ to zero. By doing so, we get an equation which is slightly different from (15) and which can be used to control the propagation of η (see (21)).

We define $\chi_\delta^n = 1_{\{1-\mathrm{tr}(A^n)/b\geq\delta\}}$. Hence, we get that for a fixed δ, $\tau^n\chi_\delta^n$ is bounded in L^∞. We also define $T_\delta^n(t,x)$ by

$$\left\{\begin{array}{ll} T_\delta^n(t,x) = h_2^n & \text{if} \quad 1-\mathrm{tr}(A^n)/b \geq \delta \\ T_\delta^n(t,x) = b\log\frac{1}{\delta} & \text{if} \quad 1-\mathrm{tr}(A^n)/b < \delta\end{array}\right. \tag{16}$$

We will introduce several defect measures which depend on δ and then send δ to zero:

$$\eta_\delta = \overline{T_\delta^n} - h_2 \tag{17}$$

$$\gamma_\delta = \overline{\nabla u^n : \tau(A^n)}^\delta - \nabla u : \overline{\tau(A^n)}^\delta. \tag{18}$$

where here and below, $\overline{F^n}^\delta$ denotes the weak limit (modulo a subsequence extraction) of $F^n\chi^n_\delta$ when n goes to infinity. Using the fact that formally, $\partial_t T^n_\delta(t,x) = \chi^n_\delta \partial_t h^n_2$ and passing to the limit in that equation, we get

$$
\begin{aligned}
(\partial_t + u.\nabla)\eta_\delta &= 2\left[\nabla u : \left(\overline{\tau(A^n)}^\delta - \tau(A)\right)\right] \\
&+ 2[\gamma_\delta - \frac{\beta}{1-\mathrm{tr}(A)/b}] - \left[\overline{\frac{\mathrm{tr}(A^n)}{(1-\mathrm{tr}(A^n)/b)^2}}^\delta - \overline{\frac{\mathrm{tr}(A^n)}{(1-\mathrm{tr}(A^n)/b)}}\frac{1}{1-\mathrm{tr}(A)/b}\right] \\
&+ D\left[\overline{\frac{1}{(1-\mathrm{tr}(A^n)/b)}}^\delta - \frac{1}{1-\mathrm{tr}(A)/b}\right]
\end{aligned} \tag{19}
$$

Since, τ^n is bounded in L^2, we deduce that meas$\{\chi^n_\delta = 0\}$ goes to zero when δ goes to zero uniformly in n and hence, if F^n is equi-integrable, we deduce that $\overline{F^n}^\delta$ goes to $\overline{F^n}$ when δ goes to zero.

Hence, sending δ to zero we deduce that all terms will go to their weak limits without the cut-off χ^n_δ except for those where we do not know that the sequence is equi-integrable. We introduce

$$
\tilde\gamma = \lim_{\delta\to 0}\gamma_\delta = \lim_{\delta\to 0}\overline{\nabla u^n : \tau(A^n)}^\delta - \nabla u : \overline{\tau(A^n)}. \tag{20}
$$

Hence, we have

$$
\begin{aligned}
(\partial_t + u.\nabla)\eta &= 2\left[\nabla u : \left(\overline{\frac{A^n}{(1-\mathrm{tr}(A^n)/b)}} - \frac{A}{1-\mathrm{tr}(A)/b}\right)\right] \\
&+ 2[\tilde\gamma - \frac{\beta}{1-\mathrm{tr}(A)/b}] - \left[\lim_{\delta\to 0}\overline{\frac{\mathrm{tr}(A^n)}{(1-\mathrm{tr}(A^n)/b)^2}}^\delta - \overline{\frac{\mathrm{tr}(A^n)}{(1-\mathrm{tr}(A^n)/b)}}\frac{1}{1-\mathrm{tr}(A)/b}\right] \\
&+ D\left[\overline{\frac{1}{(1-\mathrm{tr}(A^n)/b)}} - \frac{1}{1-\mathrm{tr}(A)/b}\right]
\end{aligned} \tag{21}
$$

Notice that (21) and (15) only differ by the fact that γ is replaced by $\tilde\gamma$ and that $\overline{\frac{\mathrm{tr}(A^n)}{(1-\mathrm{tr}(A^n)/b)^2}}$ is replaced by $\lim_{\delta\to 0}\overline{\frac{\mathrm{tr}(A^n)}{(1-\mathrm{tr}(A^n)/b)^2}}^\delta$. Indeed, these two terms are coming from sequences that are not known to be equi-integrable. In particular we note that $\tilde\gamma$ is the Chacon limit of $\nabla u^n : \tau(A^n) - \nabla u : \overline{\tau(A^n)}$ whereas γ is its weak limit in the sense of Radon measures.

Due to the presence of A^n and not only its trace on the right hand side of (21), we need to study the propagation of an other defect measure. We introduce $\varepsilon = \mathrm{tr}(\overline{(A^n)^2} - A^2) = \overline{|A^n - A|^2}$. Indeed, recall that for a symmetric matrix A, we have $|A|^2 = \sum_{i,j+1}^D a_{ij}^2 = \mathrm{tr}(A^2)$. On one hand, multiplying the second equation of (11) by $2A$, and taking the trace, we get

$$
\partial_t|A|^2 + u.\nabla|A|^2 = 2\mathrm{tr}\left[\overline{\nabla u^n A^n}A + A\overline{A^n(\nabla u^n)^T} - \overline{\frac{A^n}{1-\mathrm{tr}(A^n)/b}}A + A\right]
$$

On the other hand, passing to the limit in the equation of $(A^n)^2$, we get

$$
\partial_t\overline{|A^n|^2} + u.\nabla\overline{|A^n|^2} = 2\mathrm{tr}\left[\overline{\nabla u^n A^n A^n} + \overline{A^n A^n(\nabla u^n)^T} - \overline{\frac{A^n A^n}{1-\mathrm{tr}(A^n)/b}} + A\right]. \tag{22}
$$

Hence, we get

$$\partial_t \varepsilon + u.\nabla\varepsilon = 4\mathrm{tr}\left[\overline{\nabla u^n A^n A^n} - \overline{\nabla u^n A^n} A\right] - 2\mathrm{tr}\left[\overline{\frac{A^n A^n}{1-\mathrm{tr}(A^n)/b}} - \overline{\frac{A^n}{1-\mathrm{tr}(A^n)/b}} A\right]. \tag{23}$$

3.1. **Identification of $\tilde{\gamma}$.** In this subsection, we give a relation between $\tilde{\gamma}$ and some defect measure related to the lack of strong convergence of ∇u^n in L^2. To state the main proposition of this subsection, we introduce few notations. Let $u^n = v^n + w^n$ where v^n and w^nsolve

$$\begin{cases} \partial_t v^n - \Delta v^n + \nabla p_1^n = \nabla.\tau^n \\ v^n(t=0) = 0 \end{cases} \tag{24}$$

$$\begin{cases} \partial_t w^n - \Delta w^n + \nabla p_2^n = -u^n.\nabla u^n \\ v^n(t=0) = u^n(t=0). \end{cases} \tag{25}$$

We further split w^n into $w_1^n + w_2^n$ where w_1^n is the solution with zero initial data and w_2^n is the solution with zero right hand side.

We also define $v^{n,\delta}$ the solution of

$$\begin{cases} \partial_t v^{n,\delta} - \Delta v^{n,\delta} + \nabla p_1^{n,\delta} = \nabla.\tau^{n,\delta} \\ v^{n,\delta}(t=0) = 0 \end{cases} \tag{26}$$

where $\tau^{n,\delta} = \tau^n \chi_\delta^n$. Extracting a subsequence, we assume that $(\nabla v^{n,\delta}, \nabla v^n, \nabla w^n, \tau^{n,\delta})$ converges weakly in L^2 to some $(\nabla v^\delta, \nabla v, \nabla w, \tau^\delta)$ and that

$$\overline{|\nabla v^{n,\delta}|^2} = |\nabla v^\delta|^2 + \mu_\delta \tag{27}$$

for some defect measure $\mu_\delta \in \mathcal{M}((0,T)\times\Omega)$. Actually, since for fixed δ, we know that $\tau^{n,\delta}$ is bounded in L^∞, we deduce that $\nabla v^{n,\delta}$ is bounded in $L^q((0,T)\times\Omega)$ for all $q<\infty$ and hence $\mu_\delta \in L^q((0,T)\times\Omega)$ for all $q<\infty$, but of course without uniform bound in δ. We also introduce the measure $\mu = \lim_{\delta\to 0}\mu_\delta$.

Proposition 3.1. *We have*

$$\mu = \lim_{\delta\to 0}\mu_\delta = -\tilde{\gamma}. \tag{28}$$

Proof of Proposition 3.1. The proof is very similar to the proof of Proposition 5.4 of [41]. It is actually easier since we know that $\tau^{n,\delta}$ is uniformly bounded in L^∞ for fixed δ

We introduce the following weak limits

$$\overline{\tau^{n,\delta} : \nabla v^{n,\delta}} = W^{\delta\delta} \tag{29}$$

$$\overline{\tau^{n,\delta} : \nabla v^{n}} = W^{\delta}. \tag{30}$$

Step 1: First, we would like to prove that $W^{\delta\delta}$ and W^δ have the same limit W when δ goes to zero and that this limit is in L^1. To prove this, we introduce for $M>0$, the following weak limits

$$\overline{\tau^{n,\delta} 1_{|\tau^{n,\delta}|\le M} : \nabla v^{n,\delta}} = W_M^{\delta\delta}, \tag{31}$$

$$\overline{\tau^{n,\delta} 1_{|\tau^{n,\delta}|> M} : \nabla v^{n,\delta}} = W^{\delta\delta} - W_M^{\delta\delta}, \tag{32}$$

$$\overline{\tau^{n,\delta} 1_{|\tau^{n,\delta}|\le M} : \nabla v^{n}} = W_M^{\delta}, \tag{33}$$

and

$$\overline{|\tau^{n,\delta} 1_{|\tau^{n,\delta}|\le M}|^2} = G_M^\delta \qquad \overline{|\tau^{n,\delta}|^2} = G^\delta. \tag{34}$$

Since for a fixed δ, $|\tau^{n,\delta}|^2$ is bounded and hence equiintegrable, we deduce that G_M^δ converges to G^δ in L^1 when M goes to infinity and is monotone in M. Also, by monotone convergence, we deduce that there exists $G\in L^1$ such that G^δ converges to G in L^1 when δ goes to zero. Actually, G is the weak limit of $|\tau^n|^2$ in the sense of Chacon (see the appendix 5 and [1]).

Let us fix $\varepsilon > 0$. We choose δ_0 and M_0 such that for $\delta < \delta_0$ and $M > M_0$, we have $\|G - G^\delta\|_{L^1} + \|G - G^\delta_M\|_{L^1} \le \varepsilon$. Using the fact that

$$\overline{|\tau^{n,\delta}|^2} = \overline{|\tau^{n,\delta} 1_{|\tau^{n,\delta}| \le M}|^2} + \overline{|\tau^{n,\delta} 1_{|\tau^{n,\delta}| > M}|^2} \tag{35}$$

$$= G^\delta_M + \quad (G^\delta - G^\delta_M), \tag{36}$$

we deduce that for $\delta < \delta_0$ and $M > M_0$, we have for all n $\|\tau^{n,\delta} 1_{|\tau^{n,\delta}|>M}\|^2_{L^2} \le \varepsilon$ and hence, by Cauchy-Schwarz we deduce that $\|W^{\delta\delta} - W^{\delta\delta}_M\|_{L^1} \le C\sqrt{\varepsilon}$ and that $\|W^\delta - W^\delta_M\|_{L^1} \le C\sqrt{\varepsilon}$. Hence to prove that $\lim_\delta W^{\delta\delta} = \lim_\delta W^\delta$, it is enough to prove it for the M approximation, namely that

$$\lim_\delta W^{\delta\delta}_M = \lim_\delta W^\delta_M. \tag{37}$$

To prove (37), we first notice that $\tau^{n,\delta} - \tau^n$ goes to zero in L^p for $p < 2$ when δ goes to zero uniformly in n. Then, by parabolic regularity of the Stokes system, we deduce that $\|\nabla v^{n,\delta} - \nabla v^n\|_{L^p((0,T)\times\Omega)}$ goes to zero when δ goes to zero uniformly in n for $p < 2$. Hence, (37) holds.

Step 2: In this second step, we will compare the local energy identity of the weak limit of (26) with the weak limit of the local energy identity of (26).

On one hand, passing to the limit in (26) and multiplying by v^δ, we deduce that

$$\partial_t \frac{|v^\delta|^2}{2} - \Delta \frac{|v^\delta|^2}{2} + |\nabla v^\delta|^2 + \operatorname{div}(p_1^\delta v^\delta) = \operatorname{div}(v^\delta.\tau^\delta) - \nabla v^\delta : \tau^\delta. \tag{38}$$

On the other hand, reversing the order, we get

$$\partial_t \frac{|v^\delta|^2}{2} - \Delta \frac{|v^\delta|^2}{2} + |\nabla v^\delta|^2 + \mu_\delta + \operatorname{div}(p_1^\delta v^\delta) = \operatorname{div}(v^\delta.\tau^\delta) - W^{\delta\delta}. \tag{39}$$

Comparing (38) and (39), we deduce that

$$W^{\delta\delta} = \nabla v^\delta : \tau^\delta - \mu_\delta. \tag{40}$$

We would like now to use this to compute the limit of γ_δ when δ goes to zero.

First from the energy estimate, we recall that u^n is bounded in $L^\infty((0,T);L^2(\Omega)) \cap L^2((0,T);\dot H^1(\Omega))$ and hence by Sobolev embeddings that u^n is bounded in $L^{\frac{2(D+2)}{D}}((0,T)\times\Omega)$ and that $u^n \nabla u^n$ is bounded in $L^{\frac{D+2}{D+1}}((0,T)\times\Omega)$. By parabolic regularity of the Stokes operator applied to (25) with zero initial data, we deduce that ∇w_1^n is bounded in $L^{\frac{D+2}{D+1}}((0,T);W^{1,\frac{D+2}{D+1}}\Omega)$ and that $\partial_t w_1^n$ is bounded in $L^{\frac{D+2}{D+1}}((0,T)\times\Omega)$. Since τ^n is bounded in L^2, we deduce from (24) that ∇v^n is also bounded in $L^2((0,T)\times\Omega)$ and hence $\nabla w^n = \nabla u^n - \nabla v^n$ is also bounded in $L^2((0,T)\times\Omega)$. Moreover, it is clear that ∇w_2^n is compact in $L^2((0,T)\times\Omega)$ and hence ∇w_1^n is also bounded in L^2 and from the previous bounds on ∇w_1^n, we deduce that ∇w_1^n is compact in $L^p((0,T)\times\Omega)$ for $p < 2$. Hence, we deduce that

$$\overline{\nabla w^n : \tau(A^n)}^\delta = \nabla w : \overline{\tau(A^n)}^\delta$$

since ∇w^n is compact in $L^p((0,T)\times\Omega)$ for $p < 2$ and $\tau(A^n)\chi^n_\delta$ is bounded in L^∞ for fixed δ. Therefore, (18) yields $\gamma_\delta = \overline{\nabla v^n : \tau(A^n)}^\delta - \nabla v : \overline{\tau(A^n)}^\delta$.

On one hand, we deduce from (20) that

$$\lim_\delta W^\delta = \overline{\nabla v^n : \tau(A^n)}^\delta = \nabla v : \overline{\tau(A^n)} + \tilde\gamma.$$

On the other hand, we deduce from (40) that

$$\lim_{\delta} W^{\delta\delta} = \nabla v : \overline{\tau(A^n)} - \lim_{\delta\to 0} \mu_\delta.$$

Finally, we deduce that $\lim_{\delta\to 0}\mu_\delta = -\tilde{\gamma}$.

3.2. **Propagation of compactness.** We introduce the following measures :

$$\begin{aligned}
\alpha_1 &= \left(\overline{\frac{A^n}{(1-\mathrm{tr}(A^n)/b)}} - \frac{A}{1-\mathrm{tr}(A)/b}\right) = (\overline{\tau(A^n)} - \tau(A)) \\
\kappa &= \left[\lim_{\delta\to 0}\overline{\frac{\mathrm{tr}(A^n)}{(1-\mathrm{tr}(A^n)/b)^2}}^{\delta} - \overline{\frac{\mathrm{tr}(A^n)}{(1-\mathrm{tr}(A^n)/b)}}\frac{1}{1-\mathrm{tr}(A)/b}\right] \\
\alpha_2 &= \left[\overline{\frac{1}{(1-\mathrm{tr}(A^n)/b)}} - \frac{1}{1-\mathrm{tr}(A)/b}\right].
\end{aligned} \tag{41}$$

Hence, using Proposition 3.1, (21) becomes

$$(\partial_t + u.\nabla)\eta = \nabla u : \alpha_1 + 2[\tilde{\gamma} - \frac{\beta}{1-\mathrm{tr}(A)/b}] - \kappa + D\alpha_2. \tag{42}$$

We also introduce the defect measures appearing in (23)

$$\begin{aligned}
\alpha_3 &= \mathrm{tr}\left[\overline{\frac{A^nA^n}{1-\mathrm{tr}(A^n)/b}} - \overline{\frac{A^n}{1-\mathrm{tr}(A^n)/b}}A\right] \\
\beta_2 &= \mathrm{tr}\left[\overline{\nabla u^n A^n A^n} - \overline{\nabla u^n A^n}A\right].
\end{aligned} \tag{43}$$

Hence, (23) becomes

$$(\partial_t + u.\nabla)\varepsilon = 4\beta_2 - 2\alpha_3 \tag{44}$$

From the strong convergence of the initial data, we deduce that $\eta+\varepsilon(t=0)=0$. Indeed, to see this we have to use some time regularity. Using that $\partial_t A^n$ is bounded in $L^2(0,T;H^{-1}(\Omega))$, we deduce that for each $\phi(t,x)\in C^1([0,T);H^1(\Omega))$, we have $\int_\Omega \phi(0)A^n(0) = -\int_0^T\int_\Omega \phi\partial_t A^n + \phi\partial_t\phi A^n$ where $\int_\Omega \phi\partial_t\phi A^n$ should be understood in the sense of the H^1, H^{-1} duality. Passing to the limit we deduce that $A^n(0)$ converges weakly to $A(0)$. From the hypothesis we know that $A^n(0)$ converges strongly to A_0 in L^2. Hence $A(0)=A_0$. In a similar way and using the equation satisfied by $(A^n)^2$ as well as its weak limit (22), we deduce that $\varepsilon(t=0) = \overline{|A^n(0)|^2} - |A(0)|^2 = 0$. The same is true for $\eta(t=0)$ by noticing that $\eta_\delta(t=0) = b\inf(-\log(1-\mathrm{tr}(A(0))/b), \log\frac{1}{\delta}) + b\log(1-\mathrm{tr}(A(0))/b)$ and then sending δ to zero. Hence, $\eta(t=0)=0$.

We would like to prove that $\eta+\varepsilon$ is identically equal to zero by applying a Gronwall lemma. We have just to control all the defect measures appearing on the right hand side of (42) and (44) using μ, κ and $\eta+\varepsilon$. We start with the most difficult term

Proposition 3.2. *We have the following bound*

$$|\alpha_2| \le C\sqrt{\eta\kappa} \qquad \textit{for } i=1,2. \tag{45}$$

We denote $f_n = \mathrm{tr}(A^n)/b$. For simplicity, we take $b=1$. To prove this proposition, we introduce two other defect measures related to κ

$$\kappa_1 = \left[\lim_{\delta\to 0} \overline{\frac{1}{(1-f^n)^2}}^{\delta} - \frac{1}{(1-f)^2} \right]$$

$$\kappa_2 = \left[\lim_{\delta\to 0} \overline{\frac{1}{(1-f^n)^2}}^{\delta} - \overline{\frac{1}{(1-f^n)}} \frac{1}{1-f} \right] \tag{46}$$

Hence, the proposition is proved if we prove the following lemma

Lemma 3.3. *The following bounds hold*
1) $\alpha_2 \leq C\sqrt{\kappa_1 \eta}$
2) $\kappa_1 \leq 2\kappa_2 \leq 4\kappa$

To prove the first claim, we use a decomposition in power series. Of course, one has to invert two limits to do it and the presence of the cut-off factor χ_δ^n allows us to do it. Let

$$\alpha_2 = \sum_{k=1}^{\infty} [\overline{(f^n)^k} - f^k],$$

$$\kappa_1 = \sum_{k=1}^{\infty} (k+1)[\overline{(f^n)^k} - f^k],$$

$$\eta = \sum_{k=1}^{\infty} \frac{1}{k} [\overline{(f^n)^k} - f^k]. \tag{47}$$

We write

$$(\alpha_2)^2 = \sum_{k,l=1}^{\infty} [\overline{(f^n)^k} - f^k][\overline{(f^n)^l} - f^l]$$

$$\kappa_1 \eta = \sum_{k,l=1}^{\infty} \frac{1}{2}(\frac{l+1}{k} + \frac{k+1}{l})[\overline{(f^n)^k} - f^k][\overline{(f^n)^l} - f^l] \tag{48}$$

and the claim follows since $[\overline{(f^n)^k} - f^k] \geq 0$ for each k and $\frac{l+1}{k} + \frac{k+1}{l} \geq 2$.
To prove the second claim, we write

$$2\kappa_2 - \kappa_1 = \lim_{\delta\to 0} \overline{\frac{1}{(1-f^n)^2}}^{\delta} - 2\overline{\frac{1}{(1-f^n)}} \frac{1}{1-f} + \frac{1}{(1-f)^2}$$

$$= \lim_{\delta\to 0} \overline{\left(\frac{1}{(1-f^n)} - \frac{1}{(1-f)} \right)^2}^{\delta} \geq 0. \tag{49}$$

Hence, $\kappa_1 \leq 2\kappa_2$. For the second inequality, we use that $\kappa_2 = \kappa + \alpha$ and then decompose κ in power series

$$\kappa = \sum_{k=1}^{\infty} \left[(k+1)\overline{(f^n)^{k+1}} - \sum_{i=0}^{k} \overline{(f^n)^{i+1}} f^{k-i} \right]$$

$$\geq \sum_{k=1}^{\infty} [\overline{(f^n)^{k+1}} - f^{k+1}] = \alpha_2. \tag{50}$$

Now, we will control the other terms. To control $\beta = \overline{\nabla u^n : A^n} - \nabla u : A$, we first recall that $\nabla u^n = \nabla v^n + \nabla w^n$ and ∇w^n is compact in $L^p((0,T)\times\Omega)$ for $p<2$. Hence, $\beta = \overline{\nabla v^n : A^n} - \nabla v : A$. Now, since A^n is bounded, $\nabla v^n : A^n$ is equiintegrable and $\overline{\nabla v^n : A^n} = \lim_{\delta\to 0} \overline{\nabla v^n : A^n}^{\delta}$. Hence,

$$|\beta| \leq C\sqrt{\varepsilon\mu}. \tag{51}$$

For β_2, we have

$$\begin{aligned}
\beta_2 &= \mathrm{tr}\left[\overline{\nabla u^n A^n (A^n - A)}\right] \\
&= \mathrm{tr}\left[\overline{(\nabla u^n - \nabla u) A^n (A^n - A)} + \nabla u \overline{A^n (A^n - A)}\right] \\
&\leq C\sqrt{\varepsilon\mu} + C|\nabla u|\varepsilon.
\end{aligned}$$

For α_1 and α_3, we argue in a similar way. We have

$$\begin{aligned}
\alpha_1 &= \overline{\frac{A^n - A}{(1 - \mathrm{tr}(A^n)/b)}} + A\alpha_2 \\
&= \overline{(A^n - A)\left[\frac{1}{(1 - \mathrm{tr}(A^n)/b)} - \overline{\frac{1}{(1 - \mathrm{tr}(A^n)/b)}}\right]} + A\alpha_2.
\end{aligned}$$

Hence, $|\alpha_1| \leq C(\sqrt{\varepsilon\kappa_3} + \alpha_2)$ where

$$\kappa_3 = \left[\lim_{\delta\to 0} \overline{\frac{1}{(1 - \mathrm{tr}(A^n)/b))^2}}^{\delta} - \overline{\frac{1}{(1 - \mathrm{tr}(A^n)/b)}}^{2}\right]$$

satisfies $\kappa_3 \leq \kappa$.

Similarly, we have $\alpha_3 = \mathrm{tr}\overline{\tau(A^n)(A^n - A)}$. Hence,

$$\begin{aligned}
\alpha_3 &= \overline{\frac{A^n(A^n - A)}{(1 - \mathrm{tr}(A^n)/b)}} \\
&= \overline{A^n(A^n - A)\left[\frac{1}{(1 - \mathrm{tr}(A^n)/b)} - \overline{\frac{1}{(1 - \mathrm{tr}(A^n)/b)}}\right]} + \overline{\frac{1}{(1 - \mathrm{tr}(A^n)/b)}}\,\overline{A^n(A^n - A)}.
\end{aligned}$$

and we deduce that $|\alpha_3| \leq C\sqrt{\varepsilon\kappa_3} + \overline{\frac{1}{(1-\mathrm{tr}(A^n)/b)}}\varepsilon$.

Putting all the estimates together, we get (at least formally) that

$$(\partial_t + u.\nabla)(\eta + \varepsilon) + \kappa + 2\mu \leq C\,F[\sqrt{(\eta + \varepsilon)(\mu + \kappa)} + \varepsilon] \tag{52}$$

where F is given by $F = 1 + |\nabla u| + \overline{\frac{1}{(1-\mathrm{tr}(A^n)/b)}}$ satisfies that $F \in L^2((0,T)\times\Omega)$.

Actually, the term $F\sqrt{(\eta + \varepsilon)(\mu + \kappa)}$ does not make sense in the sense of distribution. Indeed, $\sqrt{(\mu + \kappa)}$ is in L^2 and we do not have an L^∞ bound on η. To overcome this difficulty, we use a renormalized form of (52), namely

$$(\partial_t + u.\nabla)\zeta + \frac{\kappa + 2\mu}{(1 + \eta + \varepsilon)^2} \leq C\,F[\sqrt{\zeta\frac{\kappa + 2\mu}{(1 + \eta + \varepsilon)^2}} + \zeta] \tag{53}$$

where $\zeta = \frac{\eta+\varepsilon}{1+\eta+\varepsilon}$ is bounded and satisfies $\zeta(t=0) = 0$. We also introduce the unique a.e flow X in the sense of DiPerna and Lions [10] of u, solution of

$$\partial_t X^n(t,x) = u^n(t, X(t,x)) \qquad X^n(t=0,x) = x. \tag{54}$$

Hence, we deduce that for almost all x, we have

$$\frac{\partial\zeta(t,X(t,x))}{\partial t} \leq C(1+|F(t,X(t,x))|)\zeta(t,X(t,x)) \tag{55}$$

and by Gronwall lemma, we deduce that ζ vanishes almost everywhere and hence we deduce that κ and μ also vanish. This of course allows us to pass to the limit in all the nonlinear term in (1). This concludes the proof of weak compactness.

Now, to prove Theorem 2.1, one has to reproduce this same proof for a sequence of solutions to a regularized version of (1). We do not detail this part here since it is standard.

4. The Giesekus and the PTT models

In the Giesekus model, the second equation in (1) is replaced by an equation on τ:

$$\begin{cases} \partial_t u + (u\cdot\nabla)u - \nu\Delta u + \nabla p = \mathrm{div}\tau, \quad \mathrm{div} u = 0, \\ \lambda\Big(\partial_t\tau + u.\nabla\tau - \nabla u\tau - \tau(\nabla u)^T\Big) + \tau + \alpha\tau\tau + \xi(D(u)\tau + \tau D(u)) = 2\eta D(u) \end{cases} \tag{56}$$

where $D(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$ is the symmetric part of ∇u. Here, λ is a relaxation time, $\eta > 0$ is an extra viscosity and $\alpha > 0$ is a constant which measure the effect of the extra nonlinear term $\tau\tau$ and ξ is a constant that is typically in $[0, 2\lambda]$. In the case $\xi = 0$, we get the upper convective model and in the case $\xi = \lambda$ we get the co-rotational model. We will detail the case where $\xi = 0$ and discuss in remark 4.4 the case where $\xi \neq 0$.

In the Phan-Thien and Tannes (PTT) model, the second equation in (56) is replaced by

$$\lambda(\partial_t\tau + u.\nabla\tau - \nabla u\tau - \tau(\nabla u)^T) + \tau + \alpha\mathrm{tr}(\tau)\tau = 2\eta D(u) \tag{57}$$

The main difference with the Giesekus model (56) is that the quadratic term $\alpha\tau\tau$ is replaced by $\alpha\mathrm{tr}(\tau)\tau$. In the sequel, we will only concentrate on the Giesekus model. The proofs for the Phan-Thien and Tannes model are exactly the same. They are even simpler since there is less matrix calculation involved.

4.1. **Free energy.** Using the fact that $\mathrm{tr}D(u) = 0$, we get the following estimate

$$\partial_t \int_\Omega [\mathrm{tr}\tau + \frac{|u|^2}{2}] + \int_\Omega \nu|\nabla u|^2 + \frac{1}{\lambda}\mathrm{tr}(\tau) + \frac{\alpha}{\lambda}\mathrm{tr}(\tau\tau) = 0. \tag{58}$$

Notice, that τ is not necessary a positive symmetric matrix. To overcome this problem, we consider $A = Id + \frac{\lambda}{\eta}\tau$. Hence, A solves

$$\lambda(\partial_t A + u.\nabla A - \nabla u A - A(\nabla u)^T) + (A - Id) + \frac{\alpha\eta}{\lambda}(A-Id)^2 = 0 \tag{59}$$

Taking the determinant, we get that

$$(\partial_t + u.\nabla)\mathrm{det}A + \mathrm{det}A\,\mathrm{tr}\Big[A^{-1}[(A-Id) + \frac{\alpha\eta}{\lambda}(A-Id)^2]\Big] = 0. \tag{60}$$

Using that for positive symmetric matrices, we have $\mathrm{tr}[A^{-1}(A-Id)^2] \geq 0$ and that $\frac{1}{d}\mathrm{tr}[A^{-1}] \geq (\mathrm{det}A^{-1})^{1/d}$, we deduce easily that

$$\lambda(\partial_t + u.\nabla)(\mathrm{det}A)^{1/d} \geq (1 - (\mathrm{det}A)^{1/d}). \tag{61}$$

Hence, if at $t = 0$, we have $(\mathrm{det}A)^{1/d} \geq 1$, this property will be propagated for late time and so A remains a positive symmetric matrix and $\mathrm{tr}A \geq d$ which means that $\mathrm{tr}\tau \geq 0$. We have the following decay of the free energy

$$\partial_t \int_\Omega [-\ln\det A - d + \mathrm{tr} A + \frac{|u|^2}{2}] + \int_\Omega \nu |\nabla u|^2 \\ + \frac{1}{\lambda}\int_\Omega \mathrm{tr}[(Id - A^{-1})^2 A + \frac{\alpha\eta}{\lambda}(A - Id)^2 + \frac{\alpha\eta}{\lambda} A^{-1}(A - Id)^2] = 0. \tag{62}$$

Recall for a positive symmetric matrix, we have

$$0 \leq -\ln\det A - d + \mathrm{tr} A \leq \mathrm{tr}[(Id - A^{-1})^2 A]$$
$$\mathrm{tr}[(A - Id)^2] = |(A - Id)|^2 = |\tau|^2.$$

4.2. **Statement of the result.**

Theorem 4.1. *Let $u_0(x) \in L^2(\Omega)$ be a divergence free field and $A_0(x) = \frac{1}{\eta}\tau_0(x) + Id$ a positive definite matrix function of x such that $detA_0 \geq 1$ and $A_0 \in L^1_{loc}(\Omega)$ and*

$$\int_\Omega -\log(\det A_0) - d + trA_0 < C. \tag{63}$$

Then, (56) has a global weak solution (u, A) such that $u \in L^\infty(\mathbb{R}_+; L^2) \cap L^2(\mathbb{R}_+; \dot{H}^1)$, $A \in L^\infty((0,T); L^1_{loc}(\Omega))$ and $\tau \in L^2((0,T); L^2(\Omega))$. Moreover, (62) holds with an inequality $\leq$ instead of the equality.

Remark 4.2. *If $\eta = 0$, we can perform the above computation by taking the determinant of τ:*

$$\lambda(\partial_t + u.\nabla) det\tau = -det\tau\ tr[Id + \alpha\tau]. \tag{64}$$

Since, $tr[Id + \alpha\tau]$ is in L^1_{loc}, we deduce as in the proof of ...., that if at $t = 0$, $det\tau \geq 0$, then this property will be conserved for later times almost everywhere. In this case, we can use (58) as a free energy estimate instead of (62) and theorem (4.1) becomes

Corollary 4.3. *(Case $\eta = 0$.) Let $u_0(x) \in L^2(\Omega)$ be a divergence free field and $\tau_0(x)$ a positive matrix function of x such that $\tau_0 \in L^1(\Omega)$. Then, (56) has a global weak solution (u, τ) such that $u \in L^\infty(\mathbb{R}_+; L^2) \cap L^2(\mathbb{R}_+; \dot{H}^1)$, $\tau \in L^\infty((0,T); L^1(\Omega))$ and $\tau \in L^2((0,T); L^2(\Omega))$. Moreover, (58) holds with an inequality $\leq$ instead of the equality.*

Remark 4.4. *If $\xi \neq 0$, we have to change slightly the definition of A. In particular if $\xi < \lambda$, we can take $A = A = Id + \frac{\lambda-\xi}{\eta}\tau$ and hence (59) becomes*

$$\lambda(\partial_t A + u.\nabla A - \nabla u A - A(\nabla u)^T) + (A - Id) + \frac{\alpha\eta}{\lambda - \xi}(A - Id)^2 + \xi(D(u)A + AD(u)) = 0. \tag{65}$$

The rest of the argument is the same. However, in the case $\xi \geq \lambda$, it is not clear how to adapt the same argument and we hope to come back to this problem in a forthcoming work.

4.3. **Proof of theorem 4.1.** In the sequel, the constants λ and η will be taken equal to 1 and $\xi = 0$. We will only sketch the proof since it is very similar to the proof in the FENE-P case.

We consider $(u^n, \tau^n = A^n - Id)$ a sequence of weak solutions to (56) satisfying the free energy bound (62) with an initial data $(u_0^n, \tau_0^n = A_0^n - Id)$ such that u_0^n converges strongly to u_0 in L^2 and $A_0^n(x)$ converges strongly to A_0 in L^1_{loc} and A_0^n satisfies (63) with a uniform constant.

We extract a subsequence such that u^n converges weakly to u in $L^p((0,T); L^2(\Omega)) \cap L^2((0,T); H^1_0(\Omega))$ and τ^n converges weakly to τ in $L^2((0,T) \times \Omega)$. We would like to prove that (u, τ) is still a solution of (56).

Step 1 : Case $\frac{\alpha\eta}{\lambda} = 1$. In this case we can replace (59) by

$$\lambda(\partial_t F + u.\nabla F - \nabla u F) + \frac{1}{2}(FF^T F - F) = 0 \tag{66}$$

and recover A by taking $A = FF^T$. Indeed, it is easy to see that if F solves (66) then A solves (59).

we consider $F_0^n(x)$ a matrix function such that $F_0^n(F_0^n)^T = A_0^n$ and such that F_0^n converges strongly to F_0 in $L^2_{loc}(\Omega)$. One can just take $F_0^n = (A_0^n)^{1/2}$. We assume that instead of being given (u^n, A^n), we are given (u^n, F^n) such that u^n solves the first equation of (56) with $\tau^n = F^n(F^n)^T$ and F^n solves (66). We also assume that F^n converges weakly to some F in $L^4_{loc}((0,T)\times\Omega)$. On one hand, we pass to the limit in the system solved by (u, FF^T) and get

$$\begin{cases} \partial_t u + (u\cdot\nabla)u - \nu\Delta u + \nabla p = \mathrm{div}\tau, \quad \mathrm{div} u = 0, \\ \\ \partial_t A + u.\nabla A - \overline{\nabla u^n A^n} - \overline{A^n(\nabla u^n)^T} + (\overline{A^n A^n} - A) = 0 \end{cases} \tag{67}$$

where $A = \overline{F^n(F^n)^T}$. On the other hand, we pass to the limit in (66) and get

$$\partial_t F + u.\nabla F - \overline{\nabla u^n F^n} + \frac{1}{2}(\overline{F^n(F^n)^T F^n} - F) = 0 \tag{68}$$

As in (14), we introduce the defect measures (here η has a different meaning from the one in (56))

$$\eta = \mathrm{tr}(\overline{F^n(F^n)^T} - FF^T)$$
$$\gamma = \overline{\nabla u^n : \tau^n} - \nabla u : \tau.$$

Hence, we deduce that η solves

$$\begin{aligned}(\partial_t + u.\nabla)\eta = 2\mathrm{tr}\left[\nabla u\overline{F^n(F^n)^T} - \overline{\nabla u^n F^n}F^T\right] + 2\gamma \\ - (\overline{F^n(F^n)^T F^n(F^n)^T} - \overline{F^n(F^n)^T F^n}F^T) + (\overline{F^n(F^n)^T} - FF^T)\end{aligned} \tag{69}$$

Actually, to be able to use the same argument as in subsection 3.1, we have to introduce a cut-off. Take β a C^∞ function such that $\beta(t) = t$ for $0 < t < 1/2$ and $\beta(t) = 1$ for $2 < t$ and define $\beta_\delta(t) = \frac{1}{\delta}\beta(\delta t)$. We denote $\chi^n_\delta = \beta'_\delta(\mathrm{tr}(F^n(F^n)^T)$ and denote $\overline{G^n}^\delta$ the weak limit (modulo a subsequence extraction) of $G^n\chi^n_\delta$ when n goes to infinity. We will not detail this here.

Now, we estimate the different terms on the right hand side of (69). We have

$$\beta = \nabla u\overline{F^n(F^n)^T} - \overline{\nabla u^n F^n}F^T = \nabla u\overline{(F^n - F)(F^n)^T} - \overline{(\nabla u^n - \nabla u)F^n}F^T$$

and hence $|\beta| \le C(|\nabla u|\eta + |F|\sqrt{\mu\eta})$.

$$\begin{aligned}\kappa_1 &= \lim_{\delta\to 0}\overline{F^n(F^n)^T F^n(F^n)^T}^\delta - \overline{F^n(F^n)^T F^n}F^T \\ &= \kappa - \overline{F^n(F^n)^T(F^n - F)}F^T + A\,\overline{(F^n - F)(F^n - F)^T}\end{aligned} \tag{70}$$

where we set $\kappa = \lim_{\delta\to 0}\overline{F^n(F^n)^T(F^n(F^n)^T - \overline{F^n(F^n)^T})}^\delta \ge 0$. Notice that the second term in (70) is controlled by $|F|\sqrt{\kappa\eta}$ and that the last term is controlled by $|F|^2\eta$. Finally, we get that

$$(\partial_t + u.\nabla)\eta + \frac{1}{2}(\mu + \kappa) \le C(|\nabla u| + |A|)\eta \tag{71}$$

and so if at $t = 0$, $\eta = 0$, this will be the case for later times and hence, one can pass to the limit in (66) and recover a solution of (66) and hence, $A = FF^T$ is a solution of (56).

Step 2 : General case. In general we can not use the F formulation. We introduce

$$\eta = \mathrm{tr}(A) - \overline{\sqrt{\mathrm{tr}(A^n)}}^2 \tag{72}$$

Passing to the limit in the equation of $\mathrm{tr}(A^n)$, we get

$$(\partial_t + u.\nabla)\mathrm{tr}A - 2\mathrm{tr}\overline{\nabla u^n A^n} + \mathrm{tr}(A - Id) + \alpha\mathrm{tr}\overline{(A^n - Id)^2} = 0 \tag{73}$$

As we did in section 3 and in Step1, we have to renormalize this equation before passing to the limit. Take β a C^∞ function such that $\beta(t) = t$ for $0 < t < 1/2$ and $\beta(t) = 1$ for $2 < t$ and define $\beta_\delta(t) = \frac{1}{\delta}\beta(\delta t)$. We denote $\chi_\delta^n = \beta_\delta'(\mathrm{tr}(A^n))$ and denote $\overline{G^n}^\delta$ the weak limit (modulo a subsequence extraction) of $G^n\chi_\delta^n$ when n goes to infinity. Hence, we get

$$(\partial_t + u.\nabla)\mathrm{tr}A - 2\mathrm{tr}\lim_{\delta\to 0}\overline{\nabla u^n A^n}^\delta + \mathrm{tr}(A - Id) + \alpha\lim_{\delta\to 0}\mathrm{tr}\overline{(A^n - Id)^2}^\delta = 0 \tag{74}$$

If we pass to the limit in the equation of $\sqrt{\mathrm{tr}(A^n)}$, we get

$$(\partial_t + u.\nabla)\overline{\sqrt{\mathrm{tr}(A^n)}} - \mathrm{tr}\overline{\nabla u^n\frac{A^n}{\sqrt{\mathrm{tr}(A^n)}}} + \mathrm{tr}\overline{\frac{(A^n - Id)}{2\sqrt{\mathrm{tr}(A^n)}}} + \alpha\mathrm{tr}\overline{\frac{(A^n - Id)^2}{2\sqrt{\mathrm{tr}(A^n)}}} = 0. \tag{75}$$

Hence, η solves

$$\begin{aligned}(\partial_t + u.\nabla)\eta = {} & 2\mathrm{tr}\left[\nabla uA - \overline{\nabla u^n\frac{A^n}{\sqrt{\mathrm{tr}(A^n)}}}\,\overline{\sqrt{\mathrm{tr}(A^n)}}\right] - 2\mu \\ & - \alpha\mathrm{tr}\left[\lim_{\delta\to 0}\overline{(A^n)^2}^\delta - \overline{\frac{(A^n)^2}{\sqrt{\mathrm{tr}(A^n)}}}\,\overline{\sqrt{\mathrm{tr}(A^n)}}\right] \\ & + (2\alpha - 1)\mathrm{tr}\left[A - \overline{\frac{A^n}{\sqrt{\mathrm{tr}(A^n)}}}\,\overline{\sqrt{\mathrm{tr}(A^n)}}\right] + (1-\alpha)d\left[1 - \overline{\frac{1}{\sqrt{\mathrm{tr}(A^n)}}}\,\overline{\sqrt{\mathrm{tr}(A^n)}}\right]\end{aligned} \tag{76}$$

where here we applied Proposition 3.1 to deduce that $-\mu = \tilde{\gamma} = \lim_{\delta\to 0}\overline{\nabla u^n A^n}^\delta - \nabla uA$. Due to the presence of A^n and not only its trace on the right hand side of (21), we need to study the propagation of an other defect measure. We introduce $\varepsilon = \mathrm{tr}\left[\overline{\left(\frac{A^n}{\mathrm{tr}(A^n)}\right)^2} - \left(\overline{\frac{A^n}{\mathrm{tr}(A^n)}}\right)^2\right]$. Denoting $B^n = \frac{A^n}{\mathrm{tr}(A^n)}$, we have

$$\begin{aligned}&\partial_t B^n + u.\nabla B^n - \nabla u^n B^n - B^n(\nabla u^n)^T + \frac{(A^n - Id)}{\mathrm{tr}(A^n)} + \alpha\frac{(A^n - Id)^2}{\mathrm{tr}(A^n)} \\ &\quad - B^n\left[-2\mathrm{tr}\nabla u^n B^n + \frac{\mathrm{tr}(A - Id)}{\mathrm{tr}(A^n)} + \alpha\frac{\mathrm{tr}(A^n - Id)^2}{\mathrm{tr}(A^n)}\right] = 0.\end{aligned} \tag{77}$$

Passing weakly to the limit, we get

$$\partial_t B + u.\nabla B + G - \overline{B^n\mathrm{tr}G^n} = 0 \tag{78}$$

where $G = \overline{G^n}$ is the weak limit of G^n and

$$G^n = -\nabla u^n B^n - B^n(\nabla u^n)^T + \frac{(A^n - Id)}{\mathrm{tr}(A^n)} + \alpha\frac{(A^n - Id)^2}{\mathrm{tr}(A^n)}. \tag{79}$$

Hence,

$$\partial_t |B|^2 + u.\nabla |B|^2 + 2\mathrm{tr}(BG) - 2\mathrm{tr}(B\overline{B^n \mathrm{tr} G^n}) = 0 \tag{80}$$

Moreover,

$$\partial_t \overline{|B^n|^2} + u.\nabla \overline{|B^n|^2} + 2\mathrm{tr}\overline{(B^n G^n)} - 2\mathrm{tr}\overline{(B^n B^n \mathrm{tr} G^n)} = 0 \tag{81}$$

Hence, ε solves

$$\partial_t \varepsilon + u.\nabla \varepsilon = -2\mathrm{tr}\overline{([B^n - B]G^n)} + 2\mathrm{tr}\overline{([B^n - B]B^n \mathrm{tr} G^n)}. \tag{82}$$

Now, we have to estimate the terms on the right hand sides of (76) and (82) using the defect measures $\eta + \varepsilon$, μ and κ that will be defined later.

We will use the following lemma during the estimate

Lemma 4.5. *Assume that f^n is bounded in L^2 and g^n is bounded in L^∞ by M and that f^n and g^n converges weakly to f and g. Then, the defect measure of $f^n g^n$ is controlled by :*

$$\overline{(f^n g^n - \overline{f^n g^n})^2} \le 2M^2 \overline{(f^n - f)^2} + 2\overline{(f^n)^2}\ \overline{(g^n - g)^2}. \tag{83}$$

If in addition f^n is bounded below by a positive constant, then we have

$$\overline{\frac{1}{f^n}} f - 1 \le C\overline{(f^n - f)^2}. \tag{84}$$

For the proof of the lemma, we use Cauchy-Schwarz

$$\begin{aligned}
\overline{(f^n g^n - \overline{f^n g^n})^2} &= \overline{\Big((f^n - f)g^n + fg^n - \overline{f^n g^n}\Big)^2} \\
&\le 2M^2 \overline{(f^n - f)^2} + 2\overline{(fg^n - \overline{f^n g^n})^2} \\
&\le 2M^2 \overline{(f^n - f)^2} + 2\overline{\Big(f(g^n - g) - \overline{(f^n - f)(g^n - g)}\Big)^2} \\
&\le 2M^2 \overline{(f^n - f)^2} + 2\overline{(f(g^n - g))^2} + 2\left(\overline{(f^n - f)(g^n - g)}\right)^2
\end{aligned}$$

To prove (84), we compute

$$\begin{aligned}
&\overline{\frac{1}{f^n}} f - 1 \le C\overline{(f^n - f)^2} = \overline{\frac{1}{f^n}(f - f^n)} = \overline{\Big(\frac{1}{f^n} - \frac{1}{f}\Big)(f - f^n)} = \\
&= \overline{\frac{(f - f^n)^2}{ff^n}} \le C\overline{(f^n - f)^2}.
\end{aligned}$$

One application of the lemma that will be use later is the fact that

$$\beta = \overline{\left(\frac{A^n}{\sqrt{\mathrm{tr}(A^n)}} - \overline{\frac{A^n}{\sqrt{\mathrm{tr}(A^n)}}}\right)^2} \le C\eta + C\mathrm{tr}(A)\varepsilon \tag{85}$$

We start by the first term on the right hand side of (76). We can control it by

$$\begin{aligned}
&\mathrm{tr}\left[\nabla u A - \overline{\nabla u^n \frac{A^n}{\sqrt{\mathrm{tr}(A^n)}}}\,\overline{\sqrt{\mathrm{tr}(A^n)}}\right] = \\
&= \mathrm{tr}\left[\overline{(\nabla u - \nabla u^n)\frac{A^n}{\sqrt{\mathrm{tr}(A^n)}}}\,\overline{\sqrt{\mathrm{tr}(A^n)}} + \nabla u\left(\overline{\frac{A^n}{\sqrt{\mathrm{tr}(A^n)}}(\sqrt{\mathrm{tr}(A^n)} - \overline{\sqrt{\mathrm{tr}(A^n)}})}\right)\right] \\
&\le C\sqrt{\mu\beta}\sqrt{\mathrm{tr}(A)} + C|\nabla u|\sqrt{\eta\beta}
\end{aligned}$$

The third term on the right hand side of (76) can be estimated by

$$
\begin{aligned}
&-\operatorname{tr}\left[\lim_{\delta\to 0}\overline{(A^n)^2}^{\delta}-\overline{\frac{(A^n)^2}{\sqrt{\operatorname{tr}(A^n)}}}\,\overline{\sqrt{\operatorname{tr}(A^n)}}\right]=\\
&=-\operatorname{tr}\left[\lim_{\delta\to 0}\overline{(A^n-A)^2}^{\delta}-\overline{\frac{(A^n-A)A^n}{\sqrt{\operatorname{tr}(A^n)}}}\,\overline{\sqrt{\operatorname{tr}(A^n)}}-A\left(\overline{\frac{A^n}{\sqrt{\operatorname{tr}(A^n)}}}\,\overline{\sqrt{\operatorname{tr}(A^n)}}-A\right)\right]\\
&\le -\kappa+C\sqrt{\operatorname{tr}(A)}\sqrt{\kappa\beta}+|A|\sqrt{\eta\beta}
\end{aligned}
$$

where $\kappa=\lim_{\delta\to 0}\operatorname{tr}\overline{(A^n-A)^2}^{\delta}$. The fourth term is easily controlled by $C\sqrt{\eta\beta}$ and the fifth term by $C\eta$ using (84).

Now, we control the term on the right hand side of (82). We will only treat the second one since the first one is easier. We split it into 4 terms and use that B^n is bounded:

$$
\begin{aligned}
&\operatorname{tr}\left[\overline{(B^n-B)B^n\operatorname{tr}[B^n(\nabla u^n-\nabla u)]}+\overline{(B^n-B)B^n\operatorname{tr}[B^n\nabla u]}\right]\\
&\le C(\sqrt{\mu\varepsilon}+|\nabla u|\varepsilon),
\end{aligned}
\tag{86}
$$

$$
\begin{aligned}
&\operatorname{tr}\left[\overline{(B^n-B)B^n\operatorname{tr}\Big[\frac{(A^n)^2}{\operatorname{tr}A^n}-\overline{\frac{(A^n)}{\sqrt{\operatorname{tr}A^n}}}^{2}\Big]}+\overline{(B^n-B)B^n\operatorname{tr}\Big[\overline{\frac{(A^n)}{\sqrt{\operatorname{tr}A^n}}}^{2}\Big]}\right]\\
&\le C(\beta+\operatorname{tr}A\,\varepsilon),
\end{aligned}
\tag{87}
$$

$$
\operatorname{tr}\left[\overline{(B^n-B)B^n\operatorname{tr}\Big[B^n\Big]}\right]\le C\varepsilon,
\tag{88}
$$

$$
\begin{aligned}
&\operatorname{tr}\left[\overline{(B^n-B)B^n\Big[\frac{1}{\operatorname{tr}A^n}-\overline{\frac{1}{\sqrt{\operatorname{tr}A^n}}}^{2}\Big]}+\overline{(B^n-B)B^n\overline{\frac{1}{\sqrt{\operatorname{tr}A^n}}}^{2}}\right]\\
&\le C(\varepsilon+\eta).
\end{aligned}
\tag{89}
$$

As we did in (53) and (54), we introduce $\zeta=\frac{\eta+\varepsilon}{1+\eta+\varepsilon}$ and the unique a.e flow X in the sense of DiPerna and Lions [10] of u. Hence, $\zeta(t,X(t,x))$ satisfies (55) and hence vanishes since it vanishes at $t=0$. Of course, we also deduce that κ and μ vanish. Unlike for equations (1) and (66), we can not pass to the limit directly in the second equation of (56) or in (59) due to the presence of the terms $\nabla u^n A^n$ and $(A^n)^2$. Instead, we pass to the limit in equations (77) and (75). Then, we deduce that A solves (59) by writing that $A=(\sqrt{\operatorname{tr}A})^2B$.

5. Appendix : Young measures and Chacon limit

We recall here two important weak convergence objects used in this paper, namely the Young measure and the Chacon's biting lemma. Actually, these two notions are very related as was observed in Ball and Murat [1].

Proposition 5.1. *(Young measures) If f^n is a sequence of functions bounded in $L^1(U;\mathbb{R}^m)$ where U is an open set of $\mathbb{R}^N$, then there exists a family $(\nu_x)_{x\in U}$ of probability measures on $\mathbb{R}^m$ (the Young measures), depending measurably on x and a subsequence also denoted f^n such that if $g:\mathbb{R}^m\to\mathbb{R}$ is continuous, if $A\subset U$ is measurable and*

$$g(f^n)\rightharpoonup z(x)\quad \textit{weakly in}\, L^1(A;\mathbb{R}),$$

then $g(.) \in L^1(\mathbb{R}^m;\nu_x)$ *for a.e.* $x \in A$ *and*

$$z(x) = \int_{\mathbb{R}^m} g(\lambda) d\nu_x(\lambda) \quad a.e. \quad x \in A.$$

In the case where f^n is bounded in $L^p(U;\mathbb{R}^m)$ for some $p > 1$ (or when f^n is equi-integrable), we can always take $A = U$ and we have (extracting a subsequence)

$$g(f^n) \rightharpoonup \int_{\mathbb{R}^m} g(\lambda) d\nu_x(\lambda).$$

Proposition 5.2. *(Chacon limit) If* f^n *is a sequence of functions bounded in* $L^1(U;\mathbb{R}^m)$ *where* U *is an open set of* $\mathbb{R}^N$*, then there exists a function* $f \in L^1(U;\mathbb{R}^m)$*, a subsequence* f^n *and a non-increasing sequence of measurable sets* E_k *of* U *with* $\lim_{k\to\infty} \mathcal{L}_N(E_k) = 0$ *(where* $\mathcal{L}_N$ *is the Lebesgue measure on* $\mathbb{R}^N$*) such that for all* $k \in \mathbb{N}$*,* $f^n \rightharpoonup f$ *weakly in* $L^1(U - E_k;\mathbb{R}^m)$ *as* n *goes to infinity.* f *is called the Chacon limit of* f^n*.*

It is easy to see that if f^n is equi-integrable then the Chacon limit of f^n is equal to the weak limit of f^n in the sense of distribution.

If we consider continuous functions $g_k : \mathbb{R}^m \to \mathbb{R}^m$, $k \in \mathbb{N}$ satisfying the conditions :

(a) $g_k(\lambda) \to \lambda$ when $k \to \infty$, for each $\lambda \in \mathbb{R}^m$,

(b) $|g_k(\lambda)| \leq C(1 + |\lambda|)$, for all $k \in \mathbb{N}$ and $\lambda \in \mathbb{R}^m$,

(c) $\lim_{|\lambda|\to\infty} |\lambda|^{-1}|g_k(\lambda)| = 0$ for each k,

then, under the hypotheses of Proposition 5.1, for each fixed k, the sequence of functions $g_k(f^n)$ is equi-integrable and hence (extracting a subsequence) converges weakly in $L^1(U;\mathbb{R}^m)$, to some f_k. Applying a diagonal process, as k goes to infinity, the sequence f_k converges strongly to some f in $L^1(U;\mathbb{R}^m)$. The limit f is the Chacon's limit of the subsequence f^n and it is given by

$$f(x) = \int_{\mathbb{R}^m} \lambda d\nu_x(\lambda) \quad a.e. \quad x \in U.$$

This gives an other possible definition of Chacon's limit which is equivalent to the one given in Proposition 5.2. For the proof of these results we refer to [1].

6. Acknowledgments

The work of N. M. is partially supported by NSF-DMS grant 0703145. The author would like to thank P.-L. Lions for many discussions about these models.

References

[1] J. M. Ball and F. Murat. Remarks on Chacon's biting lemma. *Proc. Amer. Math. Soc.*, 107(3):655–663, 1989.

[2] J. W. Barrett, C. Schwab, and E. Süli. Existence of global weak solutions for some polymeric flow models. *Math. Models Methods Appl. Sci.*, 15(6):939–983, 2005.

[3] J. W. Barrett and E. Süli. Existence of global weak solutions to dumbbell models for dilute polymers with microscopic cut-off. *Math. Models Methods Appl. Sci.*, 18(6):935–971, 2008.

[4] J. W. Barrett and E. Süli. Existence and equilibration of global weak solutions to finitely extensible nonlinear bead-spring chain models for dilute polymers. *preprint*, 2010.

[5] R. B. Bird, C. Curtiss, R. Amstrong, and O. Hassager. *Dynamics of polymeric liquids, Kinetic Theory Vol. 2,*. Wiley, New York, 1987.

[6] J.-Y. Chemin and N. Masmoudi. About lifespan of regular solutions of equations related to viscoelastic fluids. *SIAM J. Math. Anal.*, 33(1):84–112 (electronic), 2001.

[7] P. Constantin and N. Masmoudi. Global well-posedness for a Smoluchowski equation coupled with Navier-Stokes equations in 2D. *Comm. Math. Phys.*, 278(1):179–191, 2008.

[8] P. Constantin and G. Seregin. Global regularity of solutions of coupled Navier-Stokes equations and nonlinear Fokker Planck equations. *Discrete Contin. Dyn. Syst.*, 26(4):1185–1196, 2010.

[9] P. Degond, M. Lemou, and M. Picasso. Viscoelastic fluid models derived from kinetic equations for polymers. *SIAM J. Appl. Math.*, 62(5):1501–1519 (electronic), 2002.

[10] R. J. DiPerna and P.-L. Lions. Ordinary differential equations, transport theory and Sobolev spaces. *Invent. Math.*, 98(3):511–547, 1989.
[11] M. Doi and S. F. Edwards. *The theory of polymer Dynamics*. Oxford University press, Oxford, 1986.
[12] Q. Du, C. Liu, and P. Yu. FENE dumbbell model and its several linear and nonlinear closure approximations. *Multiscale Model. Simul.*, 4(3):709–731 (electronic), 2005.
[13] W. E, T. Li, and P. Zhang. Well-posedness for the dumbbell model of polymeric fluids. *Comm. Math. Phys.*, 248(2):409–427, 2004.
[14] E. Fernández-Cara, F. Guillén, and R. R. Ortega. Some theoretical results for viscoplastic and dilatant fluids with variable density. *Nonlinear Anal.*, 28(6):1079–1100, 1997.
[15] E. Fernández-Cara, F. Guillén, and R. R. Ortega. Some theoretical results concerning non-Newtonian fluids of the Oldroyd kind. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 26(1):1–29, 1998.
[16] E. Fernández-Cara, F. Guillén, and R. R. Ortega. *The mathematical analysis of viscoelastic fluids of the Oldryod kind*. 2000.
[17] H. Giesekus. A simple constitutive equation for polymer fluids based on the concept of deformation-dependent tensorial mobility. *J. Non-Newtonian Fluid Mech*, 1:69–109, 1982.
[18] C. Guillopé and J.-C. Saut. Existence results for the flow of viscoelastic fluids with a differential constitutive law. *Nonlinear Anal.*, 15(9):849–869, 1990.
[19] C. Guillopé and J.-C. Saut. Global existence and one-dimensional nonlinear stability of shearing motions of viscoelastic fluids of Oldroyd type. *RAIRO Modél. Math. Anal. Numér.*, 24(3):369–401, 1990.
[20] D. Hu and T. Lelièvre. New entropy estimates for Oldroyd-B and related models. *Commun. Math. Sci.*, 5(4):909–916, 2007.
[21] B. Jourdain, C. Le Bris, T. Lelièvre, and F. Otto. Long-time asymptotics of a multiscale model for polymeric fluid flows. *Arch. Ration. Mech. Anal.*, 181(1):97–148, 2006.
[22] B. Jourdain, T. Lelièvre, and C. Le Bris. Existence of solution for a micro-macro model of polymeric fluid: the FENE model. *J. Funct. Anal.*, 209(1):162–193, 2004.
[23] R. Keunings. *Simulation of Viscoelastic Fluid Flow, in Fundamentals of Computer Modeling for Polymer Processing*. C.L Tucker III (Ed.). Carl Hanser Verlag,, 1989.
[24] R. Keunings. On the Peterlin approximation for finitely extensible dumbbells. *J. Non-Newtonian Fluid Mech*, 86:85–100, 1997.
[25] O. Kreml and M. Pokorný. On the local strong solutions for the fene dumbbell model. *preprint 2010*.
[26] R. Kupferman, C. Mangoubi, and E. S. Titi. A Beale-Kato-Majda breakdown criterion for an Oldroyd-B fluid in the creeping flow regime. *Commun. Math. Sci.*, 6(1):235–256, 2008.
[27] C. Le Bris and T. Lelièvre. Multiscale modelling of complex fluids: a mathematical initiation. In *Multiscale modeling and simulation in science*, volume 66 of *Lect. Notes Comput. Sci. Eng.*, pages 49–137. Springer, Berlin, 2009.
[28] Z. Lei. Global existence of classical solutions for some Oldroyd-B model via the incompressible limit. *Chinese Ann. Math. Ser. B*, 27(5):565–580, 2006.
[29] Z. Lei, C. Liu, and Y. Zhou. Global solutions for incompressible viscoelastic fluids. *Arch. Ration. Mech. Anal.*, 188(3):371–398, 2008.
[30] Z. Lei, N. Masmoudi, and Y. Zhou. Remarks on the blowup criteria for Oldroyd models. *J. Differential Equations*, 248(2):328–341, 2010.
[31] Z. Lei and Y. Zhou. Global existence of classical solutions for the two-dimensional Oldroyd model via the incompressible limit. *SIAM J. Math. Anal.*, 37(3):797–814 (electronic), 2005.
[32] J. Leray. Etude de diverses équations intégrales nonlinéaires et de quelques problèmes que pose l'hydrodynamique. *J. Math. Pures Appl.*, 12:1–82, 1933.
[33] J. Leray. Essai sur les mouvements plans d'un liquide visqueux emplissant l'espace. *Acta. Math.*, 63:193–248, 1934.
[34] T. Li and P. Zhang. Mathematical analysis of multi-scale models of complex fluids. *Commun. Math. Sci.*, 5(1):1–51, 2007.
[35] F.-H. Lin, C. Liu, and P. Zhang. On hydrodynamics of viscoelastic fluids. *Comm. Pure Appl. Math.*, 58(11):1437–1471, 2005.
[36] F.-H. Lin, C. Liu, and P. Zhang. On a micro-macro model for polymeric fluids near equilibrium. *Comm. Pure Appl. Math.*, 60(6):838–866, 2007.
[37] P.-L. Lions and N. Masmoudi. Global solutions for some Oldroyd models of non-Newtonian flows. *Chinese Ann. Math. Ser. B*, 21(2):131–146, 2000.
[38] P.-L. Lions and N. Masmoudi. Global existence of weak solutions to micro-macro models. *C. R. Math. Acad. Sci. Paris*, 2007.
[39] C. Liu and N. J. Walkington. An Eulerian description of fluids containing visco-elastic particles. *Arch. Ration. Mech. Anal.*, 159(3):229–252, 2001.

[40] N. Masmoudi. Well-posedness for the FENE dumbbell model of polymeric flows. *Comm. Pure Appl. Math.*, 61(12):1685–1714, 2008.

[41] N. Masmoudi. Global existence of weak solutions to the FENE dumbbell model of polymeric flows. *in preparation*, 2010.

[42] N. Masmoudi, P. Zhang, and Z. Zhang. Global well-posedness for 2D polymeric fluid models and growth estimate. *Phys. D*, 237(10-12):1663–1675, 2008.

[43] L. Molinet and R. Talhouk. Newtonian limit for weakly viscoelastic fluid flows of Oldroyd type. *SIAM J. Math. Anal.*, 39(5):1577–1594, 2008.

[44] H. C. Öttinger. *Stochastic processes in polymeric fluids*. Springer-Verlag, Berlin, 1996. Tools and examples for developing simulation algorithms.

[45] F. Otto and A. E. Tzavaras. Continuity of velocity gradients in suspensions of rod-like molecules. *Comm. Math. Phys.*, 277(3):729–758, 2008.

[46] R. G. Owens and T. N. Phillips. *Computational rheology*. Imperial College Press, London, 2002.

[47] A. Peterlin. Hydrodynamics of macromolecules in a velocity field with longitudinal gradient. *Journal of Polymer Science Part B: Polymer Letters*, 4:287–291, 1966.

[48] M. Renardy. An existence theorem for model equations resulting from kinetic theories of polymer solutions. *SIAM J. Math. Anal.*, 22(2):313–327, 1991.

[49] B. Thomases and M. Shelley. Emergence of singular structures in oldroyd-b fluids. *Phys. Fluids*, 19(12):103103, 2007.

[50] B. Thomases and M. Shelley. Transition to mixing and oscillations in a stokesian viscoelastic flow. *Phys. Rev. Lett.*, 103(9):094501, Aug 2009.

[51] H. Zhang and P. Zhang. Local existence for the FENE-dumbbell model of polymeric fluids. *Arch. Ration. Mech. Anal.*, 181(2):373–400, 2006.

Courant Institute, New York University, 251 Mercer St, New York NY 10012, email:masmoudi@cims.nyu.edu