Examples of singular limits in hydrodynamics.

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Abstract This chapter is devoted to the study of some asymptotic problems in hydrodynamics. In particular, we will review results about the inviscid limit, the compressible-incompressible limit, the study of rotating fluids at high frequency, the hydrodynamic limit of the Boltzmann equation as well as some homogenization problems in fluid mechanics.

1 Introduction

Any physical system can be described by a system of equations which governs the evolution of the different physical quantities such as the density, the velocity, the temperature... The unknowns usually involve several physical units such as (m, kg, s ...). Introducing some length scale, time scale, velocity scale..., the system of equations can always be written in a dimensionless form. This dimensionless form contains some ratios between the different scales such at the Reynolds number, the Mach number or the ratio between two length scales. Indeed, the system may have different length scales. For instance, it may have a vertical length scale and a horizontal one.

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1.1 Dimensionless parameters

Writing the system in its dimensionless form allows us to compare the relative influence of the several terms appearing in the equations. Moreover, it allows us to compare different systems. For instance two incompressible flows which have the same Reynolds number have very similar properties, even if the length scales, the velocity scales and viscosities are very different. The only important factor of comparison is the ratio $Re = \frac{\underline{U}\,\underline{L}}{\nu_0}$ where $\underline{U}$ is the velocity scale, $\underline{L}$ is the length scale, and ν_0 is the kinematic viscosity.

In hydrodynamics, asymptotic problems arise when a dimensionless parameter ϵ goes to zero in a dimensionless system of equations describing the motion of some fluid. Physically, this allows a better knowledge of the system in this limit regime by describing (usually by a simpler system) the prevailing phenomenon when this parameter is small. Indeed, this small parameter, usually describes a physical reality. For instance, a slightly compressible flow is characterized by a low Mach number, whereas a slightly viscous flow is characterized by a high Reynolds number. Notice, here, that we used the terminology slightly compressible flow or slightly viscous flow instead of fluid. Indeed, this is a property of the flow rather than the fluid itself. However, we will often use the terminology slightly compressible fluid or slightly viscous fluid to mean the properties of the flow.

Let us notice that if the viscosity goes to zero, then the Reynolds number goes to infinity. But this is not the only way of getting a big Reynolds number. For instance if $\underline{L}$ or $\underline{U}$ increase then the Reynolds number also increases and we get the same properties as when the viscosity goes to zero. This is of course very important from a physical point of view since it is much easier to change $\underline{L}$ or $\underline{U}$ in a physical experiment than to change the viscosity. This shows the importance of the dimensionless parameters. So, when we speak about the inviscid limit, this should be understood as the limit when the Reynolds number goes to infinity.

Moreover, in many cases, we have different small parameters (we can be in presence of a slightly compressible and slightly viscous fluid in the same time). Depending on the way these small parameters go to zero, we can recover different systems at the limit. For instance, if $\epsilon, \delta, \nu, \eta << 1$, the limit system can depend on the magnitude of the ratio of ϵ/δ or ϵ/ν... This again shows the importance of having dimensionless quantities which can be compared.

The study of these asymptotic problems allows us to get simpler models at the limit, due to the fact that we usually have fewer variables or (and) fewer unknowns. This simplifies the numerical simulations. In fact, instead of solving the initial system, we can solve the limit system and then add a corrector.

1.2 Mathematical problems

Many mathematical problems are encountered when we try to justify the passage to the limit, which are mainly due to the change of the type of the equations the presence of many spatial and temporal scales, the presence of boundary layers (we can no longer impose the same boundary conditions for the initial system and the limit one), the presence of oscillations in time at high frequency ....

Usually, we say that we have a singular limit if there is a change of the type of the equation. For instance in the inviscid limit (Reynolds number going to infinity), we go from a parabolic equation to a hyperbolic equation. However, this terminology seems a little bit restrictive since, we can see from the examples that it is not usually easy to give a type to each system of equations. Moreover, we can say that we have a singular limit if we have a reduction of the number of variables or unknowns due to a more restrained dynamics. Different type of questions can be asked :

1) What do the solutions of the initial system (S_ϵ) converge to ? Is the convergence strong or weak ?

2) In the case of weak convergence, can we give a more detailed description of the sequences of solutions ? Can we describe the time oscillations for instance ?

3) Can we use some properties of the limit system to deduce properties for the initial system when the parameter in small.

In this chapter, we will try to answer some of these questions by studying some examples of singular limits in hydrodynamics. In the next subsection, we recall the physical equation of fluid dynamics and introduce the several dimensionless parameters.

1.3 The compressible Navier Stokes system

In this subsection, we recall the compressible Navier Stokes system for a Newtonian fluid and introduce the several dimensionless parameters used in the next sections. The CNS reads

$$\begin{cases} \dfrac{\partial\rho}{\partial t} + \mathrm{div}(\rho u) = 0\ , \qquad \rho \geq 0, \\ \\ \dfrac{\partial\rho u}{\partial t} + \mathrm{div}(\rho u \otimes u) - \mathrm{div}(2\mu D(u)) - \nabla(\lambda \mathrm{div} u) + \nabla p = f\ , \\ \\ \dfrac{\partial\rho e}{\partial t} + \mathrm{div}(\rho u e) + p\mathrm{div} u - \mathrm{div}(k\nabla T) = 2\mu|D(u)|^2 + \lambda(\mathrm{div} u)^2 \end{cases} \tag{1}$$

In the above system, t is time, div and ∇ only act in the x variable and $x \in \mathbb{R}^N$. Moreover ρ, u, p, e and T are respectively the density, the velocity, the pressure, the internal energy by unit mass and the temperature of the fluid. Besides, μ and λ are the so-called Lamé viscosity coefficients and satisfy the relation $\mu \geq 0$, $N\lambda + 2\mu \geq 0$. The coefficient k is the thermal conduction coefficient and satisfies $k \geq 0$. In general μ, λ and k can depend on the the thermodynamical functions and their gradients. Finally, f is the force term. For geophysical flows we will consider a force which is the sum of the gravitational force and the Coriolis force, namely $f = \rho\mathbf{g} + \rho\Omega\mathbf{e} \times u$ where Ω is the rotation frequency and $\mathbf{e}$ is the direction of rotation. We also denote $g = |\mathbf{g}|$.

The system (1) can be closed by the thermodynamic state equations, namely $p = P(\rho, T)$ and $e = e(\rho, T)$. For an ideal gas, these functions are given by

$$\begin{cases} e = C_v T \\ p = \rho R T \end{cases} \tag{2}$$

where $R > 0$ is the ideal gas constant and $C_v > 0$ is a constant. We also define $C_p = R + C_v$. The constant C_v and C_p are respectively the specific heats at constant volume and constant pressure. We also define the adiabatic constant $\gamma = C_p/C_v$.

The system formed by (1) and (2) is closed. There is an other important thermodynamical function, namely the entropy. It is defined by the following thermodynamic relation

$$T\,dS = \frac{\partial e}{\partial T} dT + (\frac{\partial e}{\partial \rho} - \frac{p}{\rho^2}) d\rho. \tag{3}$$

For an ideal gas, (3) yields $\frac{\partial S}{\partial T} = \frac{C_v}{T}$ and $\frac{\partial S}{\partial \rho} = -\frac{R}{\rho}$. Hence S is given by $S = C_v \log \frac{T}{\rho^{\gamma-1}}$. In particular we can replace the third equation of (1) by an equation for the entropy, namely

$$\frac{\partial \rho S}{\partial t} + \mathrm{div}(\rho u S) = \frac{1}{T}\mathrm{div}(k\nabla T) + \frac{2\mu|D(u)|^2 + \lambda(\mathrm{div} u)^2}{T}. \tag{4}$$

Let us notice that if we take $\mu = \lambda = 0$ and $k = 0$ then (4) reduces to a transport equation and that if the entropy is constant initially $S = S_0$ then it remains constant at later times. In this case, $T = e^{\frac{S}{cv}}\rho^{\gamma-1}$ and $p = Re^{\frac{S_0}{cv}}\rho^{\gamma}$. This yields the compressible isentropic Euler system. An other model we will deal with is the isentropic compressible Navier-Stokes system (69). It corresponds to the case $k = 0$, S is constant and we neglect the variation of S due to the viscous effects. However, (69) can not be rigorously derived from (1) in any asymptotic regime.

1.4 Dimensionless parameters

Let us now define the different dimensionless parameters. We take $\underline{t}$, $\underline{L}$, $\underline{U}$, $\underline{\rho}$ and $\underline{P}$ to be respectively the characteristic time scale, the characteristic length scale, the characteristic velocity scale, the characteristic density scale and the characteristic pressure scale. This means that each time or length is made dimensionless by dividing it by $\underline{t}$ or $\underline{L}$. Hence, we can define a dimensionless time and dimensionless length by $\tilde{t} = \frac{t}{\underline{t}}$ and $\tilde{x} = \frac{x}{\underline{L}}$. We can do the same for all the other quantities. We also take characteristic values of μ and k which we denote $\underline{\mu}$ and $\underline{k}$. These are equal to μ and k if they are constant.

The Strouhal number and Reynolds number are defined by

$$St = \frac{\underline{L}}{\underline{t}\,\underline{U}} \tag{5}$$

$$Re = \frac{\underline{L}\,\underline{U}}{\underline{\mu}/\underline{\rho}}. \tag{6}$$

A small Strouhal number St corresponds to the longtime behavior of a system. A large Reynolds number Re corresponds to small viscous effects.

The acoustic waves propagates at the sound speed which is given in the isentropic case by $c^2 = \frac{\partial p}{\partial \rho} = \gamma RT$. Hence we can define the Mach number as the ratio between $\underline{U}$ and c, namely

$$Ma = \frac{\underline{U}}{c} = \frac{\underline{U}}{\sqrt{\gamma R \underline{t}}}. \tag{7}$$

When $Ma < 1$, we have a subsonic flow and when $Ma > 1$, we have a supersonic flow.

The velocity and the temperature satisfy both a diffusion equation with a diffusivity given respectively by $\frac{\underline{\mu}}{\underline{\rho}}$ and $\frac{\underline{k}}{C_v\underline{\rho}}$. The ratio between this two numbers is the Prandtl number

$$Pr = \gamma \frac{C_v \underline{\mu}}{\underline{k}} = \frac{C_p \underline{\mu}}{\underline{k}}. \tag{8}$$

Now, we will introduce some other dimensionless parameters related to the gravity force and the Coriolis force. First, let us introduce a vertical length scale $\underline{H}$. Hence the gravity wave speed is given by $\sqrt{g\underline{H}}$ and we can define the Froude number which measures the importance of the gravity force. It is the ratio between $\underline{U}$ and $\sqrt{g\underline{H}}$, namely

$$Fr = \frac{\underline{U}}{\sqrt{g\underline{H}}}. \tag{9}$$

The Rossby number measures the importance of the earth's rotation. It is the ratio between the rotation time scale $t_\Omega = 1/\Omega$ and the fluid time scale $t_U = \underline{L}/\underline{U}$. It is given by

$$Ro = \frac{\underline{U}}{\Omega \underline{L}}. \tag{10}$$

Since, we have two length scale, we can define the ratio between $\underline{H}$ and $\underline{L}$, $\delta = \frac{\underline{H}}{\underline{L}}$. It measures how shallow the fluid is.

In section 2, we study the inviscid limit, namely the limit when the Reynolds number goes to infinity. We will mostly emphasis the problem of boundary layers. In section 3, we study the compressible-incompressible limit, namely the limit when the Mach number goes to infinity and the density becomes almost constant. We also study the limit when γ (the adiabatic constant) goes to ∞. We will emphasis the problem of oscillations in time. In section 4, we study rotating fluid at high frequency. In section 5, we will study the hydrodynamic limit of the Boltzmann equation and derive several compressible and incompressible fluid systems. In section 6, we will recall few results about the homogenization of the Stokes, the Euler and the compressible Navier-Stokes system. In section 7.1, we will give some other examples of singular limits which were not studied in the previous sections. Finally, in section 7.2, we will give some concluding remarks.

Let us end this introduction by giving some general references about fluid mechanics. We refer to [33, 119, 122] for mathematical results about the incompressible Euler equation. We refer to [108, 163, 39] for mathematical results about the incompressible Navier-Stokes system. We refer to [109, 117, 66, 139] for results about the compressible Navier-Stokes system. We also refer to [176, 177] for many formal asymptotic developments and to [80, 144, 118] for physical and mathematical results about the geophysical equations.

2 The inviscid limit

The Navier-Stokes system is the basic mathematical model for viscous incompressible flows. It reads

$$\left\{ \begin{array}{c} \partial_t u^\nu + u^\nu.\nabla u^\nu - \nu\Delta u^\nu + \nabla p = 0, \\ \mathrm{div}(u^\nu) = 0, \\ u^\nu = 0 \quad \text{on} \quad \partial\Omega, \end{array} \right. \tag{11}$$

where u^ν is the velocity, p is the pressure and ν is the kinematic viscosity. We can define a typical length scale L and a typical velocity U. The dimensionless parameter $Re = \frac{UL}{\nu}$ is very important to compare the properties of different flows. When Re is very large (ν very small), we can expect that the Navier-Stokes system (NS_ν) behaves like the Euler system

$$\left\{ \begin{array}{c} \partial_t \mathbf{u} + \mathbf{u}.\nabla \mathbf{u} + \nabla p = 0, \\ \mathrm{div}(\mathbf{u}) = 0, \\ \mathbf{u}.n = 0 \quad \text{on} \quad \partial\Omega. \end{array} \right. \tag{12}$$

The zero-viscosity limit for the incompressible Navier-Stokes equation in a bounded domain, with Dirichlet boundary conditions, is one of the most challenging open problems in Fluid Mechanics. This is due to the formation of a boundary layer which appears because we can not impose a Dirichlet boundary condition for the Euler equation. This boundary layer satisfies formally the Prandtl equations, which seem to be ill-posed in general. Let us first state some results in the whole space where the boundary layer problem does not occur.

2.1 The whole space case

The inviscid limit in the whole space case was performed by several authors, we can refer for instance to Swann [159] and Kato [95]. They basically prove the following result. Take the Navier-Stokes system in the whole space $\mathbb{R}^d$

$$\partial_t u^n + \mathrm{div}(u^n \otimes u^n) - \nu_n \Delta u^n = -\nabla p \quad \text{in} \quad \mathbb{R}^d \tag{13}$$

$$\mathrm{div}(u^n) = 0 \quad \text{in} \quad \mathbb{R}^d \tag{14}$$

$$u^n(t=0) = u_0 \quad \text{with} \quad \mathrm{div}(u_0) = 0 \tag{15}$$

where ν_n goes to 0 when n goes to infinity.

Theorem 2.1 *Let $s > d/2 + 1$, and $u_0 \in H^s(\mathbb{R}^d)$. If T^* is the time of existence and $u \in C_{loc}([0, T^*); H^s)$ is the solution of the Euler system*

$$\partial_t u + \mathrm{div}(u \otimes u) = -\nabla p \quad \textit{in} \quad \mathbb{R}^d \tag{16}$$

$$\mathrm{div}(u) = 0 \quad \textit{in} \quad \mathbb{R}^d \tag{17}$$

$$u(t=0) = u_0 \quad \textit{with} \quad \mathrm{div}(u_0) = 0, \tag{18}$$

then for all $0 < T < T^*$, there exists ν_0 such that for all $\nu_n \leq \nu_0$, the Navier-Stokes system (13 - 15) has a unique solution $u^n \in C([0, T]; H^s(\mathbb{R}^d))$ and for each $t \in [0, T]$, $u(t) = \lim_{n\to\infty} u^n(t)$ exists strongly in $H^s(\mathbb{R}^d)$ uniformly in $t \in [0, T]$. Moreover,

$$\|u^n - u\|_{L^\infty(0,T;H^{s-2})} \leq C\nu_n \tag{19}$$

where C depends only on u.

We point out that this result can be easily extended to the periodic case and more generally to domains without boundaries.

Idea of the proof: The proof of this theorem is based on a standard Grönwall inequality (see [159, 95, 38]). Let us start by proving (19). First, we see that we can solve the Navier-Stokes system and Euler system in $C([0, T]; H^s(\mathbb{R}^d))$ on some time interval independent of ν_n with bounds which are independent of n. This is because there is no boundary. Then, we can write an energy estimate in H^{s-2} for $w^n = u^n - u$,

$$\begin{aligned}\partial_t \|w^n\|^2_{H^{s-2}} &+ \nu_n \|\nabla w^n\|^2_{H^{s-2}} \\ &\leq \Big(C(\|u\|_{H^s} + \|w^n\|_{H^s})\|w^n\|_{H^{s-2}} + \nu_n \|\Delta u\|_{H^{s-2}}\Big)\|w^n\|_{H^{s-2}}\end{aligned} \tag{20}$$

and by Grönwall lemma, we can deduce that (19) holds. It is easy to see that the above argument holds as long as we can solve the Euler system and that we can take any T such that $T < T^*$ (see [38]). Notice that in [38], the regularity required is $s - 2 > d/2 + 1$. However, it seems that this is not necessary modulo the regularization argument given below.

Interpolating between (19) and the uniform bound for w^n in $C([0,T]; H^s(\mathbb{R}^d))$, we deduce that u^n converges to u in $H^{s'}$ for any $s' < s$ and for $s-2 < s' < s$, we have

$$\|u^n - u\|_{L^\infty(0,T;H^{s'})} \leq C\nu_n^{\frac{s-s'}{2}}. \tag{21}$$

To get the convergence in H^s requires a regularization of the initial data. For all $\delta > 0$, we take u_0^δ such that $\|u_0^\delta\|_{H^s} \leq C\|u_0\|_{H^s}$, $\|u_0^\delta\|_{H^{s+1}} \leq \frac{C}{\delta}$, $\|u_0^\delta\|_{H^{s+2}} \leq \frac{C}{\delta^2}$ and for some s' such that $d/2 < s' < s-1$, we have $\|u_0^\delta - u_0\|_{H^{s'}} \leq C\delta^{s-s'}$. Such a u_0^δ can be easily constructed by taking $u_0^\delta = \mathcal{F}^{-1}(1_{\{|\xi|\leq 1/\delta\}}\mathcal{F}u_0)$. Let v^δ be the solution of the Euler system (16,17,18) with the initial data $v^\delta(t = 0) = u_0^\delta$. Then, setting $w^\delta = v^\delta - u$, we have

$$\partial_t\|w^\delta\|_{H^s}^2 \leq C(\|u\|_{H^s} + \|v^\delta\|_{H^s})\|w^\delta\|_{H^s}^2 + C\|v^\delta\|_{H^{s+1}}\|w^\delta\|_{H^s}\|w^\delta\|_{L^\infty}. \tag{22}$$

Then, we notice that on some time interval $[0,T]$, $T < T^*$ (T depends only on $\|u_0\|_{H^s}$), we have $\|v^\delta\|_{H^{s+1}} \leq \frac{C}{\delta}$ and $\|v^\delta\|_{H^{s+2}} \leq \frac{C}{\delta^2}$. Moreover, writing (22) at the regularity s', we can prove easily that $\|w^\delta\|_{L^\infty(0,T;H^{s'})} \leq C\delta^{s-s'}$. Hence, (22) gives

$$\partial_t\|w^\delta\|_{H^s} \leq C(\|u\|_{H^s} + \|v^\delta\|_{H^s}))\|w^\delta\|_{H^s} + C\delta^{s-s'-1}. \tag{23}$$

Hence w^δ goes to zero in $L^\infty(0,T;H^s)$, namely v^δ goes to v in $L^\infty(0,T;H^s)$. Writing an energy estimate for $w^{n,\delta} = u^n - v^\delta$, we get (here we drop the n and δ)

$$\begin{aligned}\partial_t\|w\|_{H^s}^2 &+ \nu_n\|\nabla w\|_{H^s}^2 \\ &\leq C(\|w\|_{L^\infty}\|v^\delta\|_{H^{s+1}}\|w\|_{H^s} + (\|v^\delta\|_{H^s} + \|u^n\|_{H^s})\|w\|_{H^s}^2) + \\ &\qquad \nu_n\|v^\delta\|_{H^{s+2}}\|w\|_{H^s}.\end{aligned} \tag{24}$$

Hence, we get

$$\begin{aligned}\partial_t\|w\|_{H^s} \leq C\|u^n - u\|_{L^\infty}\|v^\delta\|_{H^{s+1}} &+ C\|v^\delta - u\|_{L^\infty}\|v^\delta\|_{H^{s+1}} + \\ &\nu_n\|v^\delta\|_{H^{s+2}} + C(\|v^\delta\|_{H^s} + \|u^n\|_{H^s})\|w\|_{H^s}.\end{aligned} \tag{25}$$

Since u^n converges to u is H^{s-1}, we deduce that

$$\|u^n - u\|_{L^\infty} \leq \|u^n - u\|_{H^{s-1}} \leq C(\nu_n)^{1/2}. \tag{26}$$

Taking $\delta = \delta_n$ such that $\delta = \delta_n$ and $\frac{\nu_n}{\delta_n^2}$ go to zero when n goes to infinity, we deduce that

$$\partial_t \|w^{n,\delta}\|_{H^s} \leq C(\frac{\nu^{1/2}}{\delta} + \delta^{s-s'-1} + \frac{\nu}{\delta^2} + \|w^{n,\delta}\|_{H^s}) \tag{27}$$

Hence, by Grönwall lemma, we deduce that $w^{n,\delta}$ goes to zero in $L^\infty(0,T;H^s)$ and that u^n goes to u in $L^\infty(0,T;H^s)$. □

2.1.1 The $2D$ case

We notice that the time T^* is related to the existence time for the Euler system (16). If $d = 2$ it is known [175, 171] that the Euler system (16) has a global solution and hence one can take any time $T < \infty$ in the above theorem.

Also in the $2D$ case, one can lower the regularity assumption. Indeed Yudovich [175] proved that if $\omega_0 = \mathrm{curl}(u_0) \in L^\infty \cap L^p$ for some $1 < p < \infty$ then the Euler system (16) has a unique global solution. It was proved in [34] that the solution to the Navier-Stokes system converges in $L^\infty((0,T);L^2)$ to the solution of the Euler system if we only assume that $\omega_0 = \mathrm{curl}(u_0) \in L^\infty \cap L^p$. More precisely, Chemin [34] proves that

$$\|u^n - u\|_{L^\infty(0,T;L^2)} \leq C\|\mathrm{curl}(u_0)\|_{L^\infty \cap L^2}(\nu_n T)^{\frac{1}{2}exp(-C\|\mathrm{curl}(u_0)\|_{L^\infty \cap L^2}T)}. \tag{28}$$

Notice that here, the rate of convergence deteriorates with time. This does not happen if we also know that u is in $L^\infty(0,T;Lip)$ as was proved by Constantin and Wu [40].

For vortex patches, namely the case where $\mathrm{curl}(u_0)$ is the characteristic function of a $C^{1+\alpha}$ domain $\alpha > 0$, it was proved in [32] (see also [22]) that the characteristic function of $\mathrm{curl}(u)$ remains a $C^{1+\alpha}$ domain and that the velocity u is in $L^\infty_{loc}(\mathbb{R};Lip)$. It was proved in [40, 41] that under the condition, $u \in L^\infty_{loc}(\mathbb{R};Lip)$, the estimate (28) is actually better since there is no loss for the rate of convergence, namely

$$\|u^n - u\|_{L^\infty(0,T;L^2)} \leq C(\nu_n T)^{\frac{1}{2}}. \tag{29}$$

In [41], the authors also prove some estimate in L^p spaces for the difference between the vorticities, in particular they prove for $p \geq 2$ that $\|\mathrm{curl}(u^n - u)\|_{L^\infty(0,T;L^p)} \leq C\nu_n^{1/4p-\epsilon}$ for some short time T and $\epsilon > 0$.

Concerning vortex patches one can give more precise results about the convergence. It was proved by Danchin [42] that the boundary of the patch under the Navier-Stokes flow converges to the boundary of the patch under the Euler flow. A similar result is also proved in higher dimension locally in time [43]. Also, in [1], a better rate of convergence is given for vortex patches, namely

$$\|u^n - v\|_{L^\infty(0,T;L^2)} \leq C(\nu_n T)^{\frac{3}{4}} \tag{30}$$

which is optimal (see also [129] for a similar result in $3D$).

Let us end this subsection by the vortex sheet case, namely the case where the vorticity is a measure. For the $2D$ case, it is known that we have existence of weak solutions for the Euler system if we only assume that $u_0 \in L^2$ and $\omega_0 \in L^1 \cap L^p$, $1 < p$. In this case, extracting a subsequence, we can prove the weak convergence of the solutions to the Navier-Stokes system towards a weak solution to the Euler system. Indeed, from the bound we have on the vorticity $\mathrm{curl}(u^n) \in L^\infty(0,T;L^p)$, we deduce that u^n is bounded in $L^\infty(0,T;W^{1,p})$ and since $\partial_t u^n$ is bounded in $L^\infty(0,T;H^{-1})$ we deduce that u^n is precompact in $L^2L^2_{loc}$. Then extracting a subsequence, we deduce that u^n converges to some u and u is a weak solution of the Euler system. Here, the main point is that $W^{1,p}(\mathbb{R}^2)$ is compactly injected in $L^2_{loc}(\mathbb{R}^2)$. The above argument does not work if $p = 1$. However, the best result in this direction is due to Delort [50] where he can prove the weak convergence under the assumption that the initial vorticity is compactly supported, belongs to $H^{-1}(\mathbb{R}^2)$ and can be decomposed into two parts: one being a nonnegative measure, the other belonging to some $L^q(\mathbb{R}^2)$, $q > 1$. The proof requires a precise analysis to rule out concentrations at the limit.

2.2 The case of the Dirichlet boundary condition

Let us consider the limit from (11) towards (12). In the region close to the boundary the length scale becomes very small and we can not neglect the viscous effect. In 1904, Prandtl [145] suggested that there exists a thin layer called boundary layer, where the solution u^ν undergoes a sharp transition from a solution to the Euler system to the no-slip boundary condition $u^\nu = 0$ on $\partial\Omega$ of the Navier-Stokes system. In other words, Prandtl proves formally that $u^\nu = \mathbf{u} + u^\nu_{BL}$ where u^ν_{BL} is small except near the boundary. Giving a rigorous justification of this formal expansion is still an open problem. We refer to [151, 150] for a justification in the analytic case.

There are many review papers about the inviscid limit of the Navier-Stokes in a bounded domain and the Prandtl system (see [60, 29]). We also refer to [83] for a review about boundary layers.

2.2.1 Formal derivation of Prandtl system

To illustrate this, we consider a two-dimensional (planar) flow $u^\nu = (u, v)$ in the half-space $\{(x, y) \,|\, y > 0\}$ subject to the following initial condition $u^\nu(t = 0, x, y) = u^\nu_0(x, y)$, boundary condition $u^\nu(t, x, y = 0) = 0$ and $u^\nu \to (U_0, 0)$ when $y \to \infty$. Taking the typical length and velocity of order one, the Reynolds number reduces to $Re = \nu^{-1}$. Let $\epsilon = Re^{-1/2} = \sqrt{\nu}$. Near the boundary, the Euler system is not a good approximation. We introduce new independent variables and new unknowns

$$\tilde{t} = t \quad \tilde{x} = x \quad \tilde{y} = \frac{y}{\epsilon}$$

$$(\tilde{u}, \tilde{v})(\tilde{t}, \tilde{x}, \tilde{y}) = (u, \frac{v}{\epsilon})(\tilde{t}, \tilde{x}, \epsilon\tilde{y})$$

Notice that when $\tilde{y}$ is of order one, $y = \epsilon\tilde{y}$ is of order ϵ. Rewriting the Navier-Stokes system in terms of the new variables and unknowns yields

$$\left\{ \begin{array}{c} \tilde{u}_{\tilde{t}} + \tilde{u}\tilde{u}_{\tilde{x}} + \tilde{v}\tilde{u}_{\tilde{y}} - \tilde{u}_{\tilde{y}\tilde{y}} - \epsilon^2 \tilde{u}_{\tilde{x}\tilde{x}} + p_{\tilde{x}} = 0 \\ \epsilon^2(\tilde{v}_{\tilde{t}} + \tilde{u}\tilde{v}_{\tilde{x}} + \tilde{v}\tilde{v}_{\tilde{y}} - \tilde{v}_{\tilde{y}\tilde{y}}) - \epsilon^4 \tilde{v}_{\tilde{x}\tilde{x}} + p_{\tilde{y}} = 0 \\ \tilde{u}_{\tilde{x}} + \tilde{v}_{\tilde{y}} = 0 \end{array} \right.$$

Neglecting the terms of order ϵ^2 and ϵ^4 yields

$$\left\{ \begin{array}{c} \tilde{u}_{\tilde{t}} + \tilde{u}\tilde{u}_{\tilde{x}} + \tilde{v}\tilde{u}_{\tilde{y}} - \tilde{u}_{\tilde{y}\tilde{y}} + p_{\tilde{x}} = 0 \\ p_{\tilde{y}} = 0, \qquad \tilde{u}_{\tilde{x}} + \tilde{v}_{\tilde{y}} = 0 \end{array} \right.$$

Since p does not depend on $\tilde{y}$, we deduce that the pressure does not vary within the boundary layer and can be recovered from the Euler system (12) when $y = 0$, namely $p_x(t, x) = -(U_t + UU_x)(t, x, y = 0)$, since $V(t, x, y = 0) = 0$. Going back to the old variables, we obtain

$$\left\{ \begin{array}{c} u_t + uu_x + vu_y - \nu u_{yy} + p_x = 0 \\ u_x + v_y = 0 \end{array} \right. \tag{31}$$

which is the so-called Prandtl system. It should be supplemented with the following boundary conditions

$$\left\{ \begin{array}{c} u(t,x,y=0) = v(t,x,y=0) = 0 \\ u(t,x,y) \to U(t,x,0) \quad \text{as} \quad y \to \infty. \end{array} \right. \tag{32}$$

Formally, a good approximation of u^ν should be $\mathbf{u} + u^\nu_{BL}$ where $\mathbf{u}$ is the solution of the Euler system (12) and $\mathbf{u}(t,x,0) + u^\nu_{BL}$ is the solution of the Prandtl system (31), (32).

Replacing the Navier-Stokes system by the Euler system in the interior and the Prandtl system near the boundary requires a justification. Mathematically this can be formulated as a convergence theorem when ν goes to 0, namely $u^\nu - (\mathbf{u} + u^\nu_{BL})$ goes to 0 when ν goes to 0 in L^∞ or in some energy space. In its whole generality this is still a major open problem in fluid mechanics. This is due to problems related to the well-posedness of the Prandtl system. Indeed, under some monotonicity condition on the initial data, Oleinik proved the local existence for the Prandtl system [140, 141] (see also [142]). These solutions can be extended as global weak solutions [173]. However, Weinan E and Enguist [61] proved a blow up result for the Prandtl system for some special type of initial data. For general initial data, it is not known whether we have local well-posedness or not. Moreover, even if we have existence for Prandtl system there are other problems related to the instability of some solutions to the Prandtl system [82] which may prevent the convergence.

2.2.2 The analytic case

In this subsection, we will present the result of [151, 150]. We will just give an informal statement since the result requires the definition of several spaces to keep track of the analyticity of the solution.

Theorem 2.2 *Suppose that $\mathbf{u}(t,x,y)$ and $\mathbf{u}(t,x,0)+u^\nu_{BL}$ are respectively the solutions of the Euler system (12) and the Prandtl system (31), (32) which are analytic in the space variables. Then for a short time independent of $\sqrt{\nu}$, there is an analytic solution u of the Navier-Stokes equations such that it is given by $u = \mathbf{u} + O(\sqrt{\nu})$ in the interior and $u = \mathbf{u}(t,x,0) + u^\nu_{BL} + O(\sqrt{\nu})$ inside the boundary layer.*

We refer to [29] for a sketch of the proof and to [151, 150] for the complete proof.

2.2.3 Kato's criterion of convergence

The convergence of $u^\nu - (\mathbf{u} + u^\nu_{BL})$ to 0 when ν goes to 0 in L^2 is still an open problem. Kato [96] gave a very simple criterion which is equivalent to the convergence of u^ν to $\mathbf{u}$ in L^2.

First let us notice that working with strong solutions to the Navier-Stokes system does not really help. Indeed, the existence of strong solution for $d \geq 3$ only holds on a time interval $[0, T_\nu]$ where T_ν may go to zero when ν goes to 0. Also, for $d = 2$, working with strong solutions does not help since the higher Sobolev norms blow up when ν goes to zero. This is why we consider a family of weak solutions u^ν to the Navier-Stokes system (11) with an initial data u^ν_0. We assume that $u^\nu \in C_w([0,T]; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega))$ for all $T > 0$, divu^ν=0 and (11) holds in the sense of distributions, namely

$$-\int_\Omega u^\nu_0 \phi(t=0) + \int_0^T \int_\Omega -u^\nu \otimes u^\nu : \nabla\phi + \nu\nabla u^\nu \cdot \nabla\phi - u^\nu \partial_t\phi \, dxdt = 0 \quad (33)$$

for all $\phi \in C^\infty_0([0,T) \times \Omega)$, div$\phi$ =0 and the following energy inequality holds

$$\int_\Omega |u^\nu(t,x)|^2 dx + 2\nu \int_0^T \int_\Omega |\nabla u^\nu(s,x)|^2 dxds \leq \int_\Omega |u^\nu_0(x)|^2 dx \quad (34)$$

Assume that u^ν_0 is divergence-free and converges in L^2 to some u_0 and $u_0 \in H^s$, $s > d/2 + 1$. Let $\mathbf{u}$ be the unique strong solution of the Euler system (12) with the initial data u_0 in the space $C([0,T^*); H^s)$ for some $T^* \leq \infty$ and $T^* = \infty$ if $d = 2$. We refer to [162] and [33] for this existence result. Kato proves the following convergence criterion

Theorem 2.3 *For $0 < T < T^*$, the following conditions are equivalent*

- *i) $u^\nu(t,.)$ converges to $\mathbf{u}(t,.)$ in $L^2(\Omega)$ uniformly for $t \in [0,T]$*
- *ii)*
$$\nu \int_0^T \int_{\Gamma_\nu} |\nabla u^\nu|^2 dxdt \rightarrow 0 \quad (35)$$
when ν goes to 0. Here Γ_ν is a strip of width $O(\nu)$ around the boundary $\partial\Omega$.
- *iii)*
$$\nu \int_0^T \int_\Omega |\nabla u^\nu|^2 dxdt \rightarrow 0 \quad (36)$$

Idea of the proof:

We will just give a sketch of the proof of ii) implies i). The idea is to construct a corrector or boundary layer which allows to recover the Dirichlet boundary condition for the difference $u^\nu - \mathbf{u}$ and which satisfies some natural bounds. Kato constructs such a corrector $\mathcal{B}^\nu$ which is divergence free and with a support contained in a strip of size $\mathrm{O}(\nu)$ around $\partial\Omega$. Then, considering $u^\nu - \mathbf{u} - \mathcal{B}$, he can write the following energy estimate where $\mathbf{u}^\mathcal{B} = \mathbf{u} + \mathcal{B}^\nu$

$$\begin{aligned}\frac{1}{2}\|u^\nu - \mathbf{u}\|_{L^2}^2(t) + \nu \int_0^t \|\nabla u^\nu\|_{L^2}^2 ds \\ \leq \int_0^t \int_\Omega -(u^\nu \times u^\nu) : \nabla \mathbf{u}^\mathcal{B} + \mathbf{u} \cdot \nabla \mathbf{u}.u^\nu + \nu \nabla u^\nu . \nabla \mathbf{u}^\mathcal{B} \; dxds + o(1) \quad (37)\end{aligned}$$

for $0 < t \leq T$, where $o(1)$ goes to zero when ν goes to 0. This yields

$$\begin{aligned}\frac{1}{2}\|u^\nu - \mathbf{u}\|_{L^2}^2(t) + \nu \int_0^t \|\nabla u^\nu\|_{L^2}^2 \leq \\ \int_0^t \int_\Omega -(u^\nu - \mathbf{u}) \times (u^\nu - \mathbf{u}) : \nabla \mathbf{u} - u^\nu \times u^\nu : \nabla \mathcal{B}^\nu + \\ \int_0^t \int_\Omega \nu \nabla u^\nu . \nabla \mathbf{u}^\mathcal{B} \; dxdt + o(1). \quad (38)\end{aligned}$$

Then, using some natural L^2 and L^∞ bounds satisfied by $\mathcal{B}^\nu$, the Hardy-Littlewood inequality for the second term on the right hand side of (38) and applying a Gronwall lemma, Kato gets

$$\|u^\nu - \mathbf{u}\|_{L^2}^2(t) \leq \int_0^t K\|u^\nu - \mathbf{u}\|_{L^2}^2 + R(s) \; ds + o(1) \quad (39)$$

for some constant K related to the L^∞ norm of $\nabla \mathbf{u}$ and

$$R(t) \leq K \int_0^t \nu \|\nabla u^\nu\|_{L^2(\Gamma_\nu)}^2 + K\nu \|\nabla u^\nu\|_{L^2} + K\nu^{1/2}\|\nabla u^\nu\|_{L^2(\Gamma_\nu)}.$$

This ends the proof of the uniform convergence in L^2. Notice that it also proves iii) since the total dissipation appears on the left hand side of (38). □

In the same spirit as the Kato criterion, Temam and Wang [164] give a different criterion based on the magnitude of the pressure at the boundary.

They prove that if there exists some $0 \leq \delta < 1/2$ such that

$$\text{either} \quad \nu^{\delta} \int_0^T \|p^\nu\|_{H^{1/2}(\partial\Omega)} \leq C \quad \text{or} \quad \nu^{\delta+1/4} \int_0^T \|\nabla p^\nu\|_{L^2(\partial\Omega)} \leq C \tag{40}$$

then the convergence of u^ν towards $\mathbf{u}$ holds and

$$\|u^\nu - \mathbf{u}\|_{L^2} \leq C\nu^{(1-2\delta)/5}. \tag{41}$$

Also, in [169], Wang gives a criterion which only involves the tangential derivative of the velocity, namely $\nabla_\tau u^\nu$. However, he needs a control on a strip of size bigger than ν.

Concerning bounded domain with boundary conditions other than the Dirichlet boundary condition, let us mention that in [165], Temam and Wang prove the convergence of the solutions to the Navier-Stokes system towards a solution of the Euler system in the non-characteristic case, namely the normal velocity is prescribed at the boundary. In this case a boundary layer of size ν can be constructed.

Let us also mention that in [11], Bardos treats the case of a bounded domain with a boundary condition on the vorticity, which does not engender any boundary layer. He has a result similar to theorem 2.1.

Also, in [36], the vanishing viscosity limit is considered with the Navier (friction) boundary condition.

2.2.4 Different vertical and horizontal viscosities.

One of the main ideas of Kato in the previous subsection is to take the freedom of using a corrector which does not necessary satisfy the Prandtl system. The same idea was used in [123] to get a complete convergence result without any condition on the dissipation in the case we take different vertical and horizontal viscosities. We consider the following system of equations $(NS_{\nu,\eta})$

$$\partial_t u^n + \mathrm{div}(u^n \otimes u^n) - \nu \partial_z^2 u^n - \eta \Delta_{x,y} u^n = -\nabla p \quad \text{in} \quad \Omega \tag{42}$$

$$\mathrm{div}(u^n) = 0 \quad \text{in} \quad \Omega \tag{43}$$

$$u^n = 0, \quad \text{in} \quad \partial\Omega \tag{44}$$

$$u^n(0) = u_0^n \quad \text{with} \quad \nabla . u_0^n = 0 \tag{45}$$

where $\Omega = \omega \times (0,h)$, or $\Omega = \omega \times (0,\infty)$, and $\omega = \mathbb{T}^2$, or $\mathbb{R}^2$, $\nu = \nu_n, \eta = \eta_n$. We want to point out here that this anisotropy is classical in geophysical flows. In fact instead of putting the classical viscosity $-\tilde{\nu}\Delta$ of the fluid in the equation, meteorologists often model turbulent diffusion by putting a viscosity of the form $-A_H\Delta_{x,y} - A_V\partial^2_{zz}$, where A_H and A_V are empiric constants, and where A_V is usually much smaller that A_H (for instance in the ocean, A_V ranges from 1 to $10^3 cm^2/s$ whereas A_H ranges from 10^5 to 10^8 cm^2/s. We recall that the viscosity of the water is of order 10^{-2} cm^2/s.) We refer to the book of J. Pedlovsky [144], Chapter 4 for a more complete discussion. When η, ν go to 0, we expect that u^n converges to the solution of the Euler system

$$\left\{ \begin{array}{c} \partial_t w + \operatorname{div}(w \otimes w) = -\nabla p \text{ in } \quad \Omega, \\ \operatorname{div}(w) = 0 \text{ in } \quad \Omega, \\ w.n = \pm w_3 = 0 \quad \text{on} \quad \partial\Omega, \\ w(t=0) = w^0. \end{array} \right. \tag{46}$$

It turns out that we are able to justify this formal derivation under an additional condition on the ratio of the vertical and horizontal viscosities.

Theorem 2.4 *Let $s > 5/2$, and*

$$w^0 \in H^s(\Omega)^3, \quad \operatorname{div}(w^0) = 0, \quad w^0.n = 0 \quad on \quad \partial\Omega.$$

We assume that $u^n(0)$ converges in $L^2(\Omega)$, to w^0 and $\nu, \eta, \nu/\eta$ go to 0, then any sequence of global weak solutions (à la Leray) u^n of (42- 45) satisfying the energy inequality satisfies

$$u^n - w \to 0 \quad in \quad L^\infty_{loc}([0,T^*); L^2(\Omega)),$$

$$\sqrt{\eta}\nabla_{x,y}u^n, \sqrt{\nu}\partial_z u^n \to 0 \ in \ L^2_{loc}([0,T^*); L^2(\Omega)),$$

where w is the unique solution of (46) in $C([0,T^*); H^s(\Omega)^3)$.

We give here a sketch of the proof and refer to [123] for a complete proof. The existence of global weak solutions for $(NS_{\nu,\eta})$, satisfying the energy inequality is due to J. Leray [102, 104, 103] (see also [90] and [163, 39] for some references about weak solutions of the Navier-Stokes)

$$\frac{1}{2}\|u^n(t)\|^2_{L^2} + \nu\int_0^t \|\partial_z u^n\|^2_{L^2} ds + \eta\int_0^t \|\partial_x u^n\|^2_{L^2} + \|\partial_y u^n\|^2_{L^2} \le \frac{1}{2}\|u^n_0\|^2_{L^2} \tag{47}$$

This estimate does not show that u^n is bounded in $L^2(0,T;H^1)$ and hence if we extract a subsequence still denoted by u^n converging weakly to u in $L^\infty(0,T;L^2)$, we cannot deduce that $u^n \otimes u^n$ converges weakly to $w \otimes w$. If we try to use energy estimates to show that $u^n - w$ remains small we see that the integrations by parts introduce terms that we cannot control, since $u^n - w$ does not vanish at the boundary. Hence, we must construct a boundary layer which allows us to recover the Dirichlet boundary conditions. Hence, $\mathcal{B}^n$ will be a corrector of small L^2 norm, and localized near $\partial\Omega$ (we take here the case where $\Omega = \omega \times (0,\infty)$ not to deal with boundary conditions near $z = h$)

$$\left\{ \begin{array}{l} \mathcal{B}^n(z=0) + w(z=0) = 0, \quad \mathcal{B}^n(z=\infty) = 0, \\ div(\mathcal{B}^n) = 0, \quad \mathcal{B}^n \to 0 \quad \text{in} \quad L^\infty_{loc}([0,T^*);L^2) \end{array} \right.$$

a possible choice is to take $\mathcal{B}^n$ of the form

$$\mathcal{B}^n = -w(z=0)e^{-\frac{z}{\sqrt{\nu\zeta}}} + ...$$

where ζ is a free parameter to be chosen later. We want to explain now the idea of the proof. Instead of using energy estimates on $u^n - w$, we will work with $v^n = u^n - (w + \mathcal{B}^n)$. Next we write the following equation satisfied by $w^{\mathcal{B}} = w + \mathcal{B}^n$ (in what follows, we will write $\mathcal{B}$ instead of $\mathcal{B}^n$)

$$\begin{array}{c} \partial_t w^{\mathcal{B}} + w^{\mathcal{B}}.\nabla w^{\mathcal{B}} - \nu\partial_z^2 w^{\mathcal{B}} - \eta\Delta_{x,y} w^{\mathcal{B}} = \\ \partial_t \mathcal{B} + \mathcal{B}.\nabla w^{\mathcal{B}} + w.\nabla\mathcal{B} - \nu\partial_z^2 w^{\mathcal{B}} - \eta\Delta_{x,y} w^{\mathcal{B}} - \nabla p \end{array} \tag{48}$$

which yields the following energy equality

$$\begin{array}{r} \frac{1}{2}\|w^{\mathcal{B}}(t)\|^2_{L^2} + \nu\int_0^t \|\partial_z w^{\mathcal{B}}(s)\|^2_{L^2}\,ds + \eta\int_0^t \|\partial_x w^{\mathcal{B}}\|^2_{L^2} + \|\partial_y w^{\mathcal{B}}\|^2_{L^2} = \\ \frac{1}{2}\|w^{\mathcal{B}}(0)\|^2_{L^2} + \int_0^t w^{\mathcal{B}}.[\partial_t\mathcal{B} + w.\nabla\mathcal{B} - \nu\partial_z^2 w^{\mathcal{B}} - \eta\Delta_{x,y} w^{\mathcal{B}}] \end{array} \tag{49}$$

Next, using the weak formulation of (42), we get for all t

$$\begin{array}{c} \int_\Omega u^n.w^{\mathcal{B}}(t) + \nu\int_0^t\int_\Omega \partial_z w^{\mathcal{B}}(s) u^n + \eta\int_0^t\int_\Omega \partial_x w^{\mathcal{B}}\partial_x u^n + \partial_y w^{\mathcal{B}}\partial_y u^n = \\ \int_\Omega u^n.w^{\mathcal{B}}(0) + \int_0^t\int_\Omega u^n.\nabla w^{\mathcal{B}} u^n + u^n.[\partial_t\mathcal{B} - w.\nabla w - \nu\partial_z^2 w^{\mathcal{B}} - \eta\Delta_{x,y} w^{\mathcal{B}}] \end{array} \tag{50}$$

Then adding up (47),(49) and subtracting (50), we get

$$\frac{1}{2}\|v(t)\|_{L^2}^2 + \nu\int_0^t \|\partial_z v\|_{L^2}^2 ds + \eta\int_0^t \|\partial_x v\|_{L^2}^2 + \|\partial_y u^n\|_{L^2}^2 \leq \frac{1}{2}\|v_0\|_{L^2}^2 +$$
$$\int_0^t \int_\Omega v.[\partial_t \mathcal{B} - \nu\partial_z^2 w^{\mathcal{B}} - \eta\Delta_{x,y} w^{\mathcal{B}}] + w.\nabla\mathcal{B}w^{\mathcal{B}} - u^n.\nabla w^{\mathcal{B}} u^n + w.\nabla w u^n \quad (51)$$

Finally, using that $\int (u.\nabla q)q = 0$, we get

$$\int_\Omega w.\nabla\mathcal{B}w^{\mathcal{B}} - u^n.\nabla w^{\mathcal{B}} u^n + w.\nabla w u^n = \int_\Omega -w^{\mathcal{B}}.\nabla\mathcal{B}v - \mathcal{B}.\nabla w v - v.\nabla w^{\mathcal{B}} v$$

Now, we want to use a Gronwall lemma to deduce that $\|v(t)\|_{L^2}^2$ remains small. By studying two terms among those occurring in the right hand side of the energy estimate (51), we want to show why we need the condition $\nu/\eta \to 0$. In fact

$$\begin{aligned}
\left|\int_\Omega v_3 \partial_z \mathcal{B} v\right| &\leq \int \frac{v_3}{z} \quad z^2\partial_z\mathcal{B} \quad \frac{v}{z} \\
&\leq C\|\partial_z v_3\|_{L^2} \sqrt{\nu\zeta}\|w\|_{L^\infty} \ \|\partial_z v\|_{L^2} \\
&\leq C\zeta\|\partial_z v_3\|_{L^2}^2\|w\|_{L^\infty}^2 + \frac{\nu}{4}\|\partial_z v\|_{L^2}^2
\end{aligned}$$

where, we have used the divergence-free condition $\partial_z v_3 = -\partial_x v_1 - \partial_y v_2$. We see from this term that we need the following condition to absorb the first term by the viscosity in (51) : $C\zeta\|w\|_{L^\infty}^2 \leq \eta$. On the other hand, the second term can be treated as follows

$$\begin{aligned}
\left|\nu\int_\Omega \partial_z^2\mathcal{B}v\right| &\leq \nu\|\partial_z v\|_{L^2}\|\partial_z\mathcal{B}\|_{L^2} \\
&\leq \frac{\nu}{4}\|\partial_z v\|_{L^2}^2 + \nu\|\partial_z\mathcal{B}\|_{L^2}^2 \\
&\leq \frac{\nu}{4}\|\partial_z v\|_{L^2}^2 + \nu\|w\|_{L^\infty}^2\frac{1}{\sqrt{\nu\zeta}}
\end{aligned}$$

The second term on the right hand side must go to zero, this is the case if we have $\nu/\zeta \to 0$. Finally, we see that

$$\text{If } \ \frac{\nu}{\eta} \to 0 \ \text{ then } \zeta = \frac{\eta}{C\|w\|_{L^\infty}^2}$$

is a possible choice.

2.3 Weak limit.

We want to conclude this section by mentioning an other important question in the inviscid limit of the Navier-Stokes even in the case without boundary. Consider any sequence of weak solutions to the Navier-Stokes system with viscosity ν. What can we say about this sequence when ν goes to 0. In subsection 2.1, we saw that if the initial data is regular enough then the sequence converges to the solution of the Euler system on some small time interval. Moreover, in the $2D$ case, we can take initial data such that the vorticity is a signed measure and still prove that the solutions of the Navier-Stokes system weakly converge to a solution of the Euler system [50]. Can we say more ? What can we say if we only assume that $u_0 \in L^2$? We mention here two attempts to explain what happens based on two notions of "very weak" solutions to the Euler system.

2.3.1 Measure valued solutions

In their three papers [59, 58, 57] Diperna and Majda studied the behavior of sequences of approximate solutions to the Euler system. In the introduction of [59], they state "a sequence of Leray-Hopf weak solutions of the Navier-Stokes equations converges in the high Reynolds number limit to a measure-valued solution of Euler defined for all positive times". They introduced the following notion of measure valued solutions to the Euler system.

Definition 2.5 *Let $\mathcal{O}$ be a smooth domain of $\mathbb{R}^d$, μ a nonnegative measure of $\mathcal{M}(\mathcal{O})$ and $(t,x) \to (\nu^1_{(t,x)}, \nu^2_{(t,x)})$ a $dt\, d\mu$-measurable map from $(0,T)\times\mathcal{O}$ to $\mathcal{M}^+(\mathbb{R}^d) \times Prob(\mathbb{S}^{d-1})$. We also denote $\mu = \mu_s + f dtdx$ the Lebesgue decomposition of μ into its singular and absolutely continuous parts. Then the triple (μ, ν^1, ν^2) is called a measure valued solution of the incompressible Euler system if*

$$div\Big[\big\langle \nu^1_{(t,x)}, \frac{v}{1+|v|^2}\big\rangle (1+f)\Big] = 0 \quad and$$

$$\int\int \phi_t . \Big\langle \nu^1_{(t,x)}, \frac{v}{1+|v|^2}\Big\rangle (1+f) dtdx + \nabla\phi : \Big\langle \nu^2_{(t,x)}, xi \times \xi \Big\rangle d\mu = 0 \qquad (52)$$

for all smooth divergence-free vector field $\phi(t,x)$.

Of course a weak solution u of the Euler system defines a measure valued solution by taking $f = \mu = |u|^2$, $\nu^1_{(t,x)} = \delta_{v=u(t,x)}$ and $\nu^2_{(t,x)}(\xi) = \delta_{\xi=\frac{u}{|u|}}$ if $u(t,x) \neq 0$.

They also define the notion of generalized Young measure for a sequence $\{v^\epsilon\}$ bounded in $L^2(\mathcal{O})$.

Theorem 2.6 *If $\{v^\epsilon\}$ is an arbitrary family of functions whose L^2 norm on a set $\mathcal{O}$ is uniformly bounded, then extracting a subsequence, there exist a measure $\mu \in \mathcal{M}(\mathcal{O})$ such that*

$$|v_\epsilon|^2 \to \mu \quad in \quad \mathcal{M}(\mathcal{O}), \tag{53}$$

and a μ-measurable map $x \to (\nu^1_{(x)}, \nu^2_{(x)})$ from $\mathcal{O}$ to $\mathcal{M}^+(\mathbb{R}^d) \times Prob(\mathbb{S}^{d-1})$ such that for all

$$g(v) = g_0(v)(1 + |v|^2) + g_H(\frac{v}{|v|})|v|^2,$$

where g_0 lies in the space $C_0(\mathbb{R}^d)$ of continuous function vanishing at infinity and g_H lies in the space $C(\mathbb{S}^{d-1})$ of continuous function on the unit sphere, we have

$$g(v_\epsilon) \to \Big\langle \nu^1_{(x)}, g_0(v) \Big\rangle (1+f)dx + \Big\langle \nu^2_{(x)}, g_H(v) \Big\rangle d\mu \quad in\ \mathcal{D}' \tag{54}$$

where f denotes the Radon-Nikodym derivative of μ with respect to dx. The triple (μ, ν^1, ν^2) is called the generalized Young measure of the sequence $\{v^\epsilon\}$.

The notion of generalized Young measure can be extended to the case the function v_ϵ also depend on t. The above two definitions are linked by the following theorem.

Theorem 2.7 *Assume v_ϵ is a sequence of functions satisfying* $\operatorname{div}(v_\epsilon)=0$*, v_ϵ is bounded in $L^2((0,T) \times \mathcal{O})$ and for all divergence-free test function ϕ in $C^\infty_0((0,T) \times \mathcal{O})$,*

$$\lim_{\epsilon \to 0} \int \int (\phi_t . v_\epsilon + \nabla \phi : v_\epsilon \times v_\epsilon) \quad dtdx = 0. \tag{55}$$

Then, if (μ, ν^1, ν^2) is a generalized Young measure of the sequence $\{v^\epsilon\}$ then it defines a measure-valued solution to the Euler system.

Of course, one of the main application of this theorem is the case where v_ϵ satisfies the Navier-Stokes equation with a vanishing viscosity since it implies (55).

2.3.2 Dissipative solutions

An other notion of "very weak" solutions to the Euler system was introduced by P.-L. Lions [108]. As stated by P.-L. Lions, it is not clear whether this notion is relevant. Its only merits are the fact that such solutions exist and are global and as long as a "smooth" solution exists with the same initial data, any such dissipative solution coincides with it. Let us point out that such a uniqueness property does not hold for the measure-valued solutions of the previous subsection. Before defining dissipative solutions, let us introduce few notations. For a divergence-free smooth test function v of $[0,\infty)\times\mathbb{R}^d$, we define

$$E(v) = -\frac{\partial v}{\partial t} - P(v.\nabla v) \tag{56}$$

where P is the Leray projector on divergence free vector fields. We also denote $d(v)_{ij} = \frac{1}{2}(\partial_i v_j + \partial_j v_i)$, the symmetric part of ∇v. For $t \geq 0$, let

$$||d^-||_\infty = ||\sup_{|\xi|=1} -(d\,\xi,\xi)_+||_{L^\infty(\mathbb{R}^d)} \tag{57}$$

Definition 2.8 *Let $u \in L^\infty(0,\infty;L^2)\cap C([0,\infty);L^2_w)$. Then u is a dissipative solution of the Euler system*

$$\left\{\begin{array}{c} \partial_t u + \operatorname{div}(u\otimes u) = -\nabla p \ \ in \quad \mathbb{R}^d, \\ \operatorname{div}(u) = 0 \ \ in \quad \mathbb{R}^d, \\ u(t=0) = u^0, \end{array}\right. \tag{58}$$

if $u(0) = u^0$, $\operatorname{div}(u) = 0$ and for all divergence-free smooth test function v, we have

$$\begin{aligned} ||(u-v)(t)||^2_{L^2(\mathbb{R}^d)} \leq e^{2\int_0^t ||d^-||_\infty}||(u-v)(0)||^2_{L^2(\mathbb{R}^d)} + \\ + 2\int_0^t\int_{\mathbb{R}^d} e^{2\int_s^t ||d^-||_\infty} E(v).(u-v)ds. \end{aligned} \tag{59}$$

In [108], P.-L. Lions proves the following result

Theorem 2.9 *Let u^ν be a sequence of Leray-weak solutions to the Navier-Stokes system with viscosity ν and initial data u^ν_0. In particular it satisfies*

$$\frac{d}{dt}||u^\nu||^2_{L^2(\mathbb{R}^d)} + \nu||\nabla u^\nu||^2_{L^2(\mathbb{R}^d)} \leq 0 \quad in \quad \mathcal{D}', \tag{60}$$

$u^\nu \in L^2(0,T;H^1)\cap L^\infty(0,\infty;L^2)\cap C([0,\infty);L^2_w)$ for all $T>0$ and $u^\nu(t)$ goes to u^ν_0 in $L^2(\mathbb{R}^d)$ when t goes to 0. Assume that u^ν_0 converges in L^2 to u^0 then, extracting a subsequence, u^ν converges weakly-$*$ in $L^\infty(0,\infty;L^2)$ to some u and converges weakly in L^2 uniformly in $t\in[0,T]$ to u. Moreover, u is a dissipative solution of the Euler system.

Let us give a sketch of the proof. From (60), we can deduce that for all divergence-free test function v, we have

$$\frac{d}{dt}||u^\nu - v||^2_{L^2(\mathbb{R}^d)} + \nu||\nabla u^\nu||^2_{L^2(\mathbb{R}^d)} \le 2||d^-||_\infty||u^\nu - v||^2_{L^2(\mathbb{R}^d)} + 2\int E(v).(u^\nu - v)dx + C(v)\nu||\nabla u^\nu||_{L^2} \quad (61)$$

Then, we can apply a Gronwall lemma to get

$$||(u^\nu - v)(t)||^2_{L^2(\mathbb{R}^d)} \le e^{2\int_0^t ||d^-||_\infty}||(u^\nu_0 - v(0))||^2_{L^2(\mathbb{R}^d)} + + 2\int_0^t\int_{\mathbb{R}^d} e^{2\int_s^t ||d^-||_\infty}E(v).(u^\nu - v)ds + C_T(v)\nu. \quad (62)$$

Then, we can extract a subsequence of u^ν which converges weakly-$*$ in $L^\infty(0,\infty;L^2)$. Passing to the limit in (62), we deduce that u is a dissipative solution of the Euler system.

3 Compressible-incompressible limit

It is well-known from a Fluid Mechanics viewpoint that one can derive formally incompressible models such as the Incompressible Navier-Stokes system or the Euler system from compressible ones namely compressible Navier-Stokes system (CNS) when the Mach number goes to 0 and the density becomes constant. There are several mathematical justifications of this derivation. One can put these works in two categories depending on the type of solutions considered. Indeed, one viewpoint consists on looking at local strong solutions and trying to prove existence on some time interval independent of the Mach number and then studying the limit when the Mach number goes to zero. This was initiated by Klainerman and Majda [97] (see also Ebin [62]). The second point of view consists on retrieving the Leray global weak solutions [104, 103] of the incompressible Navier-Stokes system starting from global weak solutions of the compressible Navier-Stokes system (see [111]). Let us also mention that there were many works about this limit during the last 10 years and that there are many review papers about it (see for instance [124, 70, 47, 155]).

3.1 Formal limit

We first wish to recall the general set up for such asymptotic problems. We will present it for the compressible isentropic Navier-Stokes system. The unknowns $(\tilde{\rho}, v)$ are respectively the density and the velocity of the fluid (gas) and solve on $(0, \infty) \times \mathbb{R}^N$

$$\frac{\partial \tilde{\rho}}{\partial t} + \operatorname{div}\,(\tilde{\rho} v) \;=\; 0\;, \quad \tilde{\rho} \geq 0, \tag{63}$$

$$\frac{\partial \tilde{\rho} v}{\partial t} + \operatorname{div}\,(\tilde{\rho} v \otimes v) - \tilde{\mu}\Delta v - \tilde{\xi}\nabla \operatorname{div} v + \nabla \tilde{p} \;=\; 0\;, \tag{64}$$

and

$$\tilde{p} \;=\; a\tilde{\rho}^{\gamma}\;, \tag{65}$$

where $N \geq 2$, $\tilde{\mu} > 0$, $\tilde{\mu} + \tilde{\xi} > 0$, $a > 0$ and $\gamma > 1$ are given.

From a physical view-point, the fluid should behave (asymptotically) like an incompressible one when the density is almost constant, the velocity is small and we look at large time scales. More precisely, we scale ρ and v (and thus p) in the following way

$$\tilde{\rho} = \rho(\epsilon t, x), \;\; v = \epsilon u(\epsilon t, x) \tag{66}$$

and we assume that the viscosity coefficients μ, ξ are also small and scale like

$$\tilde{\mu} = \epsilon \mu_\epsilon, \;\; \tilde{\xi} = \epsilon \xi_\epsilon \tag{67}$$

where $\epsilon \in (0,1)$ is a "small parameter " and the normalized coefficient $\mu_\epsilon, \xi_\epsilon$ satisfy

$$\mu_\epsilon \to \mu \;,\;\; \mu_\epsilon \to \xi \quad \text{as} \;\; \epsilon \text{ goes to } \;\; 0_+ \;. \tag{68}$$

We shall always assume that we have either $\mu > 0$ and $\mu + \xi > 0$ or $\mu = 0$.

With the preceding scalings, the system (63)-(65) yields

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \operatorname{div}(\rho u) = 0\;, \quad \rho \geq 0, \\[2ex] \dfrac{\partial \rho u}{\partial t} + \operatorname{div}(\rho u \otimes u) - \mu_\epsilon \Delta u - \xi_\epsilon \nabla \operatorname{div} u + \dfrac{a}{\epsilon^2}\nabla \rho^\gamma = 0\;. \end{cases} \tag{69}$$

We may now explain the heuristics which lead to incompressible models. First of all, the second equation (for the momentum ρu) indicates that ρ should be like $\bar{\rho}+O(\epsilon^2)$ where $\bar{\rho}$ is a constant. Of course, $\bar{\rho} \geq 0$ and we always assume that $\bar{\rho} > 0$ (in order to avoid the trivial case $\bar{\rho} = 0$). Obviously, we need to assume this property holds initially (at $t = 0$). And, let us also remark that by a simple (multiplicative) scaling, we may always assume without loss of generality that $\bar{\rho} = 1$.

Since ρ goes to 1, we expect that the first equation in (69) yields at the limit : div $u = 0$. And writing $\nabla \rho^\gamma = \nabla(\rho^\gamma - 1)$, we deduce from the second equation in (69) that we have in the case when $\mu > 0$

$$\frac{\partial u}{\partial t} + \operatorname{div}(u \otimes u) - \mu \Delta u + \nabla \pi = 0 \tag{70}$$

or when $\mu = 0$

$$\frac{\partial u}{\partial t} + \operatorname{div}(u \otimes u) + \nabla \pi = 0 \tag{71}$$

where π is the "limit" of $\dfrac{\rho^\gamma - 1}{\epsilon^2}$. In other words, we recover the incompressible Navier-Stokes equations (70) or the incompressible Euler equations (71), and the hydrostatic pressure appears as the limit of the "renormalized" thermodynamical pressure $(\dfrac{\rho^\gamma - 1}{\epsilon^2})$. In fact, as we shall see later on, the derivation of (70) (or (71)) is basically correct even globally in time, for global weak solutions ; but the limiting process for the pressure is much more involved and may, depending on the initial conditions, incorporate additional terms coming from the oscillations in $\operatorname{div}(\rho_\epsilon u_\epsilon \otimes u_\epsilon)$.

This section about the compressible incompressible limit is organized as follows. In the next subsection 3.2, we recall the results of Klainerman and Majda [97, 98] for the strong solutions to the isentropic compressible Navier-Stokes when the Mach number goes to zero. Then, we give several extensions of that result by taking general or "ill-prepared" initial data [166, 154]. Also we state result about long time existence for the slightly compressible system [87, 89]. We also present results in "almost" critical spaces [45, 46]. In subsection 3.3, we recall the results of convergence from the global weak solutions to the isentropic compressible Navier-Stokes towards the global weak solutions of the incompressible Navier-Stokes. In the last subsection 3.5, we state some newer results about the non-isentropic case [133, 134].

We will not mention result about the steady problem and refer to [17, 106, 111].

3.2 The case of strong solutions

The first mathematical justification of the incompressible limit is due to Ebin [62]. By using Lagrangian coordinates and a geometric description of the equations, he proved that "slightly compressible fluid motion can be described as a motion with a strong constraining force, while incompressible fluid flow is the analogous constrained motion." The first justification using PDE methods was done by Klainerman and Majda [97, 98] using the theory of singular limits of symmetric hyperbolic systems. We should also mention the work of Kreiss [99] about problems with different time scales but which requires the control of more time derivatives at time $t = 0$.

We consider the compressible Euler system which can be recovered from (69), by taking $\mu_\epsilon = \xi_\epsilon = 0$,

$$\begin{cases} \dfrac{\partial \rho_\epsilon}{\partial t} + u_\epsilon \cdot \nabla \rho_\epsilon + \rho_\epsilon \mathrm{div} u_\epsilon = 0 \ , \quad \rho_\epsilon \geq 0, \\ \\ \rho_\epsilon \left(\dfrac{\partial u_\epsilon}{\partial t} + u_\epsilon \cdot \nabla u_\epsilon \right) + \dfrac{1}{\epsilon^2} \nabla p_\epsilon = 0 \end{cases} \tag{72}$$

where p_ϵ and ρ_ϵ are related by $p_\epsilon = a\rho_\epsilon^\gamma$ where $a > 0$ and $\gamma \geq 1$ are given constants. They consider the above system in the torus or the whole space, $\Omega = \mathbb{T}^N$ or $\Omega = \mathbb{R}^N$ with the following initial data

$$u_\epsilon(t = 0, x) = u_\epsilon^0(x), \quad p_\epsilon(t = 0, x) = p_\epsilon^0(x). \tag{73}$$

Notice that we can retrieve the initial data for ρ_ϵ from the initial data for p_ϵ. Here $\|.\|_s$ will denote the H^s norm and $s_0 = [\frac{N}{2}] + 1$.

Theorem 3.1 *Assume the initial data (73) satisfies*

$$\|u_\epsilon^0(x)\|_s + \frac{1}{\epsilon} \|p_\epsilon^0(x) - \underline{p}\|_s \leq C_0 \tag{74}$$

for some constants $\underline{p} > 0$ *and* C_0 *and some* $s \geq s_0 + 1$*. Then there exists an* ϵ_0 *and a fixed time interval* $[0, T]$ *with* T *depending only upon* $\|u_\epsilon^0(x)\|_{s_0+1} + \frac{1}{\epsilon}\|p_\epsilon^0(x) - p_0\|_{s_0+1}$ *and a constant* C_s *such that for* $\epsilon < \epsilon_0$*, a classical solution of the compressible Euler system exists on* $[0, T] \times \Omega$ *and satisfies*

$$\sup_{0 \leq t \leq T} \|u_\epsilon\|_s + \frac{1}{\epsilon} \|p_\epsilon^0 - \underline{p}\|_s + \epsilon \|\frac{\partial u_\epsilon}{\partial t}\|_{s-1} + \|\frac{\partial p_\epsilon}{\partial t}\|_{s-1} \leq C_s. \tag{75}$$

Moreover if the initial data satisfies the additional condition

$$
\begin{gathered}
u_\epsilon^0(x) = u^0(x) + \epsilon u^1(x), \quad \mathrm{div} u^0 = 0, \\
p_\epsilon^0(x) = \underline{p} + \epsilon^2 p^1(x), \\
\|u^1(x)\|_s + \|p^1(x)\|_s \leq C_0
\end{gathered}
\tag{76}
$$

then, on the same time interval $[0,T]$, *we have*

$$
\sup_{0\leq t\leq T} \|\frac{\partial u_\epsilon}{\partial t}\|_{s-1} + \epsilon^{-1}\|\frac{\partial p_\epsilon}{\partial t}\|_{s-1} \leq C_s^1 \tag{77}
$$

and as ϵ *goes to* 0, u_ϵ *converges weakly in* $L^\infty([0,T];H^s)$ *and uniformly in* $C_{loc}([0,T]\times\Omega)$ *to* u^∞ *where* u^∞ *satisfies the incompressible Euler system*

$$
\begin{cases}
\dfrac{\partial u^\infty}{\partial t} + u^\infty \cdot \nabla u^\infty + \nabla p^\infty = 0 \\
u^\infty(t=0,x) = u^0(x), \quad \mathrm{div} u^\infty = 0.
\end{cases}
\tag{78}
$$

The condition (76) means that the flow is initially almost incompressible and that the density is initially almost constant. These data are called "well-prepared" initial data. The more general condition (74) will be called general initial data or "ill-prepared" initial data. Notice that we still need to assume that $p_\epsilon^0 - \underline{p}$ is of order ϵ this is because, we need to make a change a variable $q_\epsilon = \epsilon^{-1}(p_\epsilon - \underline{p})$ to write our system in a form which is suitable for energy estimates, we will denote $q_\epsilon^0 = \epsilon^{-1}(p_\epsilon^0 - \underline{p})$.

Idea of the proof: We rewrite the system in terms of the new unknowns (u_ϵ, q_ϵ) where $q_\epsilon = \epsilon^{-1}(p_\epsilon - \underline{p})$

$$
\begin{cases}
\dfrac{\partial q_\epsilon}{\partial t} + u_\epsilon \cdot \nabla q_\epsilon + \dfrac{\gamma}{\epsilon}(\underline{p} + \epsilon q_\epsilon)\mathrm{div} u_\epsilon = 0 \;, \quad \rho_\epsilon \geq 0, \\[2ex]
\dfrac{\partial u_\epsilon}{\partial t} + u_\epsilon \cdot \nabla u_\epsilon + \dfrac{1}{\epsilon(\underline{p} + \epsilon q_\epsilon)^{1/\gamma}} \nabla q_\epsilon = 0
\end{cases}
\tag{79}
$$

To prove (75), we just need to prove H^s estimates on some time interval $[0,T]$ which is independent of ϵ. For each ϵ, we denote

$$
E_s(t) = \int \sum_{|\alpha|=s} \frac{1}{(\underline{p} + \epsilon q_\epsilon)^{1/\gamma}} |\partial^\alpha q_\epsilon|^2 + \gamma(\underline{p} + \epsilon q_\epsilon)|\partial^\alpha u_\epsilon|^2. \tag{80}
$$

Then, we can prove that $\partial_t E_s \leq C(E_s)^2$ where C does not depend on $\epsilon < \epsilon_0$. This shows that there exists a time of existence T which is uniform in ϵ.

Next, we have to prove (77) and the convergence towards the incompressible system (78) under the well-prepared condition (76). We notice that, taking the time derivative of (79), we can write a hyperbolic equation for $(\partial_t u_\epsilon, \partial_t q_\epsilon)$ which is similar to (79). To prove uniform bounds for $(\partial_t u_\epsilon, \partial_t q_\epsilon)$ in H^{s-1} on some time interval $[0,T]$ we only need to have bounds in H^{s-1} initially. This follows immediately from (76). Hence, if (76) holds then (77) holds. Moreover, by simple compactness arguments, we can extract a subsequence such that (u_ϵ, q_ϵ) converges in $C([0,T]; H^{s-\kappa}_{loc})$ to some (u,q) for $\kappa > 0$. Then, it is easy to see that u satisfies the Euler system (78) by passing weakly to the limit in the different terms. Since, we have uniqueness for (78), we deduce the convergence of the whole sequence. $\square$

Remark 3.2 *1) In [97, 98], the authors also deal with the Navier-Stokes case by proving that the viscosity does not affect the leading hyperbolic behavior.*

2)For the "well-prepared" case, the convergence stated in the theorem can be improved to a convergence in $C([0,T]; H^s)$ (see Beirão da Veiga [18, 19]).

During the last 25 years there were different extensions of this result in different directions. First, there were results trying to take more general initial data. These results require some analysis of the acoustic waves. Then, there were results about more general models, namely the non-isentropic model (the entropy is not constant and is transported by the flow). Also, there were results trying to improve the minimum regularity required for the convergence.

3.2.1 General initial data

In the whole space $\mathbb{R}^N$ (see Ukai [166]) or in the exterior of a bounded domain (see Isozaki [92, 93]), the result of [98] has been extended to the case of general initial data or "ill-prepared" initial data. The convergence towards the incompressible limit holds locally in space. However, we do not have uniform convergence near $t = 0$ due to the presence of an initial layer in time. This layer comes from acoustic waves that go to infinity. We have the following result

Theorem 3.3 *($\Omega = \mathbb{R}^N$) Assume the initial data (73) satisfies (74) and that $(u^0_\epsilon(x), q^0_\epsilon(x))$ converges to some $(u^0(x), q^0(x))$ in H^s, then the solution* *constructed in theorem 3.1 satisfies*

$$(q_\epsilon, u_\epsilon) \ \to \ (0, u^\infty) \tag{81}$$

weakly in* $L^\infty((0,T); H^s)$ *and strongly in* $C^0_{loc}((0,T] \times \mathbb{R}^N)$ *where* u^∞ *is the unique solution to the incompressible Euler system (78) with the initial data* Pu^0 *where* P *is the Leray projection onto divergence free vector fields* $P = Id - \nabla\Delta^{-1}\nabla\cdot$.

In the periodic case $\mathbb{T}^N$, Schochet [154] extends the result of [98] to the case of "ill-prepared" initial data. He proves the same theorem 3.3 in the periodic case with the only difference that the $(0, u^\infty)$ is replaced by (c, u^∞) for some constant c and that the convergence is only weak due to the acoustic waves. The convergence is strong for the divergence-free part Pu_ϵ.

Theorem 3.4 *($\Omega = \mathbb{T}^N$) Assume the initial data (73) satisfies (74) and that* $(u^0_\epsilon(x), q^0_\epsilon(x))$ *converges to some* $(u^0(x), q^0(x))$ *in* H^s*, then the solution constructed in theorem 3.1 satisfies*

$$(q_\epsilon, u_\epsilon) \ \to \ (c, u^\infty) \tag{82}$$

weakly in* $L^\infty((0,T); H^s)$ *where* u^∞ *is the unique solution to the incompressible Euler system (78) with the initial data* Pu^0 *where* P *is the Leray projection onto divergence free vector fields* $P = Id - \nabla\Delta^{-1}\nabla\cdot$. *Moreover,* Pu_ϵ *converges strongly in* $C^0_{loc}([0,T] \times \mathbb{T}^N)$ *to* u^∞.

Idea of the proofs:

The idea of theorem 3.4 is to use the group method to filter the oscillations. We also would like to mention that ideas close to the group method were also developed by Joly, Métivier and Rauch [94]. We introduce the following group $(\mathcal{L}(\tau), \tau \in \mathbb{R})$ defined by $e^{\tau L}$ where L is the operator defined on $\mathcal{D}' \times (\mathcal{D}')^N$, by

$$L \begin{pmatrix} \varphi \\ v \end{pmatrix} \ = - \begin{pmatrix} \gamma \underline{p} \mathrm{div} v \\ \frac{1}{\underline{p}^{1/\gamma}} \nabla \varphi \end{pmatrix} . \tag{83}$$

It is easy to check that $e^{\tau L}$ is an isometry on each $H^s \times (H^s)^N$ for all $s \in \mathbb{R}$ and for all τ. This show that if we define $\begin{pmatrix} \varphi(\tau) \\ v(\tau) \end{pmatrix} \ = \ e^{\tau L} \begin{pmatrix} \varphi_0 \\ v_0 \end{pmatrix}$ then it solves

$$\frac{\partial \varphi}{\partial \tau} \ = \ -\gamma \underline{p} \,\mathrm{div} v \ , \ \frac{\partial v}{\partial \tau} \ = \ -\frac{1}{\underline{p}^{1/\gamma}} \nabla \varphi.$$

If we denote $U_\epsilon = {}^t(q_\epsilon, u_\epsilon)$, then $V_\epsilon = \mathcal{L}(-\frac{t}{\epsilon})U_\epsilon$ is such that $\partial_t V_\epsilon$ is bounded in $L^\infty(0,T;H^{s-1})$. Then, we can use compactness argument to extract a subsequence which converges to some V in $C([0,T];H^{s-\kappa}_{loc})$. Now, passing to the limit in the equation satisfied by V requires the study of resonances. It turns out these resonances do not affect the divergence-free flow. See also subsection 3.3.5 for more about resonances.

If we consider the whole space case, we notice that the long time behavior of the operator $e^{\tau L}$ is not the same in the whole space and in the torus. Indeed, in the whole space we have dispersion and the following Strichartz [158] type estimate holds

$$\left\| e^{\frac{t}{\epsilon}L} \begin{pmatrix} \psi \\ \nabla\phi \end{pmatrix} \right\|_{L^p(\mathbb{R};W^{s,q}(\mathbb{R}^N))} \leq C\epsilon^{1/p} \left\| \begin{pmatrix} \psi \\ \nabla\phi \end{pmatrix} \right\|_{H^{s+\sigma}} \tag{84}$$

for all $p, q > 2$ and $\sigma > 0$ such that

$$\frac{2}{q} = (N-1)\Big(\frac{1}{2} - \frac{1}{p}\Big), \quad \sigma = \frac{1}{2} + \frac{1}{p} - \frac{1}{q}.$$

This dispersion allows for the convergence in $C^0_{loc}((0,T]\times\mathbb{R}^N)$ (see also [52]).

3.2.2 Long time existence for the compressible system

In [87], Hagstrom and Lorenz give a result about the global existence of strong solutions to the slightly compressible Navier-Stokes system in $2D$ for initial data which are close to the incompressible, namely satisfying a condition of the type (76). Also, in [89], Hoff gives a similar result in dimension 2 or 3 with a force term under some assumptions about the limit system. These two results use different properties of the system. However, they both use in a critical way the presence of the viscosity. Consider the system (69) with $a\gamma = 1$, $\mu_\epsilon = \mu > 0$, $\xi_\epsilon = \xi$ and $\mu + \xi > 0$. The limit system reads

$$\begin{cases} \dfrac{\partial u^\infty}{\partial t} + u^\infty \cdot \nabla u^\infty - \mu u^\infty + \nabla p^\infty = 0 \\ u^\infty(t=0,x) = u^0(x), \quad \mathrm{div} u^\infty = 0. \end{cases} \tag{85}$$

In [87], the following result is proved

Theorem 3.5 *Let $u^0 \in C^\infty(\mathbb{T}^2)$ be an incompressible velocity field and $\pi_0(x) = p^\infty(t=0,x)$ where (u^∞, p^∞) is the solution to (85), $-\Delta\pi_0(x) = \sum_{i,j=1}^2 \partial_i u_j^0 \partial_j u_i^0$.*

There exists $\epsilon_0 = \epsilon_0(u^0, \mu, \xi)$ and $\delta_0 = \delta_0(u^0, \mu, \xi)$ such that if $0 < \epsilon < \epsilon_0$ and the initial data $(\rho^0_\epsilon, u^0_\epsilon)$ for (69) satisfies

$$\|u^0_\epsilon(x) - u^0\|_3 + \epsilon^{-1}\|\rho^0_\epsilon(x) - 1 - \epsilon^2\pi_0\|_3 \leq \delta_0 \tag{86}$$

then there exists a global solution $(\rho_\epsilon, u_\epsilon) \in C^\infty([0,\infty) \times \mathbb{T}^2)$ to (69) which locally converges to $(1, u^\infty)$ when ϵ goes to zero.

We also refer to Gallagher [69] for a similar result.

Idea of the proof : We write $u^\epsilon = u^\infty + u'$ and $\rho^\epsilon = 1 + \epsilon^2(\pi^\infty + \rho')$. Then, we denote

$$w = \begin{pmatrix} u' \\ \epsilon\rho' \end{pmatrix}.$$

Hence, w satisfies the following equation

$$w_t + (u^\infty + u') \cdot \nabla w = A_\epsilon w + G \tag{87}$$

where A_ϵ is a constant coefficient operator given by

$$A_\epsilon = -\frac{1}{\epsilon}\begin{pmatrix} 0 & 0 & \partial_x \\ 0 & 0 & \partial_y \\ \partial_x & \partial_y & 0 \end{pmatrix} + \begin{pmatrix} \mu\Delta + \xi\partial^2_{xx} & \xi\partial^2_{xy} & 0 \\ \xi\partial^2_{xy} & \mu\Delta + \xi\partial^2_{yy} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{88}$$

and G consists of nonlinear terms involving (u^∞, π^∞) and w. It turns out that this term can be controlled for long time due to the exponential decay of the incompressible Navier-Stokes solution u^∞.

Equation (87) is a coupled parabolic-hyperbolic system where the large hyperbolic part is symmetric. Even though (87) is not completely parabolic, in particular there is no viscosity in the third equation, the coupling between the three equations yields some decay for w. This cannot be seen from the standard L^2 estimate but requires the use of a different scalar product. We denote $\hat{A}_\epsilon(k)$ the symbol of A_ϵ, $k \in \mathbb{Z}^2$ which can be obtained from A_ϵ by replacing ∂_x by ik_1 and ∂_y by ik_2. Then a symmetrizer $H(k)$ can be constructed for (87) satisfying the following lemma

Lemma 3.6 *[87] There exist c_0, c_1, C_1, C_2 depending on μ, ξ, ϵ_0 such that for $0 < \epsilon < \epsilon_0$ there are Hermitian matrices $H(k,\epsilon) \in \mathbb{C}^{3\times 3}$ satisfying*

$$\begin{gathered} 0 < (I - C_1\epsilon I) \leq H \leq (I + C_1\epsilon I) \\ q^*(H\hat{A}_\epsilon(k) + \hat{A}_\epsilon(k)^* H)q \leq -c_0 q^* H q - c_1|k|^2(|q_1|^2 + |q_2|^2) \quad \forall\, q \in \mathbb{C}^3, \\ |H - I| \leq \frac{C_2\epsilon}{|k|}. \end{gathered} \tag{89}$$

Using this lemma, we can define a new inner product on $L^2(\mathbb{T}^2, \mathbb{R}^3)$ by

$$(w_1, w_2)_H = \sum_{k\in\mathbb{Z}^2} \hat{w}_1(k)^* H(k,\epsilon)\hat{w}_2(k)$$

which is used to prove the exponential decay. □

In [89], Hoff takes an other approach to prove the long time existence for the slightly compressible Navier-Stokes. He uses the effective viscous flux F given by

$$F = (\mu + \xi)\mathrm{div} u_\epsilon - \epsilon^{-2}[\rho_\epsilon^\gamma - 1] \tag{90}$$

which satisfies the following elliptic equation

$$\Delta F = \mathrm{div}(\rho_\epsilon \partial_t u_\epsilon + \rho_\epsilon u_\epsilon \cdot \nabla u_\epsilon - \rho_\epsilon f) \tag{91}$$

where f is the force term. It turns out that this equation yields some regularity for F which is not shared by $\mathrm{div} u_\epsilon$ or by $\epsilon^{-2}[\rho_\epsilon^\gamma - 1]$. Then, Hoff uses the equation for the density to deduce that

$$(\mu + \xi)\partial_t(\rho_\epsilon - 1) + \epsilon^{-2}[\rho_\epsilon^\gamma - 1] = -\rho_\epsilon F \tag{92}$$

from which we can deduce some decay for $(\rho_\epsilon - 1)$ if we have some good control on F. We refer to [89] for more details.

3.2.3 Convergence in critical spaces

The compressible Navier-Stokes system (69) is invariant, up to a change of the pressure law, under the transformation

$$\begin{aligned} (\rho(t,x), u(t,x)) &\rightarrow (\rho(l^2t, lx), lu(l^2t, lx)) & (93)\\ P(\rho) &\rightarrow l^2 P(\rho). & (94) \end{aligned}$$

Hence it seems natural to consider initial data $(\rho^0, u^0) \in \dot{H}^{d/2} \times H^{d/2-1}$. For fixed ϵ the local existence for (69) in the critical Besov space $B^{d/2}_{2,1} \times B^{d/2-1}_{2,1}$ was performed by Danchin [44]. He also proves global existence if the data is small. We refer to [44] for the precise definition of the Besov space $B^{d/2}_{2,1}$. We only recall that unlike $H^{d/2}$, $B^{d/2}_{2,1}$ is injected in L^∞.

In [46] and [45], Danchin proves the convergence of the solutions constructed in [44] towards solutions of the incompressible Navier-Stokes system.

More precisely for the critical case, namely $B^{d/2}_{2,1} \times B^{d/2-1}_{2,1}$ he proves a global existence and convergence result but only for small data. For large data he works with spaces which are slightly more regular, namely $B^{d/2+\kappa}_{2,1} \times B^{d/2-1+\kappa}_{2,1}$ or the Sobolev spaces with the same regularity. Moreover, he proves the convergence towards the incompressible Navier-Stokes system as long as the solution of the limit system exists

3.3 The case of global weak solutions

Global weak solutions to the isentropic Navier-Stokes system were constructed by P.-L. Lions [109] (see also Feireisl [64] and Novotny and Straskraba [139]). We also refer to [65] for a review paper about the isentropic Navier-Stokes system and to [66] for the existence of weak solutions to the full compressible system. In this subsection, we would like to study the behavior of the weak solutions constructed in [109] when the Mach number goes to zero. The first paper treating this question is [111]. In [111], the group method was used to pass to the limit in the nonlinear term. This yields the convergence in the periodic case. The result of [111] was then extended in [54] and [52] to deal with the case of a bounded domain or the whole space case. In [54], the presence of a boundary layer is responsible of the damping of the acoustic waves. In [52] the dispersion of the acoustic waves yields the local strong convergence towards the incompressible solution.

In the next subsection 3.3.1, we will present in some details the simple result of [113] were the convergence is proved locally in space. This proof is independent of the boundary condition. In particular it also holds for the exterior domain.

3.3.1 The local method

Let Ω be an open bounded set in $\mathbb{R}^N$. For $\epsilon \in (0,1]$, we consider $(\rho_\epsilon, u_\epsilon)$ a weak solution of

$$\left\{ \begin{array}{l} \dfrac{\partial \rho_\epsilon}{\partial t} + \mathrm{div}(\rho_\epsilon u_\epsilon) \ = \ 0 \ , \quad \rho_\epsilon \geq 0 \\ \\ \dfrac{\partial \rho_\epsilon u_\epsilon}{\partial t} + \mathrm{div}(\rho_\epsilon u_\epsilon \otimes u_\epsilon) - \mu \Delta u_\epsilon - \xi \nabla \mathrm{div} u_\epsilon + \dfrac{a}{\epsilon^2} \nabla \rho_\epsilon^\gamma = 0 \end{array} \right. \tag{95}$$

in $(0,T)\times\Omega$, $T>0$, $a>0$, $\gamma>\dfrac{N}{2}$, $\mu>0$ and $\mu+\xi>0$. We assume that $\rho_\epsilon\in L^\infty(0,T;L^\gamma)\cap C([0,T];L^1)$, $\rho_\epsilon|u_\epsilon|^2\in L^\infty(0,T;L^1)$, $u_\epsilon\in L^2(0,T;H^1)$ and that the total energy is bounded namely

$$\begin{cases} \displaystyle\int_\Omega \rho_\epsilon|u_\epsilon|^2+\frac{1}{\epsilon^2}\Big[\rho_\epsilon^\gamma-\bar\rho_\epsilon^\gamma-\gamma\bar\rho_\epsilon^{\gamma-1}(\rho_\epsilon-\bar\rho_\epsilon)\Big]\,dx\le C, \quad \text{a.e. } t\in(0,T)\\[2ex] \displaystyle\int_0^T dt\int_\Omega dx\;|Du_\epsilon|^2\;\le\;C \end{cases} \tag{96}$$

for some positive constant C independent of ϵ, where $\bar\rho_\epsilon$ is a positive constant such that $\bar\rho_\epsilon$ and $1/\bar\rho_\epsilon$ are bounded independently of ϵ.

We denote ρ_ϵ^0 and m_ϵ^0 the initial conditions for ρ_ϵ and $\rho_\epsilon u_\epsilon$. We also assume that $\dfrac{|m_\epsilon^0|^2}{\rho_\epsilon^0}$, m_ϵ^0, ρ_ϵ^0 are bounded in L^1, $L^{2\gamma/(\gamma+1)}$, L^γ respectively. Extracting subsequences, we can assume that ρ_ϵ, $\rho_\epsilon u_\epsilon$, $\sqrt{\rho_\epsilon}u_\epsilon$, u_ϵ, ρ_ϵ^0, m_ϵ^0, $\dfrac{m_\epsilon^0}{\sqrt{\rho_\epsilon^0}}$ converge weakly when ϵ goes to zero 0, towards ρ, m, w, u, ρ^0, m^0, $\tilde u^0$ (respectively in $L^\infty(0,T;L^\gamma)-w*$, $L^\infty(0,T;L^{2\gamma/(\gamma+1)})-w*$, $L^\infty(0,T;L^2)-w*$, $L^2(0,T;H^1)$, L^γ, $L^{2\gamma/(\gamma+1)}$, L^2) and that $\bar\rho_\epsilon$ converges towards $\bar\rho$. Finally, we denote $V_0=\{u\in L^2(\Omega),\displaystyle\int_\Omega u\varphi dx=0\ \forall\varphi\in C_0^\infty(\Omega),\ \mathrm{div}\ \varphi=0\ \text{ in }\ \Omega\}$ (if Ω is regular, than $V_0=\{\nabla p,\ p\in H^1(\Omega)\}$.

The main result of [113] is the following

Theorem 3.7 *Under the above conditions*

i) ρ_ϵ converges to $\bar\rho$ in $L^\infty(0,T;L^\gamma)$, and $m\equiv\sqrt{\bar\rho}w\equiv\bar\rho u$.

ii) The weak limit u is a solution of the incompressible Navier-Stokes system

$$\begin{cases} \dfrac{\partial u}{\partial t}+\mathrm{div}(u\otimes u)-\nu\Delta u+\nabla\pi=0\ ,\ \mathrm{div}u=0 \text{ in } \ \Omega\times(0,T)\\[2ex] u(t=0,x)\;=\;u^0(x) \end{cases} \tag{97}$$

with $u\in L^2(0,T;H^1)\cap L^\infty(0,T;L^2)$, $\pi\in\mathcal{D}'$ and $\nu=\dfrac{\mu}{\bar\rho}$ and $u^0\in\tilde u^0+V_0$.

Remark 3.8 *1) For the existence of solutions to the compressible Navier-Stokes satisfying the conditions stated above, we refer to [109].*

2) Theorem 3.7 does not say anything about the boundary condition satisfied by u. This is natural since there is no boundary condition for the initial system (95). This is the reason we have and initial condition $u^0 \in \tilde{u}^0 + V_0$ *which may seem vague. However, if we fix some boundary conditions, then* u^0 *will be completely determined. This will be done in the next subsections.*

Idea of the proof:

To simplify the proof, we assume that $\bar{\rho}_\epsilon$ goes to $\bar{\rho} = 1$.

<u>Convergence of ρ_ϵ to 1</u>

We claim that ρ_ϵ converges to 1 in $C([0,\infty); L^\gamma)$: indeed, for ϵ small enough $\bar{\rho}_\epsilon \in \left(\frac{1}{2}, \frac{3}{2}\right)$ and thus for all $\delta > 0$, there exists some $\nu_\delta > 0$ such that

$$x^\gamma + (\gamma - 1)(\bar{\rho}_\epsilon)^\gamma - \gamma x (\bar{\rho}_\epsilon)^{\gamma-1} \geq \nu_\delta |x - \bar{\rho}_\epsilon|^\gamma \text{ if } \ |x - \bar{\rho}_\epsilon| \ \geq \ \delta, \ x \geq 0 \ .$$

Hence,

$$\begin{aligned} \sup_{t\geq 0} \int |\rho_\epsilon - 1|^\gamma \ &\leq \ \delta^\gamma |\Omega| + \sup_{t \geq 0} \Big[\int 1_{(|\rho_\epsilon - 1| \geq \delta)} |\rho_\epsilon - \bar{\rho}_\epsilon|^\gamma \Big] + C |\overline{\rho_\epsilon} - 1|^\gamma \\ &\leq \ \delta^\gamma |\Omega| + \frac{C\epsilon^2}{\nu_\delta} + C|\overline{\rho_\epsilon} - 1|^\gamma \end{aligned}$$

and we conclude upon letting first ϵ go to 0 and then δ go to 0. Actually, we need more information about this convergence and more precisely, denoting $\varphi_\epsilon = \dfrac{\rho_\epsilon - \bar{\rho}_\epsilon}{\epsilon}$ we can prove using some convexity inequalities that φ_ϵ is bounded in $L^\infty(0,T;L^2)$ if $\gamma \geq 2$. If $\gamma < 2$, then $\varphi_\epsilon 1_{(|\rho_\epsilon - 1| \leq 1/2)}$ is bounded in $L^\infty(0,T;L^2)$ and $||\varphi_\epsilon 1_{(|\rho_\epsilon - 1| > 1/2)}||_{L^\infty(0,T;L^\gamma)} \leq C \epsilon^{\frac{2}{\gamma} - 1}$.

Next, we notice that $\tau_\epsilon = \rho_\epsilon u_\epsilon \otimes u_\epsilon$ is bounded in $L^\infty(0,T;L^1) \cap L^2(0,T;L^q)$ with $\dfrac{1}{q} = \dfrac{1}{\gamma} + \dfrac{N-2}{2N}$ if $N \geq 3$, $1 \leq q < \gamma$ if $N = 2$. Extracting a subsequence, we denote by τ a weak limit of τ_ϵ. Passing to the limit in the first equation of (95), we deduce that $u \in L^2(0,T;H^1)$ satisfies $\mathrm{div} u = 0$ in $\Omega \times]0,T[$. Passing to the limit in the second equation of (95), we get

$$\frac{\partial u}{\partial t} + \mathrm{div}\ \tau - \mu \Delta u + \nabla \pi_1 \ = \ 0 \tag{98}$$

where $\pi_1 \in \mathcal{D}'(\Omega \times (0,T))$. We just need to prove that $\mathrm{div}(\tau) = \mathrm{div}(u \times u) + \nabla \pi_2$. It turns out that in general π_2 does not vanish.

Convergence of u_ϵ in the regular case

First, we assume that φ_ϵ, $\pi_\epsilon = \frac{a}{\epsilon^2}(\rho_\epsilon^\gamma - \bar{\rho}_\epsilon^\gamma - \gamma(\rho_\epsilon - \bar{\rho}_\epsilon))$, $m_\epsilon = \rho_\epsilon u_\epsilon$, u_ϵ are regular in x, uniformly in ϵ, i.e. φ_ϵ, π_ϵ and m_ϵ, are bounded in $L^\infty(0,T;H^s)$ and u_ϵ is bounded in $L^2(0,T;H^s)$ for all $s \geq 0$. Next, we want to show that

$$\mathrm{div}(\rho_\epsilon u_\epsilon \otimes u_\epsilon) \underset{\epsilon}{\rightharpoonup} \mathrm{div}(u \otimes u) + \nabla \pi_2 \tag{99}$$

for some distribution π_2. To this end, we will pass to the limit locally in x when ϵ goes to 0. Let B be a ball in our domain Ω. We want to prove the convergence stated in (99) locally in $B \times (0,T)$. We introduce the orthogonal projections P and Q defined on $L^2(B)$ by $I = P + Q$; $\mathrm{div} Pu = 0$, $\mathrm{curl}(Qu) = 0$ in B ; $Pu.n = 0$ on ∂B where n stands for the exterior normal to ∂B.

Applying P to the second equation of (95), we deduce easily that $\dfrac{\partial}{\partial t} P m_\epsilon$ is bounded in $L^\infty(0,T;H^s)(\forall s \geq 0)$ and hence that Pm_ϵ converges to Pu in $C([0,T];H^s)$ $(\forall s \geq 0)$. Here, we have used that the injection of $H^r(B)$ in $H^s(B)$ is compact since B is bounded. We also deduce that Pu_ϵ converges to Pu in $L^2(0,T;H^s)$ since $P(u_\epsilon - u) = P\Big((1-\rho_\epsilon)u_\epsilon\Big) + P\Big(\rho_\epsilon u_\epsilon - u\Big)$.

Next, we decompose in B, m_ϵ in $u + P(m_\epsilon - u) + Q(m_\epsilon - u)$ and u_ϵ in $u + P(u_\epsilon - u) + Q(u_\epsilon - u)$. Hence, we can decompose in $\mathcal{D}'(B)$ $\mathrm{div}(\rho_\epsilon u_\epsilon \otimes u_\epsilon)$ in 8 different terms and it is easy to see that it is sufficient to show that $\mathrm{div}\Big(Q(m_\epsilon - u) \otimes Q(u_\epsilon - u)\Big)$ converges to some gradient. Moreover since $Q(m_\epsilon - u)$ and $Q(u_\epsilon - u)$ converge weakly to 0 and that $Q(m_\epsilon - u) - Q(u_\epsilon - u) = Q(\Big((1-\rho_\epsilon)u\Big)$ converges to 0 in $L^2(0,T;H^s)$ $(\forall s \geq 0)$, we see that it is equivalent to show the above requirement for the following term $\mathrm{div}\Big(Q(m_\epsilon - u) \otimes Q(m_\epsilon - u)\Big)$. Next, we introduce ψ_ϵ such that $\displaystyle\fint_B \psi_\epsilon dx = 0$, $\nabla \psi_\epsilon = Q m_\epsilon$. Besides, it is easy to see that ψ_ϵ is bounded in $L^\infty(0,T;H^s)(\forall s \geq 0)$. With the above notations, we deduce from the initial system (69) the following one

$$\frac{\partial \varphi_\epsilon}{\partial t} + \frac{1}{\epsilon}\,\Delta\psi_\epsilon = 0\ ,\ \frac{\partial \nabla\psi_\epsilon}{\partial t} + \frac{a\gamma}{\epsilon}\,\nabla\varphi_\epsilon \ = \ F_\epsilon \tag{100}$$

where $F_\epsilon = \xi\nabla\ \mathrm{div}\ u_\epsilon + \nabla\pi_\epsilon + \mu Q\Big[\Delta u_\epsilon - \mathrm{div}(\rho_\epsilon u_\epsilon \otimes u_\epsilon)\Big]$ is bounded in $L^2(0,T;H^s)(\forall s \geq 0)$.

Next, we observe that in $\mathcal{D}'(B \times]0,T[)$, we have on one hand

$$\mathrm{div}(Qu \otimes Qu) = \frac{1}{2}\nabla|Qu|^2 + (\mathrm{div} Qu)Qu \ = \ \frac{1}{2}\nabla|Qu|^2$$

and on the other hand

$$\begin{aligned}
\operatorname{div}\Big(\nabla\psi_\epsilon \otimes \nabla\psi_\epsilon\Big) \;&=\; \frac{1}{2}\nabla|\nabla\psi_\epsilon|^2 + \Delta\psi_\epsilon\nabla\psi_\epsilon \\
=\;& \frac{1}{2}\nabla\Big(|\nabla\psi_\epsilon|^2\Big) - \frac{\partial}{\partial t}(\epsilon\varphi_\epsilon\nabla\psi_\epsilon) + \epsilon\varphi_\epsilon F_\epsilon - a\gamma\varphi_\epsilon\nabla\varphi_\epsilon \\
=\;& \frac{1}{2}\nabla\Big(|\nabla\psi_\epsilon|^2 - a\gamma\varphi_\epsilon^2\Big) - \frac{\partial}{\partial t}(\epsilon\varphi_\epsilon\nabla\psi_\epsilon) + \epsilon\varphi_\epsilon F_\epsilon \;.
\end{aligned}$$

Using that $\epsilon\varphi_\epsilon\nabla\psi_\epsilon$ converges strongly to 0 in $L^2(0,T;H^s)$ $(\forall s \geq 0)$ and that $\epsilon\varphi_\epsilon F_\epsilon$ converges strongly to 0 in $L^\infty(0,T;H^s)$ $(\forall s \geq 0)$, we deduce that

$$\operatorname{div}\Big(Q(m_\epsilon - u)\otimes Q(m_\epsilon - u)\Big) \underset{\epsilon}{\rightharpoonup} \nabla q \tag{101}$$

and finally, we obtain that

$$\operatorname{div}(\rho_\epsilon u_\epsilon \otimes u_\epsilon) \underset{\epsilon}{\rightharpoonup} \operatorname{div}(u\otimes u) + \nabla q \quad \text{in } B\times(0,T) \tag{102}$$

and the theorem is proved in the regular case. We only notice here that if Ω is not simply connected, we can take C an annulus around each hole in the previous argument to make sure that the pressure is globally well defined.

Convergence in the general case:
Now, we are going to show how we can regularize in x the above quantities (uniformly in ϵ). To do so let $K_\delta = \frac{1}{\delta^N}K(\frac{\cdot}{\delta})$, where $K \in C_0^\infty(\mathbb{R}^N)$, $\int_{\mathbb{R}^N} K dz = 1$, $\delta \in (0,1)$. We can then regularize by convolution as follows $\varphi_\epsilon^\delta = \varphi_\epsilon * K_\delta$, $m_\epsilon^\delta = m_\epsilon * K_\delta$, $u_\epsilon^\delta = u_\epsilon * K_\delta$, $\pi_\epsilon^\delta = \pi_\epsilon * K_\delta$. We can then follow the same proof as in the regular case by replacing φ_ϵ, π_ϵ, m_ϵ and u_ϵ by their regularizations and we conclude by observing that $||u_\epsilon^\delta - u_\epsilon||_{L^2(0,T;L^2)} \leq C\delta$, $||u_\epsilon^\delta||_{L^2(0,T;H^1)} \leq C$ and $||u_\epsilon^\delta\otimes u_\epsilon^\delta - u_\epsilon\otimes u_\epsilon||_{L^1(0,T;L^p)} \leq C\delta$, $||u_\epsilon^\delta\otimes u_\epsilon^\delta||_{L^1(0,T;L^p)} \leq C$ ($p = \frac{N}{N-2}$ if $N \geq 3$, $1 \leq p < +\infty$ if $N = 2$). Indeed, from the above uniform bounds, we deduce that

$$\sup_{\epsilon\in]0,1]}\Big\{||\rho_\epsilon^\delta u_\epsilon^\delta - m_\epsilon^\delta||_{L^2(L^q)} + ||m_\epsilon^\delta - u_\epsilon^\delta||_{L^2(L^q)} + ||\rho_\epsilon u_\epsilon - m_\epsilon^\delta||_{L^2(L^q)}\Big\} \underset{\delta}{\longrightarrow} 0 \;,$$

$$\sup_{\epsilon\in]0,1]}\Big\{||\rho_\epsilon^\delta u_\epsilon^\delta \otimes u_\epsilon^\delta - m_\epsilon^\delta \otimes u_\epsilon^\delta||_{L^1(L^r)} + ||m_\epsilon^\delta \otimes u_\epsilon^\delta - m_\epsilon \otimes u_\epsilon||_{L^1(L^r)} +$$

$$||m_\epsilon \otimes u_\epsilon - u_\epsilon^\delta \otimes u_\epsilon^\delta||_{L^1(L^r)}\Big\} \underset{\delta}{\longrightarrow} \ 0\ ,$$

with $\dfrac{1}{q} > \dfrac{1}{\gamma} + \dfrac{N-2}{2N}$, $\dfrac{1}{r} > \dfrac{1}{\gamma} + \dfrac{N-2}{N}$, since $\dfrac{1}{\gamma} + \dfrac{N-2}{N} < 1$. Moreover, it is easy to see that for all δ and all s, we have that $||m_\epsilon^\delta - u_\epsilon^\delta||_{L^2(H^s)}$ goes to 0 when ϵ goes to 0.

In the next three subsections, we would like to specify the boundary conditions and give a more precise convergence result.

3.3.2 The periodic case

The periodic case was treated in [111]. The convergence stated in theorem 3.7 can not be improved. Indeed, the acoustic waves will oscillate indefinitely. So, we only have weak convergence. The initial condition in (97) can be specified precisely, namely $u^0 = P\tilde{u}^0$.

3.3.3 The case of Dirichlet boundary conditions

In this subsection, we will state more precise results in the case of Dirichlet boundary conditions. Indeed, depending on some geometrical property of the domain, we can prove a strong convergence result towards the incompressible Navier-Stokes system, which means that all the oscillations are damped in the limit. Let Ω be a bounded domain. We consider the system (69) with the following Dirichlet boundary condition

$$u_\epsilon = 0 \quad \text{on} \quad \partial\Omega. \tag{103}$$

For $\epsilon \in (0,1]$, we consider $(\rho_\epsilon, u_\epsilon)$ satisfying the same hypotheses as in section 3.3.1. In order to state precisely our main Theorem, we need to introduce a geometrical condition on Ω. Let us consider the following over determined problem

$$-\Delta\phi = \lambda\phi \ \text{ in } \ \Omega, \quad \frac{\partial\phi}{\partial\mathbf{n}} = 0 \ \text{ on } \ \partial\Omega, \ \text{ and } \ \phi \ \text{ is constant on } \ \partial\Omega. \tag{104}$$

A solution of (104) is said to be trivial if $\lambda = 0$ and ϕ is a constant. We will say that Ω satisfies assumption (H) if all the solutions of (104) are trivial. Schiffer's conjecture says that every Ω satisfies (H) excepted the ball (see for instance [71]). In two dimensional space, it is proved that every bounded, simply connected open set $\Omega \subset \mathbb{R}^2$ whose boundary is Lipschitz but not real analytic satisfies (H), hence property (H) is generic in $\mathbb{R}^2$. The main result reads as follows

Theorem 3.9 *Under the above conditions, ρ_ϵ converges to 1 in $C([0,T];L^\gamma(\Omega))$ and extracting a subsequence if necessary u_ϵ converges weakly to u in $L^2((0,T)\times\Omega)^N$ for all $T>0$, and strongly if Ω satisfies (H). In addition, u is a global weak solution of the incompressible Navier-Stokes equations with Dirichlet boundary conditions satisfying $u_{|t=0} = P\tilde{u}_0$ in Ω.*

For the proof of this result, we refer to [54]. We only sketch below the phenomenon going on. Let $(\lambda_{k,0}^2)_{k\geq 1}$, $(\lambda_{k,0} > 0)$, be the nondecreasing sequence of eigenvalues and $(\Psi_{k,0})_{k\geq 1}$ the orthonormal basis of $L^2(\Omega)$ functions with zero mean value of eigenvectors of the Laplace operator $-\Delta_N$ with homogeneous Neumann boundary conditions

$$-\Delta\Psi_{k,0} = \lambda_{k,0}^2\Psi_{k,0} \text{ in } \Omega, \quad \frac{\partial\Psi_{k,0}}{\partial\mathbf{n}} = 0 \text{ on } \partial\Omega. \tag{105}$$

We can split these eigenvectors $(\Psi_{k,0})_{k\in\mathbb{N}}$ (which represent the acoustic eigenmodes in Ω) into two classes : those which are not constant on $\partial\Omega$ will generate boundary layers and will be quickly damped, thus converging strongly to 0; those which are constant on $\partial\Omega$ (non trivial solutions of (104)), for which no boundary layer forms, will remain oscillating forever, leading to only weak convergence. Indeed, if (H) is not satisfied, u^ϵ will in general only converge weakly and not strongly to u (like in the periodic case $\Omega = \mathbb{T}^d$ for instance). However, if at initial time $t=0$, no modes of second type are present in the velocity, the convergence to the incompressible solution is strong in L^2.

Notice that according to Schiffer's conjecture the convergence is not strong for general initial data when Ω is the two or three dimensional ball, but is expected to be always strong in any other domain with Dirichlet boundray conditions.

3.3.4 The whole space case

In [52], the authors give a more precise result in the whole space case by using the dispersion of the acoustic waves.

Consider the system (95) in the whole space $\mathbb{R}^N$. The initial data $(\rho_\epsilon^0, m_\epsilon^0)$ satisfies

$$\int_{\mathbb{R}^N} \pi_\epsilon(t=0) + \frac{|m_\epsilon^0|^2}{2\rho_\epsilon^0}\, dx \le C \tag{106}$$

where $\pi_\epsilon = \frac{1}{\gamma(\gamma-1)\epsilon^2}(\rho_\epsilon^\gamma - 1 - \gamma(\rho_\epsilon - 1))$, $m_\epsilon = \rho_\epsilon u_\epsilon$. We also assume that $\dfrac{m_\epsilon^0}{\sqrt{\rho_\epsilon^0}}$ converges weakly in $L^2(\mathbb{R}^N)$ to some $\tilde{u}^0$. Let $L_2^p(\mathbb{R}^N)$ denote the Orlicz space $L_2^p(\mathbb{R}^N) = \{f \in L^1_{loc}(\mathbb{R}^N)/f 1_{|f|<1} \in L^2 \quad \text{and } f 1_{|f|\ge 1} \in L^p\}$. We consider global weak solutions to (95) with the initial data (106) satisfying (96) with Ω replaced by $\mathbb{R}^N$ and such that $\rho_\epsilon - 1 \in L^\infty(0,T; L_2^\gamma(\mathbb{R}^N))$.

Theorem 3.10 *Under the above assumptions, $\rho_\epsilon - 1$ converges to 0 in $L^\infty(0,T;L_2^\gamma)$. For all subsequence of u_ϵ which converges weakly to some $u \in L^2$, u is a global weak solution of the incompressible Navier-Stokes system with the initial data $u(t=0) = P\tilde{u}^0$. Moreover, the subsequence u_ϵ converges strongly to u in $L^2(0,T;L^2(\mathbb{R}^N_{loc}))$ and the gradient part Qu_ϵ converges strongly to 0 in $L^2(0,T;L^q(\mathbb{R}^N))$ for $q > 2$ when $N = 2$ and for $q \in (2,6)$ when $N = 3$.*

The proof uses the Strichartz estimate (84) to prove that the acoustic waves locally go to zero.

3.3.5 Convergence towards the Euler system

In this subsection, we study the case where μ_ϵ goes to 0 too. We will state two results in the periodic case and in the whole space case taken from [127]. The case of domains with boundaries is open even in the incompressible case (see Section 2).

The whole space case

We consider a sequence of global weak solutions $(\rho_\epsilon, u_\epsilon)$ of the compressible Navier-Stokes equations (69) and we assume that $\rho_\epsilon - 1 \in L^\infty(0,\infty; L_2^\gamma) \cap C([0,\infty), L_2^p)$ for all $1 \le p < \gamma$, where $L_2^p = \{f \in L^1_{loc},\ |f|1_{|f|\ge 1} \in L^p,\ |f|1_{|f|\le 1} \in L^2\}$, $u_\epsilon \in L^2(0,T;H^1)$ for all $T \in (0,\infty)$ (with a norm which can explode when ϵ goes to 0), $\rho_\epsilon |u_\epsilon|^2 \in L^\infty(0,\infty;L^1)$ and $\rho_\epsilon u_\epsilon \in C([0,\infty)\,;\, L^{2\gamma/(\gamma+1)} - w)$ i.e. is continuous with respect to $t \ge 0$ with values in $L^{2\gamma/(\gamma+1)}$ endowed with its weak topology. We require (69) to hold in the sense of distributions and we impose the following conditions at infinity

$$\rho_\epsilon \to 1 \quad \text{as} \quad |x| \to +\infty\ , \quad u_\epsilon \to 0 \quad \text{as} \quad |x| \to +\infty\ . \tag{107}$$

Finally, we prescribe initial conditions $\rho_\epsilon(t=0) = \rho_\epsilon^0$, $\rho_\epsilon u_\epsilon(t=0) = m_\epsilon^0$ where $\rho_\epsilon^0 \geq 0$, $\rho_\epsilon^0 - 1 \in L^\gamma$, $m_\epsilon^0 \in L^{2\gamma/(\gamma+1)}$, $m_\epsilon^0 = 0$ a.e. on $\{\rho_\epsilon^0 = 0\}$ and $\rho_\epsilon^0 |u_\epsilon^0|^2 \in L^1$, denoting by $u_\epsilon^0 = \dfrac{m_\epsilon^0}{\rho_\epsilon^0}$ on $\{\rho_\epsilon^0 > 0\}$, $u_\epsilon^0 = 0$ on $\{\rho_\epsilon^0 = 0\}$. We also introduce the following notation $\rho_\epsilon = 1 + \epsilon\varphi_\epsilon$. Notice that if $\gamma < 2$, we cannot deduce any bound for φ_ϵ in $L^\infty(0,T;L^2)$. This is why we introduce the following approximation which belongs to L^2

$$\Phi_\epsilon = \frac{1}{\epsilon}\sqrt{\frac{2a}{\gamma-1}(\rho_\epsilon^\gamma - 1 - \gamma(\rho_\epsilon - 1))}.$$

Furthermore, we assume that $\sqrt{\rho_\epsilon^0}\; u_\epsilon^0$ converges strongly in L^2 to some $\widetilde{u}^0$. Then, we denote by $u^0 = P\tilde{u}^0$, where P is the projection on divergence-free vector fields, we also define Q (the projection on gradient vector fields), hence $\tilde{u}^0 = P\tilde{u}^0 + Q\tilde{u}^0$. Moreover, we assume that Φ_ϵ^0 converges strongly in L^2 to some φ^0. This also implies that φ_ϵ^0 converges to φ^0 in L_2^γ. We also assume that $(\rho_\epsilon, u_\epsilon)$ satisfies the energy inequality. Our last requirement on $(\rho_\epsilon, u_\epsilon)$ concerns the total energy : we assume that we have

$$E_\epsilon(t) + \int_0^t D_\epsilon(s)ds \leq E_\epsilon^0 \quad \text{a.e. } t, \ \frac{dE_\epsilon}{dt} + D_\epsilon \leq 0 \quad \text{in } \ \mathcal{D}'(0,\infty) \tag{108}$$

where $E_\epsilon(t) = \displaystyle\int_\Omega \frac{1}{2}\rho_\epsilon |u_\epsilon|^2(t) + \frac{a}{\epsilon^2(\gamma-1)}((\rho_\epsilon)^\gamma - 1 - \gamma(\rho_\epsilon - 1))(t)$, $D_\epsilon(t) = \displaystyle\int_\Omega \mu_\epsilon |Du_\epsilon|^2(t) + \xi_\epsilon\, (\text{div} u_\epsilon)^2(t)$ and $E_\epsilon^0 = \displaystyle\int_\Omega \frac{1}{2}\rho_\epsilon^0 |u_\epsilon^0|^2 + \frac{a}{\epsilon^2(\gamma-1)}((\rho_\epsilon^0)^\gamma - 1 - \gamma(\rho_\epsilon^0 - 1))$. The existence of solutions satisfying the above requirement was proved in [109].

When ϵ goes to zero and μ_ϵ goes to 0, we expect that u_ϵ converges to v, the solution of the Euler system

$$\begin{cases} \partial_t v + \ div\ (v \otimes v) + \nabla \pi = 0 \\ \\ div\ v = 0 \quad v_{|t=0} = u^0 \end{cases} \tag{109}$$

in $C([0,T^*); H^s)$. We have the following theorem

Theorem 3.11 *We assume that* $\mu_\epsilon \underset{\epsilon}{\to} 0$ *(such that* $\mu_\epsilon + \xi_\epsilon > 0$ *for all* ϵ*) and that* $P\tilde{u}^0 \in H^s$ *for some* $s > N/2 + 1$*, then* $P(\sqrt{\rho_\epsilon} u_\epsilon)$ *converges to* v

in $L^\infty(0,T;L^2)$ for all $T < T^*$, where v is the unique solution of the Euler system in $L^\infty_{loc}([0,T^*);H^s)$ and T^* is the existence time of (109). In addition $\sqrt{\rho_\epsilon} u_\epsilon$ converges to v in $L^p(0,T;L^2_{loc})$ for all $1 \le p < +\infty$ and all $T < T^*$.

The periodic case

Now, we take $\Omega = \mathbb{T}^N$ and consider a sequence of solutions $(\rho_\epsilon, u_\epsilon)$ of (69), satisfying the same conditions as in the whole space case (the functions are now periodic in space and all the integration are performed over $\mathbb{T}^N$). Of course, the conditions at infinity are removed and the spaces L^p_2 can be replaced by L^p. Here, we have to impose more conditions on the oscillating part (acoustic waves), namely we have to assume that $Q\tilde{u}^0$ is more regular than L^2. In fact, in the periodic case, we do not have a dispersion phenomenon as in the case of the whole space and the acoustic waves will not go to infinity, but they are going to interact with each other. This is why, we have to include them in the energy estimates to show our convergence result. This requires an analysis of the possible resonances between the different modes.

For the next theorem, we assume that $Q\tilde{u}^0$, $\varphi^0 \in H^{s-1}$ and that there exists a nonnegative constant ν such that $\mu_\epsilon + \xi_\epsilon \ge 2\nu > 0$ for all ϵ. For simplicity, we assume that $\mu_\epsilon + \xi_\epsilon$ converges to 2ν.

Theorem 3.12 *: (The periodic case) We assume that $\mu_\epsilon \underset{\epsilon}{\rightarrow} 0$ (such that $\mu_\epsilon + \xi_\epsilon \to 2\nu > 0$) and that $P\tilde{u}^0 \in H^s$ for some $s > N/2+1$, and $Q\tilde{u}^0$, $\varphi^0 \in H^{s-1}$ then $P(\sqrt{\rho_\epsilon} u_\epsilon)$ converges to v in $L^\infty(0,T;L^2)$ for all $T < T^*$, where v is the unique solution of the Euler system in $L^\infty_{loc}(0,T^*;H^s)$ and T^* is the existence time of (109). In addition $\sqrt{\rho_\epsilon} u_\epsilon$ converges weakly to v in $L^\infty(0,T;L^2)$*

Idea of the proofs

The proofs of theorems 3.11 and 3.12 are based on energy estimates, since we loose the compactness in x from the viscosity at the limit. Indeed, using the energy bounds, we deduce that $\rho_\epsilon - 1$ converges to 0 in $L^\infty(0,T;L^\gamma_2)$ and that there exists some $u \in L^\infty(0,T;L^2)$ and a subsequence $\sqrt{\rho_\epsilon} u_\epsilon$ converging weakly to u. Hence, we also deduce that $\rho_\epsilon u_\epsilon$ converges weakly to u in $L^{2\gamma/(\gamma+1)}$. Here we are in a situation where we do not have compactness in time and we do not have compactness in space. This is why we have to use an energy method. For this, we have to describe the oscillations in time and incorporate them in the energy estimates. It turns out that in the whole space case the acoustic waves disperse to infinity as can be deduced from the Strichartz estimate (84). We also refer to [166] and theorem 3.3 in the framework of strong solutions and [52] and theorem 3.10 in the framework of weak solutions. In the sequel, we will concentrate more on the periodic case. The operators L and $\mathcal{L}$ were defined in (83). Let

$$U^\epsilon = (\varphi_\epsilon, Q(\rho_\epsilon u_\epsilon)) \quad \text{and} \quad V^\epsilon = \mathcal{L}(-t/\epsilon)(\varphi_\epsilon, Q(\rho_\epsilon u_\epsilon)).$$

Using that

$$\epsilon\,\frac{\partial \varphi_\epsilon}{\partial t} + \mathrm{div}\; Q(\rho_\epsilon u_\epsilon) \;=\; 0, \quad \epsilon\,\frac{\partial}{\partial t} Q(\rho_\epsilon u_\epsilon) + \nabla \varphi_\epsilon \;=\; \epsilon F_\epsilon \tag{110}$$

for some F_ϵ which is bounded in $L^2 H^{-r}$ for some $r \in \mathbb{R}$, we deduce that $\partial_t U^\epsilon = \dfrac{1}{\epsilon} L U^\epsilon + (0, F_\epsilon)$, and hence that $\partial_t V^\epsilon = \mathcal{L}(-t/\epsilon)(0, F_\epsilon)$. This means that V^ϵ is compact in time since the oscillations have been canceled by $\mathcal{L}(-t/\epsilon)$. If we had enough compactness in space we could pass to the limit in this equation and recover the following limit system for the oscillating part

$$\partial_t \bar{V} + \mathcal{Q}_1(u, \bar{V}) + \mathcal{Q}_2(\bar{V}, \bar{V}) - \nu \Delta \bar{V} = 0, \tag{111}$$

where $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are respectively a linear and a bilinear forms in $\bar{V}$ defined by

Definition 3.13 *For all divergence-free vector field $u \in L^2(\Omega)^N$ and all $V = (\psi, \nabla q) \in L^2(\Omega)^{N+1}$, we define the following linear and bilinear symmetric forms in V*

$$\mathcal{Q}_1(u, V) = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau \mathcal{L}(-s) \begin{pmatrix} 0 \\ div\; \Big(u \otimes \mathcal{L}_2(s) V + \mathcal{L}_2(s) V \otimes u \Big) \end{pmatrix} ds \tag{112}$$

and

$$\mathcal{Q}_2(V, V) = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau \mathcal{L}(-s) \begin{pmatrix} 0 \\ div\; \Big(\mathcal{L}_2(s) V \otimes \mathcal{L}_2(s) V \Big) + \frac{\gamma-1}{2} \nabla (\mathcal{L}_1(s) V)^2 \end{pmatrix} ds. \tag{113}$$

The convergences stated above take place in $W^{-1,1}$ and can be shown by using almost-periodic functions (see [125] and the references therein). We also notice that

$$-\nu \Delta V = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau -\mathcal{L}(-s) \begin{pmatrix} 0 \\ 2\nu \Delta \mathcal{L}_2(s) V \end{pmatrix} ds \tag{114}$$

To recover compactness in space, we will use the regularity of the limit system. Let V^0 be the solution of the following system

$$\left\{ \begin{array}{l} \partial_t V^0 + \mathcal{Q}_1(v, V^0) + \mathcal{Q}_2(V^0, V^0) - \nu \Delta V^0 = 0 \\ \\ V^0_{|t=0} = (\varphi^0, Q\tilde{u}^0) \end{array} \right. \tag{115}$$

where v is the solution of the incompressible Euler equations with initial data u^0. The existence of global strong solutions for the system (115) (and local solutions if the viscosity term is removed) can be deduced from the exact computations of the two forms $\mathcal{Q}_1$ and $\mathcal{Q}_2$. We point out that in the case $\nu > 0$, the existence of a global solution to the system (115) is an important property of (115) which is not shared by the Navier-Stokes system from which it is derived. Indeed, the nonlinear term $\mathcal{Q}_2(V^0, V^0)$ can be decomposed into a countable number of Burgers equations. We refer to [127] for more details.

Finally, the energy method is based on the fact that we can apply a Gronwall lemma to the following quantity

$$||\sqrt{\rho_\epsilon} u_\epsilon - v - \mathcal{L}_2(\frac{t}{\epsilon})V||^2_{L^2} + ||\Phi_\epsilon - \mathcal{L}_1(\frac{t}{\epsilon})V||^2_{L^2}. \tag{116}$$

Notice indeed, that from the analysis given above, we expect that $\sqrt{\rho_\epsilon} u_\epsilon$ behaves like $v + \mathcal{L}_2(\frac{t}{\epsilon})V$ and that ϕ_ϵ and Φ_ϵ behave like $\mathcal{L}_1(\frac{t}{\epsilon})V$. For the details, we refer to [127].

We want to point out that the method of proof is the same for the whole space case and is simpler since we do not have to study all the resonances (the acoustic waves go to infinity). So, we just need to apply a Gronwall lemma to the quantity given in (116) where V is replaced by $V(t = 0)$.

Remark 3.14 *In theorem 3.12, one can remove the condition $2\nu > 0$. In that case, we still have the result of theorem 3.12 but only on an interval of time $(0, T^{**})$ which is the existence interval for the equation governing the oscillating part (115). Indeed, it is easy to see using the particular form of $\mathcal{Q}_1$ and Q_2 that if $\nu > 0$ and $V(t = 0) \in H^{s-1}$ then, we have (as long as v exists) a global solution in $L^\infty(H^{s-1})$ which satisfies $\nabla V \in L^1(0, T; L^\infty)$. On the other hand if $\nu = 0$ and $V(t = 0) \in H^{s-1}$ then we can only construct a local (in time) solution in $L^\infty(H^{s-1})$ which satisfies $\nabla V \in L^1(0, T; L^\infty)$ for all $T < T^{**}$.*

3.4 Study of the limit $\gamma \to \infty$

In this subsection we are going to study the limit γ going to infinity. Depending on the total mass, we will recover at the limit either a mixed model, which behaves as a compressible one if $\rho < 1$ and as an incompressible one if $\rho = 1$ or the classical incompressible Navier-Stokes system. We start with the first case and define the limit system, namely

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho u) = 0 \text{ in } \ (0,T) \times \Omega, \ 0 \le \rho \le 1 \text{ in } \ (0,T) \times \Omega \ , \tag{117}$$

$$\frac{\partial \rho u}{\partial t} + \operatorname{div}(\rho u \otimes u) - \mu \Delta u - \xi \nabla \operatorname{div} u + \nabla \pi = 0 \text{ in } \ (0,T) \times \Omega \ , \tag{118}$$

$$\operatorname{div} u = 0 \text{ a.e. on } \ \Big\{ \rho = 1 \Big\} \ , \tag{119}$$

$$\pi = 0 \text{ a.e. on } \ \Big\{ \rho < 1 \Big\} \ , \ \pi \ge 0 \text{ a.e. on } \ \Big\{ \rho = 1 \Big\} \tag{120}$$

In all this section, Ω is taken to be the torus, the whole space or a bounded domain with Dirichlet boundary conditions. Indeed, the proofs given in [112] can also apply to the case of Dirichlet boundary conditions, by using the bounds given in [110] and [67].

Let γ_n be a sequence of nonnegative real numbers that goes to infinity. Let (ρ_n, u_n) be a sequence of weak solutions to the isentropic compressible Navier-Stokes equations

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \operatorname{div}(\rho u) = 0 \ , \quad \rho \ge 0, \\ \\ \dfrac{\partial \rho u}{\partial t} + \operatorname{div}(\rho u \otimes u) - \mu \Delta u - \xi \nabla \operatorname{div} u + \nabla \rho^{\gamma_n} = 0 \end{cases} \tag{121}$$

where $\mu > 0$ and $\mu + \xi > 0$. We recall that global weak solutions of the above system are known to exist, if we assume in addition that $\gamma_n > \dfrac{N}{2}$. This holds for n large enough. The sequence (ρ_n, u_n) satisfies in addition the following initial conditions and the following bounds,

$$\rho_n u_n(t=0) = m_n^0 \ , \ \rho_n(t=0) = \rho_n^0, \tag{122}$$

where $0 \le \rho_n^0$ a.e. , ρ_n^0 is bounded in $L^1(\Omega)$ and $\rho_n^0 \in L^{\gamma_n}$ with $\displaystyle\int (\rho_n^0)^{\gamma_n} \le C\gamma_n$ for some fixed C, $m_n^0 \in L^{2\gamma_n/(\gamma_n+1)}(\Omega)$, and $\rho_n^0 |u_n^0|^2$ is bounded in L^1, denoting by $u_n^0 = \dfrac{m_n^0}{\rho_n^0}$ on $\{\rho_n^0 > 0\}$, $u_n^0 = 0$ on $\{\rho_n^0 = 0\}$. In the periodic case or in the Dirichlet boundary condition case, we also assume that $\int \rho_n^0 = M_n$, for some M_n such that $0 < M_n \leq M < 1$ and $M_n \to M$. Furthermore, we assume that $\rho_n^0 u_n^0$ converges weakly in L^2 to some m^0 and that ρ_n^0 converges weakly in L^1 to some ρ^0. The last requirement concerns the following energy bounds we impose on the sequence of solutions we consider,

$$E_n(t) + \int_0^t D_n(s)ds \leq E_n^0 \quad \text{a.e. } t, \ \frac{dE_n}{dt} + D_n \leq 0 \quad \text{in} \ \mathcal{D}'(0,\infty) \qquad (123)$$

where $E_n(t) = \displaystyle\int \frac{1}{2}\rho_n |u_n|^2(t) + \frac{a}{\gamma_n - 1}(\rho_n)^{\gamma_n}(t)$, $D_n(t) = \displaystyle\int \mu |Du_n|^2(t) + \xi$ $(\mathrm{div} u_n)^2(t)$ and $E_n^0 = \displaystyle\int \frac{1}{2}\rho_n^0 |u_n^0|^2 + \frac{a}{\gamma_n - 1}(\rho_n^0)^{\gamma_n}$.

Without loss of generality, extracting subsequences if necessary, we can assume that (ρ_n, u_n) converges weakly to (ρ, u). More precisely we can assume that $\rho_n \rightharpoonup \rho$ weakly in $L^p((0,T) \times \Omega)$ for any $1 \leq p \leq \infty$ and that $\rho \in L^\infty(0,T;L^p)$ (in fact we will show that ρ actually satisfies $0 \leq \rho \leq 1$), $u_n \rightharpoonup u$ weakly in $L^2(0,T;H^1_{loc})$.

Before stating the main theorem, we have to define precisely the notion of weak solutions for the limit system. (ρ, u, π) is called a weak solution of the limit system (117-120) if

$$\rho \in L^\infty(0,T;L^\infty \cap L^1(\Omega)) \cap C(0,T;L^p) \text{ for any } 1 \leq p < \infty \qquad (124)$$

$$\nabla u \in L^2(0,T,L^2) \text{ and } u \in L^2(0,T;H^1(B)), \qquad (125)$$

where $B = \Omega$ if $\Omega = \mathbb{T}^N$ or if Ω is a bounded domain (with Dirichlet boundary conditions) and B is any ball in $\mathbb{R}^N$ if $\Omega = \mathbb{R}^N$, in this last case we also impose that $u \in L^2(0,T,L^{2N/N-2}(\mathbb{R}^N))$, if in addition $N \geq 3$.
Moreover,

$$\rho|u|^2 \in L^\infty(0,\infty;L^1) \text{ and } \rho u \in L^\infty(0,\infty;L^2) \qquad (126)$$

Next, equations (117), (118) must be satisfied in the distributional sense. This can be written using a weak formulation (which also incorporate the initial conditions in some weak sense), namely we require that the following identities hold for all $\phi \in C^\infty([0,\infty) \times \Omega)$ and for all $\Phi \in C^\infty([0,\infty) \times \Omega)^N$

compactly supported in $[0,\infty)\times\Omega$ (i.e. vanishing identically for t large enough)

$$-\int_0^\infty dt\int_\Omega \rho\partial_t\phi-\int_\Omega\rho^0\phi(0)-\int_0^\infty dt\int_\Omega \rho u.\nabla\phi=\ 0, \tag{127}$$

$$\begin{aligned}-\int_0^\infty dt\int_\Omega \rho u.\partial_t\Phi-\int_\Omega m^0.\Phi(0)-\int_0^\infty dt\int_\Omega \rho(u.\nabla\Phi).u+ \\ +\int_0^\infty dt\Big\{\int_\Omega \mu Du.D\Phi+\xi\mathrm{div}u\mathrm{div}\Phi\Big\}-\pi\ \mathrm{div}\Phi\ =0.\end{aligned} \tag{128}$$

On the other hand, the equation (120) should be understood in the following way $\rho\pi=\pi\geq 0$. Of course, we have to define the sense of the product $\rho\pi$ since, we only require that $\pi\in\mathcal{M}$. Indeed, the product can be defined by using that

$$\left\{\begin{array}{l}\rho\in C([0,T];L^p)\cap C^1([0,T];H^{-1}),\\[2ex] \pi\in W^{-1,\infty}(H^1)+L^1(L^{N/(N-2)})\cap L^\alpha(L^\beta)+L^2(L^2).\end{array}\right. \tag{129}$$

where $1<\alpha,\beta<\infty$ and $\dfrac{1}{\beta}=\dfrac{1}{\alpha}\dfrac{N-2}{N}+(1-\dfrac{1}{\alpha})$.

Finally, equation (119) is just a consequence of (117), however we incorporate it in the limit system to emphasis the fact that it is a mixed system which behaves like a compressible one if $\rho<1$ and as an incompressible one if $\rho=1$.

Theorem 3.15 *Under the above conditions, we have $0\leq\rho\leq 1$ and*

$$(\rho_n-1)_+\to 0 \quad \textit{in } L^\infty(0,T;L^p) \textit{ for any } 1\leq p<+\infty.$$

Moreover, $(\rho_n)^{\gamma_n}$ is bounded in L^1 (for n such that $\gamma_n\geq N$). Then extracting subsequences again, there exists $\pi\in\mathcal{M}((0,T)\times\Omega)$ such that

$$(\rho_n)^{\gamma_n} \underset{n}{\rightharpoonup} \ \pi. \tag{130}$$

If in addition ρ_n^0 converges in L^1 to ρ^0 then (ρ,u,π) is a weak solution of (117-120) and the following strong convergences hold

$$\rho_n\to\rho \quad \textit{in} \quad C(0,T;L^p(\Omega)) \textit{ for any } 1\leq p<+\infty$$

$$\rho_n u_n\to\rho u \quad \textit{in} \quad L^p(0,T;L^q(\Omega)) \textit{ for any } 1\leq p<+\infty,\ 1\leq q<2$$

$$\rho_n u_n\otimes u_n\to\rho u\otimes u \quad \textit{in} \quad L^p(0,T;L^1(\Omega)) \textit{ for any } 1\leq p<+\infty.$$

The second result concerns the case $M > 1$. Let (ρ_n, u_n) be a sequence of solutions of (121) satisfying the above requirement but where we assume now that $\int \rho_n^0 = M > 1$, $\int (\rho_n^0)^{\gamma_n} \leq M^{\gamma_n} + C\gamma_n$ for some fixed C.

Theorem 3.16 *Under the above assumptions, ρ_n converges to M in $C([0,T]; L^p(\Omega))$ for $1 \leq p < +\infty$, $\sqrt{\rho_n}\, u_n$ converges weakly to $\sqrt{M}u$ in $L^\infty(0,T; L^2(\Omega))$ and Du_n converges weakly to Du in $L^2(0,T; L^2(\Omega))$ for all $T \in (0,\infty)$ where u is a solution of the incompressible Navier-Stokes system*

$$\frac{\partial u}{\partial t} + \mathrm{div}(u \otimes u) - \frac{\mu}{M}\Delta u + \nabla p = 0,$$
$$\mathrm{div}(u) = 0, \quad u_{|t=0} = P(m^0).$$

For the proof of these two theorems we refer to [112] and to [124] for the Dirichlet boundary condition case.

3.5 The Non-isentropic case

We consider the non-isentropic compressible Euler system. This can be written after some simple change of variable in the following form (see [133])

$$\begin{cases} a(\partial_t q + v \cdot \nabla q) + \frac{1}{\epsilon}\nabla \cdot v = 0 \\ r(\partial_t v + v \cdot \nabla v) + \frac{1}{\epsilon}\nabla q = 0 \\ \partial_t S + v \cdot \nabla S = 0 \end{cases} \tag{131}$$

where $a = a(S, \epsilon q)$ and $r = r(S, \epsilon q)$ are positive given function of S and ϵq. In (131), S is the entropy, $P = \underline{P} e^{\epsilon q}$ is the pressure for some constant $\underline{P}$ and v is a rescaled velocity. The equation of state is given by the density $\rho = R(S, P)$ from which we can deduce the function a and r by

$$a(S, \epsilon q) = \frac{P}{R}\frac{\partial R(S,P)}{\partial P} \quad r(S, \epsilon q) = \frac{R(S,P)}{P}. \tag{132}$$

Formally when ϵ goes to zero, we expect that the solution $(q_\epsilon, v_\epsilon, S_\epsilon)$ to the system (131) converges to a solution of the following limit system

$$\begin{cases} r_0(S)(\partial_t v + v \cdot \nabla v) + \nabla \pi = 0 \\ \mathrm{div}(v) = 0 \\ \partial_t S + v \cdot \nabla S = 0 \end{cases} \tag{133}$$

where $r_0(S) = r(S, 0)$. The limit system (133) is an inhomogeneous incompressible Euler system (see [108] for some remarks about this system). This convergence was first proved in the "well-prepared" case in [153].

For general initial data, there are two major questions we can ask about the system (131). Can we solve (131) on some time interval which is independent of ϵ ? and can we characterize the limit of $(q_\epsilon, v_\epsilon, S_\epsilon)$ when ϵ goes to zero ? For the first question a full satisfactory answer is given in [133]. For the second equation, Métivier and Schochet [133] prove the convergence towards the limit system (133) in the whole space by using the dispersion for a wave equation with non constant coefficients. For the periodic case the problem is much more involved due to the oscillations in time. In [134], the same authors give some partial results. The case of the exterior domain is treated in [2]. Before stating the result of [133], let us mention the reference [26] where a formal computation is made in the periodic case and the recent paper [3] where the full compressible Navier-Stokes is considered in the whole space.

Let us take some initial data for (131) $(q_\epsilon, v_\epsilon, S_\epsilon)(t = 0) = (q^0_\epsilon, v^0_\epsilon, S^0_\epsilon)$. The following result is proved in [133]. The first part applies to the case $\Omega = \mathbb{T}^N$ and $\Omega = \mathbb{R}^N$ (see also [2] for domains with boundary). The second part is only for the whole space case (see [2] for the case of an exterior domain).

Theorem 3.17 *i) Assume that* $\|(q^0_\epsilon, v^0_\epsilon, S^0_\epsilon)\|_{H^s} \leq M_0$ *where* $s > N/2 + 1$. *There exists* $T = T(M_0)$ *such that for all* $0 < \epsilon \leq 1$, *the Cauchy problem with the initial data* $(q^0_\epsilon, v^0_\epsilon, S^0_\epsilon)$ *has a unique solution* $(q_\epsilon, v_\epsilon, S_\epsilon) \in C([0, T]; H^s)$.
ii) Moreover if $\Omega = \mathbb{R}^N$ *and* $(v^0_\epsilon, S^0_\epsilon)$ *converges in* $H^s(\mathbb{R}^N)$ *to some* (v^0, S^0) *and* S^0_ϵ *decays at infinity in the sense*

$$|S^\epsilon_0(x)| \leq C|x|^{-1-\delta} \quad |\nabla S^\epsilon_0(x)| \leq C|x|^{-2-\delta}$$

then $(q_\epsilon, v_\epsilon, S_\epsilon)$ *converges weakly in* $L^\infty(0, T; H^s)$ *and strongly in* $L^2(0, T; H^{s'}_{loc})$ *for all* $s' < s$ *to a limit* $(0, v, S)$. *Moreover,* (v, S) *is the unique solution in* $C([0, T]; H^s)$ *of the limit system (133) with the initial data* (w_0, S_0) *where* w_0 *is the unique solution in* $H^s(\mathbb{R}^N)$ *of*

$$\mathrm{div}(w_0) = 0, \quad \mathrm{curl}(r_0 w_0) = \mathrm{curl}(r_0 v_0), \quad \textit{where } r_0 = r(S_0, 0). \tag{134}$$

The difficulty in proving the convergence towards the limit system is that the acoustic waves satisfy a wave equation with variable coefficients. The proof of the convergence is based on the use of the $H-$measures (which were introduced by Gérard [72] and Tartar [161]) to analysis the oscillating part and actually prove that it disperses to infinity as was the case in the isentropic case.

4 Study of rotating fluids at high frequency

In this section, we will study rotating fluids when the frequency of rotation goes to zero. This is a singular limit which has many similarities with the compressible-incompressible limit. We will not detail all the known results for this system. We consider the following system of equations

$$\partial_t u^{\mathbf{n}} + \operatorname{div}(u^{\mathbf{n}} \otimes u^{\mathbf{n}}) - \nu \partial_z^2 u^{\mathbf{n}} - \eta \Delta_{x,y} u^{\mathbf{n}} + \frac{e_3 \ \times \ u^{\mathbf{n}}}{\epsilon} = -\frac{\nabla p}{\epsilon} + F \text{ in } \quad \Omega \tag{135}$$

$$\operatorname{div}(u^{\mathbf{n}}) = 0 \quad \text{in} \quad \Omega \tag{136}$$

$$u^{\mathbf{n}}(0) = u_0^{\mathbf{n}} \quad \text{with} \quad \operatorname{div}(u_0^{\mathbf{n}}) = 0 \tag{137}$$

$$u^{\mathbf{n}} = 0 \quad \text{on} \quad \partial\Omega \tag{138}$$

where for example $\Omega = \mathbb{T}^2 \times]0, h[$ or $\Omega = \mathbb{T}^3$, $\nu = \nu^{\mathbf{n}}$ and $\eta = \eta^{\mathbf{n}}$ are respectively the vertical and horizontal viscosities and $\epsilon = \epsilon^{\mathbf{n}}$ is the Rossby number. This system describes the motion of a rotating fluid as the Ekman and Rossby numbers go to zero (see Pedlovsky [144], and Greenspan [80]). It can model the ocean, the atmosphere, or a rotating fluid in a container. As for the compressible-incompressible limit the limit system can depend on the boundary conditions in a non trivial way.

4.1 The periodic case

When there is no boundary ($\Omega = \mathbb{T}^3$ for instance) and when $\nu = \eta = 1$ (the Navier-Stokes case) or $\nu = \eta = 0$ (the Euler case), the problem was studied by several authors ([81], [33], [8], [9], [10], [63], [68], [143]...) by using the group method of [154] and [81]. This method was first introduced to treat the compressible incompressible limit (see subsections 3.2.1 and 3.3.5). Basically, denoting $Lu = -P(e_3 \times u)$ and $\mathcal{L}(\tau) = e^{\tau L}$, we see that $v^{\mathbf{n}} = \mathcal{L}(-t/\epsilon)u^{\mathbf{n}}$ satisfies

$$\partial_t v^{\mathbf{n}} + \mathcal{L}(-t/\epsilon)\left[\operatorname{div}(u^{\mathbf{n}} \otimes u^{\mathbf{n}}) - \nu\partial_z^2 u^{\mathbf{n}} - \eta\Delta_{x,y}u^{\mathbf{n}}\right] = -\nabla q \text{ in } \quad \Omega \quad (139)$$

which gives compactness in time for $v^{\mathbf{n}}$.

The special structure of the limit system which is similar to (159) allows to prove results about long time existence for the Navier-Stokes system when ϵ goes to zero. This means in some sense that the rotation has a regularizing effect. This regularizing effect also appear when we deal with boundary layers (see the next subsection).

The method introduced in [154] fails when Ω has a boundary (except in very particular cases where there is no boundary layer, or where boundary layers can be eliminated by symmetry [23]).

4.2 Ekman boundary layers in $\Omega = \mathbb{T}^2\times]0,h[$

In domains with boundaries (for instance $\Omega = \mathbb{T}^2\times]0,h[$), the case of "well-prepared" initial data was treated in [37], [84], [123], [73]. Here "well-prepared" initial data means that $Lu_0 = 0$ which implies that the initial data is bi-dimensional and only depends on the horizontal variables. Notice that this implies that there are no oscillations in time. In this case a boundary layer appears at $z = 0$ and $z = h$ to match the non-slip boundary condition with the interior flow. This boundary layer is responsible of the so-called Ekman damping. Let us give a formal expansion leading to the Ekman boundary layer in the well-prepared case (see [84])

4.2.1 Formal expansion

For convenience we will take here $\epsilon = \nu$, otherwise there is not such a formal development. Let us write $u^{\mathbf{n}}$, p and F in the following form

$$U = U^0(t,x,y,z,\frac{z}{l},\frac{h-z}{l}) + \epsilon U^1 + ...$$

where l is the length of the boundary layer. Notice here that we do not have a dependence on $\frac{t}{\epsilon}$ since we are concerned here with the well-prepared case. U^0 is decomposed as

$$U^0 = \underline{U}^0(t,x,y,z) + \tilde{U}^0(t,x,y,\theta) + \breve{U}^0(t,x,y,\lambda),$$

is the sum of an interior term $\underline{U}^0$ and of two boundary layer terms $\tilde{U}^0$ and $\breve{U}^0$ respectively near $z = 0$ and $z = h$, where we set $\theta = z/l$ and $\lambda = (h - z)/l$. We enforce

$$\lim_{\theta \to \infty} \tilde{U} = 0 \text{ and } \lim_{\lambda \to \infty} \breve{U} = 0,$$

and, to get the good limit conditions at $z = 0$ and $z = h$,

$$\underline{u}^0(t, x, y, z = 0) + \tilde{u}^0(t, x, y, \theta = 0) = 0, \tag{140}$$

$$\underline{u}^0(t, x, y, z = h) + \breve{u}^0(t, x, y, \lambda = 0) = 0. \tag{141}$$

Since the Ekman boundary layers come from the interaction between the viscosity $\nu \partial_z^2 u$ and the Coriolis force $\epsilon^{-1}(e_3 \times\ u)$, we take $l = \sqrt{\epsilon \nu}$, hence $l = \epsilon$, in this section. Let us focus on the boundary layer near $z = 0$. At the leading order ϵ^{-2} , one gets

$$\partial_\theta \tilde{p}^0 = 0 \quad \text{ hence } \quad \tilde{p}^0 = 0.$$

The pressure does not change in the boundary layer, which is classical in fluid mechanics. One also has from (135)

$$-\underline{u}_2^0 = -\partial_x \underline{p}^0, \tag{142}$$

$$\underline{u}_1^0 = -\partial_y \underline{p}^0, \tag{143}$$

$$0 = -\partial_z \underline{p}^0, \tag{144}$$

$$-\partial_\theta^2 \tilde{u}_1^0 - \tilde{u}_2^0 = 0, \tag{145}$$

$$-\partial_\theta^2 \tilde{u}_2^0 + \tilde{u}_1^0 = 0, \tag{146}$$

$$-\partial_\theta^2 \tilde{u}_3^0 = -\partial_\theta \tilde{p}^1, \tag{147}$$

and from (136)

$$\partial_\theta \tilde{u}_3^0 = 0 \quad \text{ hence } \quad \tilde{u}_3^0 = 0, \tag{148}$$

$$\partial_x \underline{u}_1^0 + \partial_y \underline{u}_2^0 + \partial_z \underline{u}_3^0 = 0, \tag{149}$$

$$\partial_x \tilde{u}_1^0 + \partial_y \tilde{u}_2^0 + \partial_\theta \tilde{u}_3^1 = 0. \tag{150}$$

Then we obtain from (144) that $\underline{p}^0$ does not depend on z, and from (147) and (148) that $\partial_\theta \tilde{p}^1 = 0$ and hence that $\tilde{p}^1 = 0$. Therefore (142) and (143) give that $\underline{u}_1^0$ and $\underline{u}_2^0$ do not depend on z, and that

$$\partial_x \underline{u}_1^0 + \partial_y \underline{u}_2^0 = 0. \tag{151}$$

Subtracting this from (149), one gets that $\underline{u}_3^0$ does not depend on z, and since $\tilde{u}_3^0 = 0$, (140) leads to $\underline{u}_3^0 = 0$.

Hence $\underline{u}^0$ satisfies an equation of 2-D Navier-Stokes' type. To find this equation, one must take the next order of (42), which gives

$$\partial_t \underline{u}^0 + \nabla(\underline{u}^0 \otimes \underline{u}^0) - \Delta_{x,y}\underline{u}^0 + \begin{pmatrix} -\underline{u}_2^1 \\ \underline{u}_1^1 \\ 0 \end{pmatrix} = -\nabla p^1 + F^0(t,x,y,z) \quad \text{in } \omega (152)$$

We will suppose that F^0 does not depend on z, and that $F_3^0(t,x,y) = 0$. The third component gives that p^1 does not depend on z. Combining this with (152), one finds that $\underline{u}_1^1$ and $\underline{u}_2^1$ do not depend on z. Hence the divergence-free condition for $\underline{u}^1$ shows that $\underline{u}_3^1$ is affine.

Let $\zeta^0 = \text{ curl } \underline{u}^0$. We have

$$\partial_t \zeta^0 + (\underline{u}^0.\nabla)\zeta^0 - \Delta_{x,y}\zeta^0 - \text{ curl } F^0 = -\partial_x u_1^1 - \partial_y u_2^1 = \partial_z u_3^1.$$

Integrating this equation with respect to z, we obtain

$$\partial_t \zeta^0 + (\underline{u}^0.\nabla)\zeta^0 - \Delta_{x,y}\zeta^0 - \text{ curl } F^0 = h^{-1}(u_3^1(z=h) - u_3^1(z=0)).$$

Therefore there is a source term in the equation of the vorticity, term which is given by the vertical velocity of the fluid just outside the Ekman boundary layer. So let us compute the boundary layer $\tilde{u}^0$, which satisfies

$$\begin{cases} \partial_\theta^2 \tilde{u}_1^0 = -\tilde{u}_2^0 \\ \partial_\theta^2 \tilde{u}_2^0 = +\tilde{u}_1^0 \\ \tilde{u}_1^0(\theta=0) = -\underline{u}_1^0, \qquad \lim_{\theta\to\infty} \tilde{u}_1^0 = 0 \\ \tilde{u}_2^0(\theta=0) = -\underline{u}_2^0, \qquad \lim_{\theta\to\infty} \tilde{u}_2^0 = 0 \end{cases}$$

The solution is given by

$$\begin{cases} \tilde{u}_1^0 = -e^{-\frac{\theta}{\sqrt{2}}} \left(\underline{u}_1^0 \cos(\frac{\theta}{\sqrt{2}}) + \underline{u}_2^0 \sin(\frac{\theta}{\sqrt{2}}) \right) \\ \tilde{u}_2^0 = -e^{-\frac{\theta}{\sqrt{2}}} \left(\underline{u}_2^0 \cos(\frac{\theta}{\sqrt{2}}) - \underline{u}_1^0 \sin(\frac{\theta}{\sqrt{2}}) \right) \end{cases}$$

Reporting this in (150), and using (151), one gets

$$\partial_\theta \tilde{u}_3^1 = e^{-\frac{\theta}{\sqrt{2}}} (\partial_x \underline{u}_2^0 - \partial_y \underline{u}_1^0) \sin(\frac{\theta}{\sqrt{2}})$$

Integrating this equation,

$$\tilde{u}_3^1 = -\frac{e^{-\frac{\theta}{\sqrt{2}}}}{\sqrt{2}}(\partial_x \underline{u}_2^0 - \partial_y \underline{u}_1^0)(\sin(\frac{\theta}{\sqrt{2}}) + \cos(\frac{\theta}{\sqrt{2}})) \tag{153}$$

The integration constant is 0, because $\lim_{\theta\to\infty} \tilde{u}_3^0 = 0$.

The same calculus holds for the boundary layer at $z = h$, if we change θ by λ and ∂_θ by $-\partial_\lambda$

$$\begin{cases} \breve{u}_1^0 = -e^{-\frac{\lambda}{\sqrt{2}}} \left(\underline{u}_1^0 \cos(\frac{\lambda}{\sqrt{2}}) + \underline{u}_2^0 \sin(\frac{\lambda}{\sqrt{2}}) \right) \\ \breve{u}_2^0 = -e^{-\frac{\lambda}{\sqrt{2}}} \left(\underline{u}_2^0 \cos(\frac{\lambda}{\sqrt{2}}) - \underline{u}_1^0 \sin(\frac{\lambda}{\sqrt{2}}) \right) \\ \breve{u}_3^1 = \frac{e^{-\frac{\lambda}{\sqrt{2}}}}{\sqrt{2}} (\partial_x \underline{u}_2^0 - \partial_y \underline{u}_1^0) \left(\sin(\frac{\lambda}{\sqrt{2}}) + \cos(\frac{\lambda}{\sqrt{2}}) \right) \end{cases}$$

Using the limit conditions, and the fact that $\underline{u}_3^1$ is affine, one gets

$$\underline{u}_3^1 = \frac{(\partial_x \underline{u}_2^0 - \partial_y \underline{u}_1^0)}{\sqrt{2}} \left(1 - \frac{2z}{h} \right) \tag{154}$$

$$\partial_z \underline{u}_3^1 = -\frac{\sqrt{2}}{h}(\partial_x \underline{u}_2^0 - \partial_y \underline{u}_1^0) \tag{155}$$

Coming back to (152), we find the limit system

$$\partial_t \underline{u}^0 + \nabla(\underline{u}^0 \otimes \underline{u}^0) - \Delta_{x,y} \underline{u}^0 + \frac{\sqrt{2}}{h} \underline{u}^0 = -\nabla q + F^0 \quad \text{in} \quad \omega \tag{156}$$

Hence $(\underline{u}_1^0, \underline{u}_2^0)$ satisfies a 2-D Navier-Stokes system with a damping term (we recall $\underline{u}_3^0 = 0$).

4.2.2 The "ill-prepared" case

We want here to present the result of [125] where $\Omega = \mathbb{T}^2 \times]0, h[$ and we consider "ill-prepared" initial data. Here, we have to study the oscillations in time and show that they do not affect the averaged flow. We can apply the same formal expansion as in the previous subsection taking into account the oscillations in time, namely

$$U = U^0(\frac{t}{\epsilon}, t, x, y, z, \frac{z}{l}, \frac{h-z}{l}) + \epsilon U^1 + ... \tag{157}$$

$$U^0 = \underline{U}^0(\tau, t, x, y, z) + \tilde{U}^0(\tau, t, x, y, \theta) + \breve{U}^0(\tau, t, x, y, \lambda). \tag{158}$$

We do not detail this expansion here and refer to [125]. We only point out that there are two extra difficulties here. Indeed, there is an oscillating boundary layer for each mode which has a vertical component. Moreover, we have to deal with the resonances between the different modes as in the works cited in the periodic case.

To write down the limit system, we introduce the spaces V^s_{sym} consisting of functions of H^s with some extra conditions on the boundary (see [125]). We also set $Lu = -P(e_3 \times u)$, where P is the projection onto divergence-free vector fields such that the third component vanishes on the boundary and $\mathcal{L}(\tau) = e^{\tau L}$. Let us denote w the solution in $L^\infty(0, T^*, V^s_{sym})$ of the following system

$$\left\{ \begin{array}{c} \partial_t w + \overline{Q}(w, w) - \Delta_{x,y} w + \gamma \overline{S}(w) = -\nabla p \text{ in } \quad \Omega, \\ \mathrm{div}(w) = 0 \text{ in } \quad \Omega, \\ w.n = \pm w_3 = 0 \quad \text{ on } \quad \partial\Omega, \\ w(t=0) = w^0. \end{array} \right. \tag{159}$$

where $\overline{Q}(w, w)$, $\overline{S}(w)$ are respectively a bilinear and a linear operators of w, given by

$$\overline{Q}(w, w) = \sum_{\substack{l,m,k \\ k \in \mathcal{A}(l,m) \\ \lambda(l)+\lambda(m)=\lambda(k)}} b(t, l) b(t, m) \alpha_{lmk} N^k(X) \tag{160}$$

where the N^k are the eigenfunctions of L and $i\lambda(k)$ are the associated eigenvalues, α_{lmk} are constants which depends on (l, m, k) and $\mathcal{A}(l, m) = \{l + m,\ Sl + m,\ l + Sm,\ Sl + Sm\}$, $(Sl = (l_1, l_2, -l_3))$ is the set of possible resonances. The bilinear term $\overline{Q}$ is due to the fact that only resonant modes in the advective term $w.\nabla w$ are present in the limit equation.

$$\overline{S}(w) = \sum_k \frac{1}{h}(D(k) + iI(k)) b(t, k) N^k(X)$$

where

$$D(k) = \sqrt{2}\left\{(1 - \lambda(k)^2)^{\frac{1}{2}}\right\}, \quad I(k) = \sqrt{2}\left\{\lambda(k)(1 - \lambda(k)^2)^{\frac{1}{2}}\right\}.$$

In fact $\overline{S}(w)$ is a damping term that depends on the frequencies $\lambda(k)$, since $D(k) \geq 0$. It is due to the presence of a boundary layer which creates a second flow of order ϵ responsible of this damping (called damping of Ekman).

Theorem 4.1 *Let $s > 5/2$, and $w^0 \in V^s_{sym}(\Omega)^3$, $\nabla.w^0 = 0$. We assume that u_0^n converges in $L^2(\Omega)$ to w^0, $\eta = 1$ and ϵ, ν go to 0 such that $\sqrt{\frac{\nu}{\epsilon}} \to \gamma$. Then any sequence of global weak solutions (à la Leray) u^n of (135- 138) satisfying the energy inequality satisfies*

$$u^n - \mathcal{L}(\frac{t}{\epsilon})w \to 0 \quad in \quad L^\infty(0, T^*, L^2(\Omega)),$$

$$\nabla_{x,y}(u^n - \mathcal{L}(\frac{t}{\epsilon})w), \sqrt{\nu}\partial_z u^n \to 0 \ in \ L^2(0, T^*, L^2(\Omega))$$

where w is the solution in $L^\infty(0, T^*, V^s_{sym})$ of (159)

The above theorem gives a precise description of the oscillations in the sequence u^n. We can also show that the oscillations do not affect the averaged flow (also called the quasi-geostrophic flow). We see then that $\overline{w}$ (the weak limit of u^n) satisfies a 2-D Navier-Stokes equation with a damping term, namely

$$\left\{ \begin{array}{c} \partial_t \overline{w} + \overline{w}.\nabla\overline{w} - \eta\Delta_{x,y}\overline{w} + \gamma\frac{\sqrt{2}}{h}\overline{w} = -\nabla p \text{ in } \quad \mathbb{T}^2, \\ \mathrm{div}(\overline{w}) = 0 \text{ in } \quad \mathbb{T}^2, \\ \overline{w}(t=0) = \mathcal{S}(w^0) = \overline{w}^0, \end{array} \right. \tag{161}$$

where $\mathcal{S}$ is the projection onto the slow modes, namely that do not depend on z, $\overline{w}(t, x, y) = \mathcal{S}(w) = (1/h)\int_0^h w(t, x, y, z)dz$.

This can be proved by studying the operator Q and showing that if $k \in \mathcal{A}(l, m)$ with $k_3 = 0$ and $l_3 m_3 \neq 0$ than $\alpha_{lmk} + \alpha_{mlk} = 0$.

4.2.3 Non flat bottom

In [125], we also deal with other boundary conditions, and construct Ekman layers near a non flat bottom

$$\Omega_\delta = \{(x, y, z), \quad \text{where } (x, y) \in \mathbb{T}^2, \quad \text{and} \quad \delta f(x, y) < z < h\},$$

with the following boundary conditions

$$u(x, y, \delta f(x, y)) = 0. \tag{162}$$

We also treat the case of a free surface,

$$u_3^n(z = h) = 0 \quad \partial_z \begin{pmatrix} u_1^n \\ u_2^n \end{pmatrix}_{|z=h} = \frac{1}{\beta}\sigma(\frac{t}{\epsilon}, t, x, y) \tag{163}$$

where σ describes the wind (see [144]). Next, we have the following theorem

Theorem 4.2 *Let $u^{\mathbf{n}}$ be global weak solutions of (135,136,137, 162,163). If $\eta = 1$ and $(\epsilon, \nu, \beta, \delta) \to (0,0,0,0)$ then*

$$u^\nu - \mathcal{L}(\frac{t}{\epsilon})w \to 0 \quad in \quad L^\infty(0,T^*;L^2(\Omega)),$$

$$\nabla_{x,y}(u^\nu - \mathcal{L}(\frac{t}{\epsilon})w), \sqrt{\nu}\partial_z u^\nu \to 0 \ in \ L^2(0,T^*;L^2(\Omega))$$

where w is the solution of the following system ($\sqrt{\frac{\nu}{\epsilon}}, \frac{\nu}{\beta}, \frac{\delta}{\epsilon}$ stand for the limit of these quantities when n goes to infinity)

$$\begin{cases} \partial_t w + \overline{Q}(w,w) - \Delta_{x,y} w + \frac{1}{2}\sqrt{\frac{\nu}{\epsilon}}\overline{S}(w) + \frac{\nu}{\beta}\overline{S}_1(\sigma) + \frac{\delta}{\epsilon}\overline{S}_2(f,w) = -\nabla p \\ \mathrm{div}(w) = 0 \ in \quad \Omega, \\ w.n = \pm w_3 = 0 \quad on \quad \partial\Omega, \\ w(t=0) = w^0. \end{cases} \tag{164}$$

where $\overline{S}_1(\sigma)$, and $\overline{S}_2(f,w)$ are source terms that are due respectively to the wind , and to the non flat bottom.

The proofs of the above two theorems are based (as in the previous section) on energy estimates and use a more complicated corrector due to the presence of oscillations in time as well as the presence of different types of boundary layers. For more details about the proof, we refer to the original paper [125].

4.3 The case of other geometries

In the whole space case or in a domain $\Omega = \mathbb{R}^2\times]0,h[$ the oscillations disperse to infinity as was the case for the acoustic waves in the compressible-incompressible limit. Let us state the following result for $\Omega = \mathbb{R}^2\times]0,h[$ taken from [35]. We take η to be constant and $\nu = \epsilon$.

Theorem 4.3 *Let u_0 be a divergence free vector field is L^2, $u_0.n = u_{03} = 0$ on $\partial\Omega$. Let u^ϵ be a family of weak solutions of (135- 138) written in $\Omega = \mathbb{R}^2\times]0,h[$. Let $\overline{w}$ be the global solution of the 2D Navier-Stokes system (161) in $\mathbb{R}^2$ with the initial data $\mathcal{S}(w^0)$. Then, we have*

$$\|u^\epsilon - (\overline{w},0)\|_{L^\infty(\mathbb{R}_+;L^2_{loc}(\mathbb{R}^2\times]0,h[))} + \|\nabla(u^\epsilon - (\overline{w},0))\|_{L^2(\mathbb{R}_+;L^2_{loc}(\mathbb{R}^2\times]0,h[))} \to 0 \tag{165}$$

when ϵ goes to zero.

The proof of this theorem uses the Ekman layer constructed in subsection 4.2.1 and some Strichartz type estimate for the oscillating part.

Let us also mention that the study of other geometries such as cylindrical domains were also studied [25].

4.4 Other related problems

We would like to end this section on rotating fluids by mentioning few related results. First, other physical systems present very similar properties to the rotating fluids. For instance there are several singular limits coming from magneto-hydrodynamic which have similar properties as the rotating fluids. We refer to [51] and [20]

An other important question concerns the stability of boundary layers. Indeed, in the previous subsection, we dealt with the case the horizontal viscosity was not going to zero. We can also study the case where η goes to zero. For the case without rotation we are lead to the inviscid limit which was studied in section 2. It was proved that if ν, η and ν/η go to zero then we have convergence towards the Euler system. In other words the horizontal viscosity has a regularizing effect which is not shared by the vertical one. In the case with rotation and when $\nu = \eta$, we can prove [123] (see also [125] for the ill-prepared case) that if

$$\|w\|_{L^\infty} \leq C\frac{\nu}{\epsilon}, \tag{166}$$

for some small enough constant C, then we have convergence towards the Euler system with damping, namely (161) with $\eta = 0$. This means that the rotation has a regularizing effect. Condition (166) is a stability condition. It was proved in [53] that the boundary layer can be instable if (166) is not satisfied. More precisely, Desjardins and Grenier [53] prove the instability of the Ekman boundary layer under a more precise spectral condition. The stability condition (166) can also be refined to match the spectral condition. This was done by Rousset [148] for the case of Ekman boundary layers and [147] for the case of Ekman-Hartmann boundary layers.

5 Hydrodynamic limit of the Boltzmann equation

From a physical point of view, we expect that a gas can be described by a fluid equation when the mean free path (Knudsen number) goes to zero. During the last two decades this problem got a lot of interest and specially after DiPerna and Lions constructed their renormalized solutions [56]. In this section, we present some of the most recent results concerning these (rigorous) derivations. We will present results for the three most classical equations of fluid mechanics in the incompressible regime, namely the incompressible Navier-Stokes equation, the Stokes equation and the Euler equation. We will also present some derivation of Fluid Mechanic boundary conditions starting from kinetic boundary conditions [132].

5.1 Scalings and formal asymptotics

In his sixth problem, Hilbert asked for a full mathematical justification of fluid mechanics equations starting from particle systems [88]. If we take the Boltzmann equation as a starting point, this problem can be stated as an asymptotic problem. Namely, starting from the Boltzmann equation, can we derive fluid mechanics equations and in which regime ?

A program in this direction was initiated by Bardos, Golse and Levermore [12] who, using the the renormalized solutions to the Boltzmann equation constructed by DiPerna and Lions, set an asymptotic regime where one can derive different fluid equations (and in particular incompressible models) depending on the chosen scaling.

5.1.1 The Boltzmann equation

The Boltzmann equation describes the evolution of the particle density of a rarefied gas. Indeed, the molecules of a gas can be modeled by hard spheres that move according to the laws of classical mechanics. However, due to the enormous number of molecules (about 2.7 10^{19} molecules in a cubic centimeter of gas at 1 atm and 0^0 C), it seems difficult to describe the state of the gas by giving the position and velocity of each individual particle. Hence, we must use some statistics and instead of giving the position and velocity of each particle, we specify the density of particles $F(x, v)$ at each point x and velocity v. This means that we describe the gas by giving for each point x and velocity v the number of particles $F(x,v)\ dx\ dv$ in the volume $(x, x+dx) \times (v, v+dv)$.

Under some assumptions (rarefied gas, ...), it is possible to derive (at least formally) the Boltzmann equation from the classical Newton laws in an asymptotic regime where the number of particles goes to infinity. (see [101], [157] and [31] for some rigorous results about the derivation of the Boltzmann equation starting from the N particle system).

The Boltzmann equation reads

$$\partial_t F + v.\nabla_x F = B(F,F) \tag{167}$$

where the collision kernel $B(F,F)$ is a quadratic form which acts only on the v variable. It describes the possible interaction between two different particles and is given by

$$B(F,F)(v) = \int_{\mathbb{R}^D} \int_{S^{D-1}} (F_1' F' - F_1 F) b(v - v_1, \omega) dv_1 d\omega \tag{168}$$

where we have used the following notation for all function ϕ

$$\phi' = \phi(v'), \quad \phi_1 = \phi(v_1), \quad \phi_1' = \phi(v_1'), \tag{169}$$

and where the primed speeds are given by

$$v' = v + \omega[\omega.(v_1 - v)], \qquad v_1' = v - \omega[\omega.(v_1 - v)]. \tag{170}$$

Moreover, the Boltzmann cross-section $b(z,\omega)$ ($z \in \mathbb{R}^D, \omega \in S^{D-1}$) depends on the molecular interactions (intermolecular potential). It is a nonnegative, locally integrable function (at least when grazing collisions are neglected). The Galilean invariance of the collisions implies that b depends only on $v - v_1, \omega$ and that

$$b(z,\omega) = |z| \mathcal{S}(|z|, |\mu_c|), \quad \mu_c = \frac{\omega.(v_1 - v)}{|v_1 - v|}, \tag{171}$$

where $\mathcal{S}$ is the specific differential cross-section. We also insist on the fact that the relations (170) are equivalent to the following conservations

$$v' + v_1' = v + v_1 \quad \text{(conservation of the moment)} \tag{172}$$

$$|v'|^2 + |v_1'|^2 = |v|^2 + |v_1|^2 \quad \text{(conservation of the kinetic energy)} \tag{173}$$

We notice that the fact that two particles give two particles after the interaction translates the conservation of mass. For a more precise discussion about the Boltzmann equation, we refer to [30], [31] and [168]. For some numerical works on the hydrodynamic limit, we refer to [156].

5.1.2 Compressible Euler

We start here by explaining how one can derive (at least formally) the Compressible Euler equation from the Boltzmann equation. A rigorous derivation can be found in Caflisch [28]. If F satisfies the Boltzmann equation, we deduce by integration in the v variable (at least formally) the following local conservations

$$\begin{cases} \partial_t\big(\int_{\mathbb{R}^D} F\ dv\big) + \nabla_x.\big(\int_{\mathbb{R}^D} v\ F\ dv\big) = 0 \\ \\ \partial_t\big(\int_{\mathbb{R}^D} vF\ dv\big) + \nabla_x.\big(\int_{\mathbb{R}^D} v\otimes v\ F\ dv\big) = 0 \\ \\ \partial_t\big(\int_{\mathbb{R}^D} |v|^2F\ dv\big) + \nabla_x.\big(\int_{\mathbb{R}^D} v|v|^2\ F\ dv\big) = 0 \end{cases} \tag{174}$$

These three equations describe respectively the conservation of mass, momentum and energy. They present a great resemblance with the compressible Euler equation. However, the third moment $\int_{\mathbb{R}^D} v|v|^2\ F\ dv$ is not a function of the others and depends in general on the whole distribution $F(v)$. In the asymptotic regimes we want to study, the distribution $F(v)$ will be very close to a Maxwellian due to the fact that the Knudsen number is going to 0. If we make the assumption that $F(v)$ is a Maxwellian for all t and x, then the third moment $\int_{\mathbb{R}^D} v|v|^2\ F\ dv$ can be given as a function of $\rho = \int_{\mathbb{R}^D} F\ dv$, $\rho u = \int_{\mathbb{R}^D} vF\ dv$ and $\rho(\frac{1}{2}|u|^2 + \frac{D}{2}\theta) = \int_{\mathbb{R}^D} \frac{1}{2}|v|^2F\ dv$. Moreover, for all i and j, $\int_{\mathbb{R}^D} v_iv_jF\ dv$ can also be expressed as a function of ρ, u and θ.

We recall that a Maxwellian $M_{\rho,u,\theta}$ is completely defined by its density, bulk velocity and temperature

$$M_{\rho,u,\theta} = \frac{\rho}{(2\pi\theta)^{D/2}}\exp(-\frac{1}{2\theta}|v-u|^2) \tag{175}$$

where ρ, u and θ depend only on t and x. If, we assume that for all t and x, F is a Maxwellian given by $F = M_{\rho(t,x),u(t,x),\theta(t,x)}$ then (174) reduces to

$$\begin{cases} \partial_t\rho + \nabla_x.\rho u = 0 \\ \\ \partial_t(\rho u) + \nabla_x.(\rho u\otimes u) + \nabla_x(\rho\theta) = 0 \\ \\ \partial_t\big(\frac{1}{2}\rho|u|^2 + \frac{D}{2}\rho\theta\big) + \nabla_x.\big(\rho u(\frac{1}{2}|u|^2 + \frac{D+2}{2}\theta)\big) = 0 \end{cases} \tag{176}$$

which is the compressible Euler system for a mono-atomic perfect gas. This derivation can become rigorous, if we take a sequence of solutions F_ϵ of

$$\partial_t F_\epsilon + v.\nabla_x F_\epsilon = \frac{1}{\epsilon} B(F_\epsilon, F_\epsilon) \tag{177}$$

where ϵ is the Knudsen number which goes to 0 (see R. Caflisch [28]). Formally the presence of the term $\frac{1}{\epsilon}$ in front of $\frac{1}{\epsilon}B(F_\epsilon, F_\epsilon)$ implies (at the limit) that $B(F, F) = 0$ which means that F is a Maxwellian (see [30], [31] or [168] for a proof of this fact).

5.1.3 Incompressible scalings

In the last subsection, we explained how we can derive the compressible Euler equation. It turns out that using different scalings, one can also derive incompressible models. We will explain what these scalings mean concerning the the Knudsen, Reynolds and Mach numbers. We consider the following global Maxwellian M which corresponds to $\rho = \theta = 1$ and $u = 0$.

$$M(v) = \frac{1}{(2\pi)^{D/2}} \exp(-\frac{1}{2}|v|^2). \tag{178}$$

Let $F_\epsilon = MG_\epsilon = M(1 + \epsilon^m g_\epsilon)$ be a solution of the following Boltzmann equation

$$\epsilon^s \partial_t F_\epsilon + v.\nabla F_\epsilon = \frac{1}{\epsilon^q} B(F_\epsilon, F_\epsilon) \tag{179}$$

which is also equivalent to

$$\epsilon^s \partial_t G_\epsilon + v.\nabla G_\epsilon = \frac{1}{\epsilon^q} Q(G_\epsilon, G_\epsilon) \tag{180}$$

where

$$Q(G, G)(v) = \int_{\mathbb{R}^D} \int_{S^{D-1}} (G_1' G' - G_1 G) b(v - v_1, \omega) M_1 dv_1 d\omega. \tag{181}$$

With this scaling, we can define

$$Ma = \epsilon^m, \quad Kn = \epsilon^q, \quad Re = \epsilon^{m-q}. \tag{182}$$

Here ϵ^s is a time scaling which is related to the Strouhal number. We recall that $St = \frac{L}{TU}$ and hence $St = \epsilon^{s-m}$. This scaling in time allows us to choose the phenomenon we want to emphasize. By varying m, q and s, we can formally derive the following systems (see the references below for some rigorous mathematical results). A part from the first case where the compressible Euler system is satisfied by the moments of F, the fluid equations are recovered for the moments of the fluctuation g and we can show at least formally that $g = \rho + u.v + \theta(\frac{|v|^2}{2} - \frac{D}{2})$ where (ρ, u, θ) satisfies one of the above equations

1) $q = 1,\ m = 0,\ s = 0$ Compressible Euler system [28, 100, 167]

2) $q = 1,\ m > 0,\ s = 0$ Acoustic waves [14]

$$\left\{\begin{array}{l} \partial_t \rho + \nabla_x . u = 0 \\ \partial_t u + \nabla_x(\rho + \theta) = 0 \\ \partial_t(\rho + \theta) + \frac{D+2}{D} \nabla_x . u = 0 \end{array}\right. \tag{183}$$

We notice here that for these two first cases, we have $St\,Ma = 1$ which is the condition to see some acoustic effects at the limit.

3) $q = 1,\ m = 1,\ s = 1$ Incompressible Navier-Stokes-Fourier system [49, 12, 16, 114, 78]

$$\left\{\begin{array}{lcl} \partial_t u + u.\nabla u - \nu \Delta u + \nabla p & = & 0, \quad \nabla_x . u = 0 \\ & & \\ \partial_t \theta + u.\nabla \theta - \kappa \Delta \theta & = & 0, \quad \rho + \theta = 0 \end{array}\right.$$

4) $q = 1,\ m > 1,\ s = 1$ Stokes-Fourier system [13, 14, 115, 74, 132]

$$\left\{\begin{array}{lcl} \partial_t u - \nu \Delta u + \nabla p & = & 0, \quad \nabla_x . u = 0 \\ & & \\ \partial_t \theta - \kappa \Delta \theta & = & 0, \quad \rho + \theta = 0 \end{array}\right.$$

5) $q > 1,\ m = 1,\ s = 1$ Incompressible Euler-Fourier system [115, 149]

$$\left\{\begin{array}{lcl} \partial_t u + u.\nabla u + \nabla p & = & 0, \quad \nabla_x . u = 0 \\ & & \\ \partial_t \theta + u.\nabla \theta & = & 0, \quad \rho + \theta = 0. \end{array}\right.$$

Note that the compressible Navier-Stokes system (with a viscosity of order 1) can not be derived in this manner because of the following physical relation

$$Re = C \frac{Ma}{Kn}. \tag{184}$$

However, the compressible Navier-Stokes system with a viscosity of order ϵ can be considered as a better approximation than the Compressible Euler system in the case $q = 1,\ m = 0,\ s = 0$.

5.1.4 Formal development

Here, we want to explain (at least formally) how we can derive the incompressible Navier-Stokes system for the bulk velocity and the Fourier equation for the temperature starting from the Boltzmann system with the scalings $q = 1,\ m = 1,\ s = 1$. A simple adaptation of the argument also yields a formal derivation of the Stokes-Fourier system (which is the linearization of the Navier-Stokes-Fourier system) as well as the Euler. Rewriting the equation satisfied by g_ϵ, we get

$$\partial_t g_\epsilon + \frac{1}{\epsilon} v.\nabla_x g_\epsilon = -\frac{1}{\epsilon^2} L g_\epsilon + \frac{1}{\epsilon} Q(g_\epsilon, g_\epsilon) \tag{185}$$

where L is the linearized collision operator given by

$$Lg = \int_{\mathbb{R}^D} \int_{S^{D-1}} (g + g_1 - g_1' - g') b(v - v_1, \omega) M_1 dv_1\ d\omega \tag{186}$$

We assume that g_ϵ can be decomposed as follows $g_\epsilon = g + \epsilon h + \epsilon^2 k + O(\epsilon^3)$ and we make the following formal development

$$\frac{1}{\epsilon^2} : \quad Lg = 0. \tag{187}$$

A simple study of the operator L shows that it is formally self-adjoint, non negative for the following scalar product $< f, g >= \langle f\ \ g \rangle$ where we use the following notation $\langle g \rangle = \int_{\mathbb{R}^D} gM dv$ and $Ker(L) = \{g,\ g = \alpha + \beta.v + \gamma |v|^2, \quad \text{where} \quad (\alpha, \beta, \gamma) \in \mathbb{R} \times \mathbb{R}^D \times \mathbb{R}\}$. Hence, we deduce that $g = \rho + u.v + \theta(\frac{|v|^2}{2} - \frac{D}{2})$.

$$\frac{1}{\epsilon} : \ v.\nabla g = -Lh + Q(g, g). \tag{188}$$

Integrating over v, we infer that $u = \langle vg \rangle$ is divergence-free (div $u = 0$). Moreover, multiplying by v and taking the integral over v, we infer that $\nabla(\rho + \theta) = 0$ which is the Boussinesq relation. Besides, at order 1, we have

$$\frac{1}{\epsilon^0} \quad : \quad \partial_t g + v.\nabla_x h = -Lk + 2Q(g, h), \tag{189}$$

from which we deduce that

$$\frac{1}{\epsilon^0} \quad : \quad \partial_t \langle vg \rangle + \nabla_x.\langle v \otimes vh \rangle = 0, \tag{190}$$

$$\frac{1}{\epsilon^0} \quad : \quad \partial_t \langle (\frac{|v|^2}{D+2} - 1) g \rangle + \nabla_x.\langle v(\frac{|v|^2}{D+2} - 1) h \rangle = 0. \tag{191}$$

To get a closed equation for g, we have to inverse the operator L. We define the matrix $\phi(v)$ and the vector $\psi(v)$ as the unique solutions of

$$L\phi(v) = v \otimes v - \frac{1}{D}|v|^2 I, \qquad L\psi(v) = (\frac{|v|^2}{D+2} - 1)v \tag{192}$$

which are orthogonal to $Ker(L)$ for the scalar product $< \cdot, \cdot >$. We also define the viscosity ν and the heat conductivity κ by

$$\nu = \frac{1}{(D-1)(D+2)}\langle \phi : L\phi \rangle, \tag{193}$$

$$\kappa = \frac{2}{D(D+2)}\langle \psi . L\psi \rangle. \tag{194}$$

We notice that ν and κ only depend on b. Using that L is formally self-adjoint, we deduce that

$$\partial_t \langle g v_i \rangle + \nabla_x . \Big\langle \phi_{ij}(Q(g,g) - v.\nabla g) \Big\rangle + \nabla \ \langle \frac{|v|^2}{N} h \rangle = 0 \tag{195}$$

$$\partial_t \langle g (\frac{|v|^2}{D+2} - 1) \rangle + \nabla_x . \Big\langle \psi(Q(g,g) - v.\nabla g) \Big\rangle = 0 \tag{196}$$

A simple (but long) computation gives the Navier-Stokes equation and the Fourier equation, namely

$$\partial_t u + u.\nabla u - \nu \Delta u + \nabla p = 0 \tag{197}$$

$$\partial_t \theta + u.\nabla \theta - \kappa \Delta \theta = 0 \tag{198}$$

where $u = \langle g v \rangle$, $\theta = -\rho = \langle (\frac{|v|^2}{D+2} - 1) g \rangle$ and the pressure p is the sum of different contributions.

5.1.5 Mathematical difficulties

Here, we want to explain the major mathematical difficulties encountered in trying to give a rigorous justification of any of the above asymptotic problems starting from renormalized solutions.

D1. The local conservation of momentum is not known to hold for the renormalized solutions of the Boltzmann equation. Indeed, the solutions constructed by R. DiPerna and P.-L. Lions [56] only hold in the renormalized sense which means that

$$\partial_t\beta(F) + v.\nabla\beta(F) = Q(F,F)\beta'(F), \tag{199}$$
$$\beta(F)(t=0) = \beta(F^0) \tag{200}$$

where β is given, for instance, by $\beta(f) = Log\ (1+f)$.

D2. The lack of a priori estimates. Indeed, all we can deduce from the entropy inequality and the conservation of energy is that g_ϵ is bounded in $L\log L$ and that $g_\epsilon|v|^2$ is bounded in L^1. However, we need a bound in L^2 to define all the product involved in the formal development. In [74], the authors used the entropy dissipation estimate to deduce some information on the structure of the fluctuation g_ϵ and get some new a priori estimates by using some Caflisch-Grad estimates.

To pass to the limit in the different products (and specially in the case we want to recover the Navier-Stokes-Fourier system or the Euler system), one has also to prove that g_ϵ is compact in space and time, namely that $g_\epsilon \in K$ where K is a compact subset of some $L^p(0,T;L^1(\Omega))$. We split this in two difficulties

D3. The compactness in space of g_ϵ. This was achieved in the stationary case by C. Bardos, F. Golse and D. Levermore [15], [12] using averaging lemma [76, 75] and proving that g_ϵ is in some compact subset of $L^1(\Omega)$. However, a newer version of the averaging lemma [78] was needed in [79] to prove some equiintegrability and hence the absence of concentration.

D4. The compactness in time for g_ϵ. It turns out that in general g_ϵ is not compact in time. Indeed, g_ϵ presents some oscillations in time which can be analyzed and described precisely. Using this description and some compensation (due to a remarkable identity satisfied by the solutions to the wave equation), it is possible to pass to the limit in the whole equation. This was done by P.-L. Lions and the author [114] using some ideas coming from the compressible-incompressible limit [111, 113].

D5. An other difficulty is that is that in [12], very restrictive conditions on the Boltzmann kernel were imposed. These conditions were slightly relaxed in [74] to treat some general hard potentials in the Stokes-Fourier scaling and in [79] to treat Maxwellian potential. The case of general potentials including soft potentials was treated in [105].

5.2 The convergence towards the incompressible Navier-Stokes-Fourier system

The first paper dealing with the rigorous justification of the formal development 5.1.4 goes back to the work of C. Bardos, F. Golse and D. Levermore [12] where the stationary case was handled under different assumptions and restrictions (see also A. De Masi, R. Esposito, and J. L. Lebowitz [49] for a similar result in a different setting). There are however some aspects of the analysis performed in [12] that can be improved. First, the heat equation was not treated because the heat flux terms could not be controlled. Second, local momentum conservation was assumed because DiPerna-Lions solutions are not known to satisfy the local conservation law of momentum (or energy) that one would formally expect. Third, the discrete-time case was treated in order to avoid having to control the time regularity of the acoustic modes. Fourth, unnatural technical assumptions were made on the Boltzmann kernel. Finally, a mild compactness assumption was required to pass to the limit in certain nonlinear terms.

During the last few years, there appeared several results trying to improve the result of [12] and give a rigorous justification of the derivation. In [114] and under two assumptions (the conservation of the momentum and a compactness assumption), it was possible to treat the time dependent case and derive the incompressible Navier-Stokes equation. In [74], Golse and Levermore gave a rigorous derivation of Stokes-Fourier system (the linearization of the Navier-Stokes-Fourier system) without any assumption. In [79], Golse and Saint-Raymond gave the first derivation of the Navier-Stokes-Fourier system without any compactness or momentum assumption. However, their result only applies to a small class of collision kernels. In a recent work in collaboration with Levermore [105], we give a derivation of the Navier-Stokes-Fourier system for a very general class of Boltzmann kernels which includes in particular soft potentials.

In what follows, we assume that Ω is the whole space or the torus to avoid dealing with the boundary. First, let us specify the conditions we impose on the initial data. It is supposed that G_ϵ^0 satisfies (we recall that $F_\epsilon^0 = MG_\epsilon^0$)

$$H(G_\epsilon^0) = \int_\Omega \int_{\mathbb{R}^D} (G_\epsilon^0 log G_\epsilon^0 - G_\epsilon^0 + 1) M \ dxdv \leq C\epsilon^2 \tag{201}$$

This shows that we can extract a subsequence of the sequence g_ϵ^0 (defined by $G_\epsilon^0 = 1 + \epsilon g_\epsilon^0$) which converges weakly in L^1 towards g^0 such that $g^0 \in L^2$. We also notice that (201) is equivalent to the fact that $\int_\Omega \langle h(\epsilon g_\epsilon^0)\rangle \ dx \leq C\epsilon^2$, where $h(z) = (1+z)\log(1+z) - z$ which is almost an L^2 estimate for g_ϵ^0. This shows at least that $g^0 \in L^2$. Then, we consider a sequence G_ϵ of renormalized solutions of the Boltzmann equation (180) with $s = q = 1$, satisfying the entropy inequality and we want to prove that g_ϵ converges to some $g = u.v + \theta(\frac{|v|^2}{2} - \frac{D+2}{2})$.

Before stating the new result of Golse and Saint-Raymond [77], we want to explain the kind of assumptions that were made in previous works. The convergence result proved in [114] (which only deals with the u component) requires the following two hypotheses (A1) and (A2) on the sequence G_ϵ which allow to circumvent the difficulties D1 and D2

(A1). The solution G_ϵ satisfies the projection on divergence-free vector fields of the local momentum conservation law

$$\partial_t P\langle vG_\epsilon\rangle + \frac{1}{\epsilon} P\nabla_x.\langle v \otimes vG_\epsilon\rangle = 0. \tag{202}$$

(A2). The family $(1 + |v|^2)g_\epsilon^2/N_\epsilon$ is relatively compact for the weak topology of $L^1(dt \ M \ dv \ dx)$ which we denote $w - L^1(dt \ M \ dv \ dx)$, where $N_\epsilon = 1 + \frac{\epsilon}{3} g_\epsilon$.

In the sequel, we denote the weak topology of $L^1(dt \ M \ dv \ dx)$ by $w - L^1(dt \ M \ dv \ dx)$. The assumption (A2) enforces the $L \log L$ estimate we have on g_ϵ, namely $\int_\Omega \langle h(\epsilon g_\epsilon)\rangle \ dx \leq C\epsilon^2$ to prevent some type of concentration.

Now, we state the result of Golse and Saint-Raymond [79] where no assumptions on the solutions is made. This result was extended by Levermore and the author [105] to treat the case of a larger class of Boltzmann kernels which includes all the classical kernels in particular soft potentials.

Under some assumptions on the Boltzmann kernel (see [79, 105]), we have

Theorem 5.1 *Let G_ϵ be a sequence of renormalized solutions of the Boltzmann equations (180) with initial condition G_ϵ^0 and satisfying the entropy inequality. Then, the family $(1+|v|^2)g_\epsilon$ is relatively compact in $w-L^1(dt \ M dv \ dx)$. If g is a weak limit of a subsequence (still denoted g_ϵ) then $Lg = 0$ and*

$g = \rho + u.v + \theta(\frac{|v|^2}{2} - \frac{D}{2})$ satisfies the limiting dissipation inequality

$$\frac{1}{2}\int_\Omega |\rho(t)|^2 + |u(t)|^2 + \frac{D}{2}|\theta(t)|^2 \; dx + \tag{203}$$

$$+ \int_0^t \int_\Omega \frac{1}{2}\nu|\nabla_x u +^t \nabla_x u|^2 + \kappa|\nabla\theta|^2$$

$$\leq \liminf_{\epsilon\to 0} \frac{1}{\epsilon^2}\int_\Omega \langle h(\epsilon g_\epsilon)\rangle dx = C^0 \tag{204}$$

Moreover, $\theta + \rho = 0$ and $(u, \theta) = (\langle vg\rangle, \langle(\frac{|v|^2}{D+2} - 1)g\rangle)$ is a weak solution of the Navier-Stokes-Fourier system (NSF)

$$(NSF)\begin{cases} \partial_t u + u.\nabla u - \nu\Delta u + \nabla p = 0, \quad \nabla.u = 0 \\ \partial_t\theta + u.\nabla\theta - \kappa\Delta\theta = 0, \\ u(t=0,x) = u^0(x) \qquad \theta(t=0,x) = \theta^0(x) \end{cases}$$

with the initial condition $u^0 = P\langle vg^0\rangle$ and $\theta^0 = \langle(\frac{|v|^2}{D+2} - 1)g^0\rangle$ and where the viscosity ν and heat conductivity κ are given by (193) and (194).

Idea of the proof:

Now, we give an idea of the proof of theorem 5.1 (see [79] and [105] for a complete proof). We start by recalling a few a prior estimates taken from [12]

Proposition 5.2 *We have*

i) The sequence $(1 + |v|^2)g_\epsilon$ is bounded in $L^\infty(dt; L^1(Mdv\; dx))$ and relatively compact in $w - L^1(dt\; Mdv\; dx)$. Moreover, if g is the weak limit of any converging subsequence of g_ϵ, then $g \in L^\infty(dt; L^2(Mdv\; dx))$ and for almost every $t \in [0, \infty)$, we have

$$\frac{1}{2}\int_\Omega \langle g^2(t)\rangle \; dx \leq \liminf_{\epsilon\to 0} \frac{1}{\epsilon^2}\int_\Omega \langle h(\epsilon g_\epsilon(t))\rangle dx \leq C^0. \tag{205}$$

ii)Denoting $q_\epsilon = \frac{1}{\epsilon^2}(G'_{\epsilon 1}G'_\epsilon - G_{\epsilon 1}G_\epsilon)$, we have that the sequence $(1 + |v|^2)q_\epsilon/N_\epsilon$ is relatively compact in $w - L^1(dt\; d\mu\; dx))$ where $d\mu = b(v - v_1, \omega)d\omega M_1\; dv_1 M\; dv$. Besides, if q is the weak limit of any converging subsequence of q_ϵ/N_ϵ then $q \in L^2(dt; L^2(d\mu\; dx))$ and q inherits the same symmetries as q_ϵ, namely $q(v, v_1, \omega) = q(v_1, v, \omega) = -q(v', v'_1, \omega)$.

iii) In addition, for almost all (t, x), $Lg = 0$, which means that g is of the form

$$g(t, x, v) = \rho(t, x) + u(t, x).v + \theta(t, x)(\frac{1}{2}|v|^2 - \frac{D}{2}), \tag{206}$$

where $\rho, u, \theta \ \in L^\infty(dt; L^2(dx))$.

iv) Finally, from the renormalized equation, we deduce that

$$v.\nabla_x g = \int\int qb(v_1 - v, \omega) d\omega M_1 dv_1 \tag{207}$$

which yields the incompressibility and Boussinesq relations, namely

$$\nabla_x.u = 0, \quad \nabla_x(\rho + \theta) = 0. \tag{208}$$

The rest of the proof is based on a new averaging lemma [78] as well as a better use of the entropy dissipation to get some estimate on the non hydrodynamic part of g_ϵ. The final passage to the limit uses the same local method of subsection 3.3.1 to deal with the acoustic waves.

Remark 5.3 *Let us also mention a new work of Y. Guo [86] where he proves that the next order terms in the formal development also hold for the case of regular solutions to the Boltzmann equation.*

5.3 The convergence towards the Stokes system

The convergence towards the Stokes system is easier than the Navier-Stokes case for two reasons. Indeed, we do not have to pass to the limit in the nonlinear terms. Besides, the control we get from the entropy dissipation is better. In this section, we want to present the result of [115] where a new notion of renormalized solution was used. In [74], the whole Stokes-Fourier system was also recovered by using a different method.

5.3.1 Defect measures

In [115], the difficulty D1 was overcome by showing that the conservation of momentum can be recovered in the limit by a very simple argument. Indeed by looking at the construction of the renormalized solutions of DiPerna-Lions [56], one sees that one can write a kind of conservation of moment (with a defect measure) which also intervenes in the energy inequality. Indeed, the solutions F_ϵ built by DiPerna and Lions satisfy in addition

$$\partial_t \int_{\mathbb{R}^D} vF_\epsilon \, dv + \frac{1}{\epsilon}\text{div} \int_{\mathbb{R}^D} (v \otimes v) F_\epsilon \, dv + \frac{1}{\epsilon}\text{div}(M_\epsilon) = 0. \tag{209}$$

Besides, the following energy equality holds

$$\frac{1}{2}\int_\Omega \int_{\mathbb{R}^D} |v|^2 F_\epsilon(t,x,v) dx \, dv + \frac{1}{2}\int_\Omega \text{tr}(M_\epsilon) \, dx = \frac{1}{2}\int_\Omega \int_{\mathbb{R}^D} |v|^2 F_\epsilon^0(x,v) dx \, dv \tag{210}$$

which can be rewritten (with $\epsilon^m m_\epsilon = M_\epsilon$)

$$\partial_t \langle g_\epsilon v\rangle + \nabla_x . \langle g_\epsilon v \otimes v\rangle + \frac{1}{\epsilon}\nabla . m_\epsilon = 0, \tag{211}$$

$$\int_\Omega \langle |v|^2 g_\epsilon \rangle dx + \int_\Omega \text{tr}(m_\epsilon) \, dx = 0 \tag{212}$$

5.3.2 Entropy inequality

One can write the entropy inequality for G_ϵ (as in the case of the limit towards the Navier-Stokes system) or write it for F_ϵ as well. It turns out that the second choice gives a better estimate for the defect measure. Indeed starting from the entropy inequality for F_ϵ, we can deduce

$$\begin{aligned}
&\int_\Omega \int_{\mathbb{R}^D} h(\epsilon^m g_\epsilon) dx \ M \ dv(t) - \int_\Omega \int_{\mathbb{R}^D} \epsilon^m \frac{|v|^2}{2} g_\epsilon dx \ M \ dv(t) + \\
&+ \frac{1}{4\epsilon^2} \int_0^t ds \int_\Omega dx \int_{\mathbb{R}^D} \int_{\mathbb{R}^D} M \ dv \ M_1 \ dv_1 \int_{S^{D-1}} d\omega b(v - v_1, \omega) \\
&\Big(G'_{\epsilon 1} G'_\epsilon - G_{\epsilon 1} G_\epsilon\Big) \log\Big(\frac{G'_{\epsilon 1} G'_\epsilon}{G_{\epsilon 1} G_\epsilon}\Big) \le \int_\Omega \int_{\mathbb{R}^D} h(\epsilon^m g_\epsilon^0) dx \ M \ dv
\end{aligned} \tag{213}$$

Let us now state the result. We take initial data satisfying

$$\int_{\mathbb{T}^D} \int_{\mathbb{R}^D} F_\epsilon^0 dx dv = 1, \quad \int_{\mathbb{T}^D} \int_{\mathbb{R}^D} vF_\epsilon^0 dx dv = 0, \quad \int_{\mathbb{T}^D} \int_{\mathbb{R}^D} |v|^2 F_\epsilon^0 dx dv = D \tag{214}$$

and

$$\int_\Omega \int_{\mathbb{R}^D} F_\epsilon^0 log F_\epsilon^0 \ dx \ dv \le -\frac{D}{2} + C\epsilon^{2m} \tag{215}$$

We also assume that b satisfies $(A0)$.

Theorem 5.4 *If F_ϵ is a sequence of renormalized solutions of the Boltzmann equations (179), $s = q = 1$ and $m > 1$, with initial condition F_ϵ^0 and satisfies the entropy inequality as well as the refined momentum equation, then the family $(1 + |v|^2)g_\epsilon$ is relatively compact in $w - L^1(dt\ Mdv\ dx)$. And, if g is a weak limit of a subsequence (still denoted g_ϵ) then $Lg = 0$ and $g = \rho + u.v + \theta(\frac{|v|^2}{2} - \frac{N}{2})$ satisfies the limiting dissipation inequality*

$$\frac{1}{2}\int_\Omega |\rho(t)|^2 + |u(t)|^2 + \frac{D}{2}|\theta|^2\ dx + \int_0^t \int_\Omega \frac{1}{2}\nu|\nabla_x u +^t \nabla_x u|^2 \\ \leq \liminf_{\epsilon\to 0} \frac{1}{\epsilon^{2m}} \int_\Omega \langle h(\epsilon^m g_\epsilon)\rangle dx = C^0 \tag{216}$$

Moreover $u = \langle vg\rangle$ is the solution of the Stokes system (S) with the initial condition $u^0 = P\langle vg^0\rangle$ and where the viscosity ν is given by (193). Besides, we have the following strong Boussinesq relationship

$$\rho + \theta = 0. \tag{217}$$

We only explain here briefly how we can recover the conservation of momentum at the limit. Indeed, starting from the entropy inequality, one deduces that

$$\int_\Omega \langle h(\epsilon^m g_\epsilon)\rangle dx + \epsilon^m \mathrm{tr}(m_\epsilon) + D(G_\epsilon) \leq C\epsilon^{2m} \tag{218}$$

and since $m > 1$, we deduce

$$\frac{1}{\epsilon}\mathrm{tr}(m_\epsilon) \quad \text{and} \quad \frac{1}{\epsilon}m_\epsilon \quad \to \quad 0 \tag{219}$$

in $L^\infty(0, T; L^1(\Omega))$ since m_ϵ is a positive matrix. This yields the local conservation of momentum in (211) at the limit.

5.4 The case of a bounded domain

In this subsection, we want to present the derivation of fluid mechanics boundary conditions starting form kinetic boundary condition. For simplicity, we will present the result in the Stokes scaling though the proof works as well for the Navier-Stokes scaling using the result of the previous sections. We also refer to [27] for a derivation of the Navier condition for the primitive equations.

Let Ω be a smooth bounded domain of $\mathbb{R}^D$ and $\mathcal{O} = \Omega \times \mathbb{R}^D$ the space-velocity domain. Let $n(x)$ be the outward unit normal vector at $x \in \partial\Omega$. We denote by $d\sigma_x$ the Lebesgue measure on the boundary $\partial\Omega$ and we define the outgoing/incoming sets Σ_+ and Σ_- by

$$\Sigma_\pm = \{(x,v) \in \Sigma, \quad \pm n(x).v > 0\} \text{ where } \Sigma = \partial\Omega \times \mathbb{R}^D.$$

We consider the Boltzmann equation in $\mathbb{R}_+ \times \mathcal{O}$ with a scaling where $q = s = 1$ and $m > 1$.

5.4.1 The Maxwell boundary condition

The boundary condition we will consider express the balance between the incoming and outgoing part of the trace of F, namely $\gamma_\pm F = 1\!\!1_{\Sigma_\pm} \gamma F$. We will use the following Maxwell reflection condition

$$\gamma_- F = (1-\alpha) L(\gamma_+ F) + \alpha K(\gamma_+ F) \quad \text{on } \Sigma_- \tag{220}$$

where α is a constant also called accommodation coefficient. The local reflection operator L is given by

$$L\phi(x,v) = \phi(x, R_x v) \tag{221}$$

where $R_x v = v - 2(n(x).v)n(x)$ is the velocity before the collision with the wall. The diffuse reflection operator K is given by

$$K\phi(x,v) = \sqrt{2\pi}\tilde{\phi}(x) M(v) \tag{222}$$

where $\tilde{\phi}$ is the outgoing mass flux

$$\tilde{\phi}(x) = \int_{v.n(x)>0} \phi(x,v)\, n(x).v dv. \tag{223}$$

We notice that

$$\int_{v.n(x)>0} n(x).v\sqrt{2\pi} M(v)\ dv = \int_{v.n(x)<0} |n(x).v| \sqrt{2\pi} M(v)\ dv = 1,$$

which expresses the conservation of mass at the boundary. Here, we are taking the temperature of the wall to be constant and equal to 1. For the existence of renormalized solutions to the Boltzmann equation in a bounded domain we refer to [136].

5.4.2 A priori estimate

Let $\mathcal{E}(\gamma_+ G_\epsilon)$, the so-called Darrozès-Guiraud information [85], be given by

$$\mathcal{E}(\gamma_+ G_\epsilon) = \int_{\partial\Omega} \left(\big\langle h(\delta_\epsilon \gamma_+ g_\epsilon) \big\rangle_{\partial\Omega} - h\big(\langle \delta_\epsilon \gamma_+ g_\epsilon \rangle_{\partial\Omega} \big) \right) d\sigma_x . \tag{224}$$

In the case of a bounded domain, the entropy inequality reads

$$H(G_\epsilon(t)) + \int_0^t \left(\frac{1}{\epsilon^2} E(G_\epsilon(s)) + \frac{\alpha_\epsilon}{\sqrt{2\pi}\epsilon} \mathcal{E}_\epsilon(\gamma_+ G_\epsilon(s)) \right) ds \le H(G_\epsilon^{in}) , \tag{225}$$

where $H(G)$ is the relative entropy functional

$$H(G) = \int_\Omega \langle (G \log(G) - G + 1) \rangle \, dx , \tag{226}$$

and $E(G)$ is the entropy dissipation rate functional

$$E(G) = \int_\Omega \left\langle\!\!\left\langle \frac{1}{4} \log\left(\frac{G_1' G'}{G_1 G} \right) (G_1' G' - G_1 G) \right\rangle\!\!\right\rangle dx . \tag{227}$$

Notice the presence of the extra positive term due to the boundary. It is easy to see that due to Jensen inequality the extra term $\mathcal{E}_\epsilon(\gamma_+ G_\epsilon(s)) \ge 0$. This also gives a bound on $\gamma_+ G_\epsilon$ which is useful.

Now, we present two results taken from [132] which hold for a wide range of collision kernels

Theorem 5.5 (Navier boundary condition) *Let $F_\epsilon^{in} = G_\epsilon^{in} M$ be a family of initial data satisfying*

$$\frac{1}{\delta_\epsilon^2} H(G_\epsilon^{in}) + \iint_{\mathcal{O}} |v|^2 F_\epsilon^{in} \, dx dv \le C^{in} \tag{228}$$

for some $C^{in} < \infty$ and

$$\begin{aligned} &\frac{1}{\delta_\epsilon} \Pi \langle v \, G_\epsilon^{in} \rangle \to u \quad \text{in } \mathcal{D}'(\Omega; \mathbf{R}^D) , \\ &\frac{1}{\delta_\epsilon} \langle (\frac{1}{D+2} |v|^2 - 1) \, G_\epsilon^{in} \rangle \to \theta \quad \text{in } \mathcal{D}'(\Omega; \mathbf{R}^D) , \end{aligned} \tag{229}$$

for some $(u^{in}, \theta^{in}) \in L^2(dx; \mathbf{R}^D \times \mathbf{R})$. *Denote by* G_ϵ *any corresponding family of renormalized solutions of the Boltzmann equation satisfying the entropy inequality (225), where the accommodation coefficient satisfies*

$$\frac{\alpha_\epsilon}{\sqrt{2\pi\epsilon}} \to \lambda \quad \textit{when } \epsilon \to 0. \tag{230}$$

Then, as $\epsilon \to 0$, *the family of fluctuations satisfies*

$$\begin{aligned}
&g_\epsilon \to v{\cdot}u + \big(\frac{1}{2}|v|^2 - \frac{D+2}{2}\big)\theta \quad \textit{in } w\text{-}L^1_{loc}(dt; w\text{-}L^1((1+|v|^2)M dv\, dx))\,,\\
&\Pi\langle v\, g_\epsilon\rangle \to u \quad \textit{in } C([0,\infty); \mathcal{D}'(\Omega; \mathbf{R}^D))\,,\\
&\langle(\frac{1}{D+2}|v|^2 - 1)\, g_\epsilon\rangle \to \theta \quad \textit{in } C([0,\infty); \mathcal{D}'(\Omega; \mathbf{R}^D))\,,
\end{aligned} \tag{231}$$

where Π *is the orthogonal projection from* $L^2(dx; \mathbf{R}^D)$ *onto divergence-free vector fields with zero normal velocity, namely the set*

$$H = \{u \in L^2(\Omega),\ \nabla_x{\cdot}u = 0,\ u.n = 0 \textit{ on } \partial\Omega\}.$$

Furthermore, $(u, \theta) \in C([0,\infty); H \times L^2(\Omega)) \cap L^2(dt; H^1(\Omega) \times H^1(\Omega))$ *and it satisfies the Stokes-Fourier system with Navier boundary condition*

$$\left\{\begin{array}{lll} \partial_t u + \nabla_x p - \nu\Delta_x u = 0\,, & div(u) = 0 & \textit{on } \mathbb{R}^+ \times \Omega,\\ (2\nu d(u)\cdot n + \lambda u) \wedge n = 0\,, & u{\cdot}n = 0 & \textit{on } \mathbb{R}^+ \times \partial\Omega, \end{array}\right.$$

$$\left\{\begin{array}{ll} \partial_t\theta - \kappa\Delta_x\theta = 0 & \textit{on } \mathbb{R}^+ \times \Omega,\\ \kappa\partial_n\theta + \lambda\frac{D+1}{D+2}\,\theta = 0 & \textit{on } \mathbb{R}^+ \times \partial\Omega, \end{array}\right. \tag{232}$$

$$u(0,x) = u^{in}(x), \quad \theta(0,x) = \theta^{in}(x) \qquad \textit{on } \Omega.$$

where $d(u)$ *denotes the symmetric part of the stress tensor* $d(u) = \frac{1}{2}(\nabla u +^t \nabla u)$.

The second result treats the case of Dirichlet boundary conditions. We will make the same assumptions as in the previous theorem but instead of assuming that $\frac{\alpha_\epsilon}{\epsilon\sqrt{2\pi}} \to \lambda$, we assume that $\frac{\alpha_\epsilon}{\epsilon} \to +\infty$.

Theorem 5.6 (Dirichlet boundary condition) *We make the same assumptions as in Theorem 5.5, except that we replace condition (230) by*

$$\frac{\alpha_\epsilon}{\epsilon} \to \infty, \quad \textit{when } \epsilon \to 0. \tag{233}$$

Then, as $\epsilon \to 0$, we have the same convergences (231) as in Theorem 5.5 with $(u, \theta) \in C([0, \infty); H \times L^2(\Omega)) \cap L^2(dt; V \times H^1_0(\Omega))$ where

$$V = \{u \in H^1(\Omega),\ \nabla_x \cdot u = 0,\ u = 0 \text{ on } \partial\Omega\}.$$

Furthermore, (u, θ) satisfies the Stokes-Fourier system with Dirichlet boundary condition

$$\begin{array}{ll} \partial_t u + \nabla_x p - \nu \Delta_x u = 0\,, \quad div(u) = 0 & \text{on } \mathbb{R}^+ \times \Omega, \\ \partial_t \theta - \kappa \Delta_x \theta = 0 & \text{on } \mathbb{R}^+ \times \Omega, \\ u = 0, \quad \theta = 0 & \text{on } \mathbb{R}^+ \times \partial\Omega, \\ u(0, x) = u^{in}(x), \quad \theta(0, x) = \theta^{in}(x) & \text{on } \Omega. \end{array} \tag{234}$$

Idea of the proof

The interior convergence can be deduced easily from the work of Golse and Levermore [74]. We just want to explain the convergence at the boundary. We prove two types of control on the trace γg_ϵ of g_ϵ on the boundary. The first control comes from the inside, it uses the interior estimates to deduce an estimate on the trace

Lemma 5.7 *We have for all $p > 0$,*

$$\gamma \hat{g}_\epsilon \to \gamma g \text{ in } w\text{-}L^1_{loc}(dt; w\text{-}L^1(M(1 + |v|^p)|v \cdot n(x)| dv\, d\sigma_x)) \tag{235}$$

$$\epsilon^m \gamma g_\epsilon \to 0 \text{ a.e. on } \mathbb{R}^+ \times \partial\Omega \times \mathbb{R}^d. \tag{236}$$

The second control comes from the boundary term appearing in the entropy dissipation. It does not give an estimate on g_ϵ but rather on g_ϵ minus its average in v. We get

Lemma 5.8 *Define $\gamma_\epsilon = \gamma_+ g_\epsilon - 1\!\!1_{\Sigma_+} \langle \gamma_+ g_\epsilon \rangle_{\partial\Omega}$ and*

$$\gamma^{(1)}_\epsilon = \gamma_\epsilon 1\!\!1_{\gamma_+ G_\epsilon \le 2\langle \gamma_+ G_\epsilon \rangle_{\partial\Omega} \le 4\gamma_+ G_\epsilon}, \quad \gamma^{(2)}_\epsilon = \gamma_\epsilon - \gamma^{(1)}_\epsilon. \tag{237}$$

Then

$$\sqrt{\frac{\alpha_\epsilon}{\epsilon}}\, \frac{\gamma^{(1)}_\epsilon}{(1 + \frac{\delta_\epsilon}{3} \gamma_+ g_\epsilon)^{1/2}} \text{ is bounded in } L^2_{loc}(dt; L^2(M|vn(x)| dv\, d\sigma_x)); \tag{238}$$

$$\sqrt{\frac{\alpha_\epsilon}{\epsilon}}\, \frac{\gamma^{(1)}_\epsilon}{(1 + \frac{\delta_\epsilon}{3} \langle \gamma_+ g_\epsilon \rangle_{\partial\Omega})^{1/2}} \text{ is bounded in } L^2_{loc}(dt; L^2(M|vn(x)| dv\, d\sigma_x)); \tag{239}$$

$$\frac{\alpha_\epsilon}{\epsilon \delta_\epsilon}\, \gamma^{(2)}_\epsilon \text{ is bounded in } L^1_{loc}(dt; L^1(M|vn(x)| dv\, d\sigma_x)). \tag{240}$$

5.5 Convergence towards the Euler system

We present, here, a method of proof based on an energy method or more precisely the relative entropy method (see Yau [174]). Indeed contrary to the two preceding cases, we suppose here the existence of a strong solution to the Euler system and we show the convergence towards this solution. The technique used is based on a Gronwall lemma. In [115] (in collaboration with P.-L. Lions), we show this convergence with an assumption on high velocities (A2). This assumption was removed in Saint-Raymond [149]. We will present the result of [149]. We introduce a defect measure (as in the Stokes case) which disappears at the limit. We take well prepared initial data (i.e. there are no acoustic waves) and the temperature fluctuation is equal to 0 initially.

5.5.1 Entropic convergence

In addition to the assumptions on G_ϵ^0 which we imposed in the case of convergence towards the Navier-Stokes system, we suppose that g_ϵ^0 converges entropically towards g^0 and that $g^0 = u^0.v$ (with $\text{div} u^0 = 0$) i.e. that

$$g_\epsilon^0 \to g^0 \quad \text{in} \quad w - L^1(M\ dvdx), \quad \text{and} \tag{241}$$

$$\lim_\epsilon \frac{1}{\epsilon^2} \int_\Omega \langle h(\epsilon g_\epsilon^0) \rangle dx = \frac{1}{2} \int_\Omega \langle (g^0)^2 \rangle dx. \tag{242}$$

It is also supposed that u^0 is regular enough (for example $u^0 \in H^s, s > \frac{D}{2}+1$) to be able to build a strong solution $\tilde{u}$ of the Euler system with the initial data u^0. Then, we have $\tilde{u} \in L^\infty_{loc}([0, T^*); H^s)$ for some $T^* > 0$.

5.5.2 Relative entropy

We want to show that the distribution F_ϵ is close to a Maxwellian $M_{(0,\epsilon\tilde{u},0)} = M\tilde{G}_\epsilon$. But as F_ϵ is only in $L\log L$, we have to estimate the difference between F_ϵ and $M_{(0,\epsilon\tilde{u},0)}$ using the relative entropy

$$H(G_\epsilon, \tilde{G}_\epsilon) = \int_\Omega \Big\langle G_\epsilon\ log\Big(\frac{G_\epsilon}{\tilde{G}_\epsilon}\Big) - G_\epsilon + \tilde{G}_\epsilon \Big\rangle. \tag{243}$$

Using the improved entropy inequality (213), we get

$$H(G_\epsilon, \tilde{G}_\epsilon) + \epsilon \int_\Omega \mathrm{tr}(m_\epsilon) + \int_0^t ds D(G_\epsilon) + \leq H(G_\epsilon^0, \tilde{G}_\epsilon^0)$$
$$+ \quad \int_0^t \int_\Omega < G_\epsilon \partial_t \log \tilde{G}_\epsilon > + \epsilon^2 \partial_t < g_\epsilon v > .\tilde{u} + \epsilon^3 \partial_t < g_\epsilon > \frac{|\tilde{u}|^2}{2} ds$$

where m_ϵ denotes the sequence of defect measures appearing in the conservation of momentum

Theorem 5.9 *Under some assumption of the collision kernel, if G_ϵ is a sequence of renormalized solutions of the Boltzmann equations with initial condition G_ϵ^0, and such that g_ϵ^0 converges entropically to $g^0 = u^0.v$, where $u^0 \in H^s$, ($s > \frac{D}{2} + 1$). Then, for all $0 \leq t < T^*$*

$$g_\epsilon(t) \to \tilde{u}(t).v \quad \textit{entropically} \tag{244}$$

where $\tilde{u}(t)$ is the unique solution of the Euler system in $L^\infty_{loc}([0, T*); H^s)$ with the initial condition u^0. Moreover, the convergence is locally uniform in time.

Let us explain here the idea of the proof of the above result. It is based on a Gronwall lemma. Indeed, after some non trivial computations, one can rewrite the entropy inequality as follows

$$\frac{1}{\epsilon^2}\Big[H(G_\epsilon, \tilde{G}_\epsilon) + \epsilon \int_\Omega \mathrm{tr}(m_\epsilon)\Big](t) + \frac{1}{\epsilon^2}\int_0^t ds D(G_\epsilon) \leq \frac{1}{\epsilon^2} H(G_\epsilon^0, \tilde{G}_\epsilon^0)$$
$$+ \quad \int_0^t ||\nabla \tilde{u}||_{L^\infty} \frac{1}{\epsilon^2}\Big[H(G_\epsilon, \tilde{G}_\epsilon) + \epsilon \int_\Omega \mathrm{tr}(m_\epsilon)\Big](s)\; ds + A_\epsilon$$

where A_ϵ converges to 0. Hence, we deduce that $H(G_\epsilon, \tilde{G}_\epsilon)$ goes to 0 in $L^\infty_{loc}([0, T^*))$.

We want to point out that the same type of argument can be used to prove the convergence towards the Navier-Stokes system in the case a regular solution is known to exist.

6 Some homogenization problems

In this section, we would like to present some homogenization problems. We will only consider examples which are related to fluid mechanics.

The homogenization of the Stokes and of the incompressible Navier-Stokes equations in a porous medium (open set perforated with tiny holes) has been studied in many works from the formal point of view as well as the rigorous one. We refer the interested reader to [21, 152, 107] for some formal developments and to [160, 4, 135] for some rigorous mathematical results.

Let us start by giving a definition of a porous medium. Let Ω be a smooth bounded domain of $\mathbb{R}^N$ and define $\mathcal{Y} =]0,1[^N$ to be the unit open cube of $\mathbb{R}^N$. Let $\mathcal{Y}_s$ (the solid part) be a closed smooth subset of $\mathcal{Y}$ with a strictly positive measure. The fluid part is then given by $\mathcal{Y}_f = \mathcal{Y} - \mathcal{Y}_s$ and we define $\theta = |\mathcal{Y}_f|$ the Lebesgue measure of $\mathcal{Y}_f$ and we assume that $0 < \theta < 1$. The constant θ is called the porosity of the porous medium. Repeating the domain $\mathcal{Y}_f$ by $\mathcal{Y}$-periodicity we get the fluid domain E_f which can also be defined as

$$E_f = \{y \in \mathbb{R}^N \mid \exists k \in \mathbb{Z}^N, \text{ such that } \; y - k \in \mathcal{Y}_f \}. \tag{245}$$

In the same way, we can define $E_s = \mathbb{R}^N - E_f$

$$E_s = \{y \in \mathbb{R}^N \mid \exists k \in \mathbb{Z}^N, \text{ such that } \; y - k \in \mathcal{Y}_s \}. \tag{246}$$

It is easy to see that E_f is a connected domain, while E_s is formed by separate smooth subsets. In the sequel, we denote for all $k \in \mathbb{Z}^N$, $\mathcal{Y}^k = \mathcal{Y} + k$ the translate of the cell $\mathcal{Y}$ by the vector k, we also denote $\mathcal{Y}_s^k = \mathcal{Y}_s + k$ and $\mathcal{Y}_f^k = \mathcal{Y}_f + k$. Hence, for all ϵ, we can define the domain Ω_ϵ as the intersection of Ω with the fluid domain rescaled by ϵ, namely $\Omega_\epsilon = \Omega \cap \epsilon E_f$. However, to get a smooth connected domain, we will not remove the solid parts of the cells which intersect the boundary of Ω. We define

$$\Omega_\epsilon = \Omega - U\{\epsilon \mathcal{Y}_s^k, \quad \text{where, } k \in \mathbb{Z}^N, \; \epsilon \mathcal{Y}^k \subset \Omega\}.$$

We also denote $K_\epsilon = \{k \mid k \in \mathbb{Z}^N \text{ and } \epsilon \mathcal{Y}^k \subset \Omega\}$.

Remark 6.1 *We can also consider more general domains, especially the more physical case where E_s is a connected set of $\mathbb{R}^N$ which can be achieved by allowing $\mathcal{Y}_s$ to be a closed subset of $\bar{\mathcal{Y}}$ (this is not possible in $N = 2$ since we also want that Ω_ϵ is connected). We refer the interested reader to the paper of G. Allaire [4] where the so-called "oscillating test function" method of Tartar is extended to the case of a connected E_s.*

Due to the presence of the holes $\epsilon \mathcal{Y}_s^k$, the domain Ω_ϵ depends on ϵ and hence to study the convergence of a sequence of functions, we have to extend the functions defined in Ω_ϵ to the whole domain Ω. This can be done in two different ways.

Definition 6.2 *For any function $\phi \in L^1(\Omega_\epsilon)$, we define*

$$\widetilde{\phi} = \begin{cases} \phi & in \quad \Omega_\epsilon \\ 0 & in \quad \Omega - \Omega_\epsilon \end{cases} \tag{247}$$

the extension by 0 *of* ϕ *and*

$$\widehat{\phi} = \begin{cases} \phi & in \quad \Omega_\epsilon \\ \frac{1}{\epsilon|\mathcal{Y}_f|}\int_{\epsilon\mathcal{Y}_f^k} \phi \, dy & in \quad \epsilon\mathcal{Y}_s^k \qquad \forall k \in K_\epsilon. \end{cases} \tag{248}$$

We will also need the restriction operator constructed by Tartar [160] for the case of a solid part $\mathcal{Y}_s$ strictly included in $\mathcal{Y}$ and by Allaire [4] for more general conditions on the solid part.

Lemma 6.3 *There exists a linear operator R_ϵ from $H_0^1(\Omega)^N$ to $H_0^1(\Omega_\epsilon)^N$ (called restriction operator) such that*

(i) $\forall \phi \in H_0^1(\Omega_\epsilon)^N$, *we have* $R_\epsilon\widetilde{\phi} = \phi$.

(ii) $\nabla \cdot u = 0$ *in* Ω *implies that* $\nabla \cdot R_\epsilon u = 0$ *in* Ω_ϵ.

(iii) There exists a constant C such that for all $u \in H_0^1(\Omega)^N$, we have

$$||R_\epsilon u||_{L^2(\Omega_\epsilon)} + \epsilon||\nabla(R_\epsilon u)||_{L^2(\Omega_\epsilon)} \leq C\Big[||u||_{L^2(\Omega)} + \epsilon||\nabla u||_{L^2(\Omega)}\Big]. \tag{249}$$

The operator R_ϵ defined above also acts from $W_0^{1,r}(\Omega)$ into $W_0^{1,r}(\Omega_\epsilon)$ for all $1 < r < \infty$ and we have an estimate similar to (249) where the L^2 norms are replaced by L^r norms.

Due to the presence of the holes in the domain Ω_ϵ, the Poincare's inequality reads

Lemma 6.4 *There exists a constant C which depends only on $\mathcal{Y}_s$ such that for all $u \in W_0^{1,p}(\Omega_\epsilon)$, we have*

$$||u||_{L^p(\Omega_\epsilon)} \leq C\epsilon||\nabla u||_{L^p(\Omega_\epsilon)} \tag{250}$$

We refer to [160] for a proof of this lemma. By a simple duality argument we also have the following relation for all $1 < p < \infty$

$$||u||_{W^{-1,p}(\Omega_\epsilon)} \leq C\epsilon||u||_{L^p(\Omega_\epsilon)}. \tag{251}$$

Finally, we define the permeability matrix $\bar{A}$. For all i, $1 \leq i \leq N$, let $(v_i, q_i) \in H^1(\mathcal{Y}_f)^N \times L^2(\mathcal{Y}_f)/\mathbb{R}$ be the unique solution of the following system

$$(S_i) \qquad \begin{cases} -\Delta v_i + \nabla q_i = e_i \quad \text{in } \mathcal{Y}_f \\ \text{div } v_i = 0 \quad \text{in } \mathcal{Y}_f \\ v_i = 0 \quad \text{on } \partial\mathcal{Y}_s \qquad \text{and} \quad v_i,\ q_i \text{ are } \mathcal{Y}-\text{periodic.} \end{cases}$$

Using regularity results of the Stokes problem, we infer that v_i and q_i are smooth. We extend v_i to the whole domain $\mathcal{Y}$ by setting $v_i(y) = 0$ if $y \in \mathcal{Y}_s$. Then, for all $y \in \mathcal{Y}_f$, $A(y)$ is taken to be the matrix composed of the column vectors $v_i(y)$ and $\bar{A} = \int_{\mathcal{Y}_f} A(y)dy$. It is easy to see that $\bar{A}$ is a symmetric positive definite matrix. Indeed, multiplying the first equation in (S_i) by v_j and the first equation in (S_j) by v_i, we get that $\int_{\mathcal{Y}_f} \nabla v_i \cdot \nabla v_j = \int_{\mathcal{Y}_f} v_{ji} = \bar{A}_{ji}$ and $\int_{\mathcal{Y}_f} \nabla v_j \cdot \nabla v_i = \int_{\mathcal{Y}_f} v_{ij} = \bar{A}_{ij}$ where we wrote $v_i(y) = \sum_{j=1}^N v_{ji}(y)e_j$. Then to prove that $\bar{A}$ is positive definite, we just notice that for all vector $X = \sum_{j=1}^N x_i e_i$, we have $\sum_{ij} x_i \bar{A}_{ij} x_j = ||\nabla \sum_{j=1}^N x_j v_j||^2_{L^2(\mathcal{Y}_f)}$ and that $\{v_i,\ 1 \leq i \leq N\}$ is an independent family.

6.1 Darcy law

Let us start by recalling the derivation of the Darcy law [48]. We consider the Stokes problem in the domain Ω_ϵ,

$$\begin{cases} -\Delta u_\epsilon + \nabla p_\epsilon \ = \ f \ , \\ \\ \text{div} u_\epsilon = 0, \qquad u_\epsilon = 0 \text{ on} \partial\Omega_\epsilon. \end{cases} \tag{252}$$

Theorem 6.5 *Prolonging u_ϵ by zero in the holes, we have the following convergence*

$$\widetilde{u}_\epsilon \quad \to \quad u \quad \textit{weakly in} \quad (L^2(\Omega)) \tag{253}$$

where $u = \bar{A}(f - \nabla p)$ and satisfies $\text{div} u = 0$. *This is the Darcy law*

The proof uses the "oscillating test function" method of Tartar [160]. Indeed, testing (252) with $\phi(x)v_i(\frac{x}{\epsilon})$ where $\phi \in C_0^\infty(\Omega)$, we can pass to the weakly to the limit in the different terms to deduce (253). Actually some non trivial work should be done to pass to the limit in the pressure term and we refer to [160] and [4].

6.2 Homogenization of a compressible model

Here, we give a derivation of the porous medium equation. We start with the following semi-stationary model

$$\begin{cases} \epsilon^2 \partial_t \rho_\epsilon + \mathrm{div}(\rho_\epsilon u_\epsilon) \;=\; 0 \;, \\ \\ -\mu \Delta u_\epsilon - \xi \nabla \mathrm{div} u_\epsilon + \nabla \rho_\epsilon^\gamma = \rho_\epsilon f + g \end{cases} \tag{254}$$

complemented with the boundary condition $u_\epsilon = 0$ on $\partial\Omega_\epsilon$ and the initial condition $\rho_\epsilon(t=0) = \rho_{\epsilon 0}$. The force term is such that $f \in L^\infty((0,T)\times\Omega_\epsilon)$ and $g \in L^2((0,T)\times\Omega_\epsilon)$. We also assume that $\gamma \geq 1$ and that $||f||_{L^\infty}$ is small enough if $\gamma = 1$.

We assume that the initial data is such that $\rho_{\epsilon 0} \in L^1 \cap L^\gamma(\Omega_\epsilon)$ if $\gamma > 1$, that $\int_{\Omega_\epsilon} \rho_{\epsilon 0} |\log \rho_{\epsilon 0}| < C$ if $\gamma = 1$ and that $\widehat{\rho}_{\epsilon 0}$ converges weakly to ρ_0 in $L^\gamma(\Omega)$.

We consider a sequence of weak solutions $(\rho_\epsilon, u_\epsilon)$ of the semi-stationary model (254) such that for all $T > 0$, $\rho_\epsilon \in C([0,T); L^1(\Omega_\epsilon)) \cap L^\infty(0,T; L^\gamma(\Omega_\epsilon)) \cap L^{2\gamma}((0,T)\times\Omega_\epsilon)$ and $\rho_\epsilon|\log\rho_\epsilon| \in L^\infty(0,T; L^1(\Omega_\epsilon))$ if $\gamma = 1$. Moreover, u_ϵ is such that $\frac{u_\epsilon}{\epsilon} \in L^2(0,T; H_0^1(\Omega_\epsilon))$ and $\frac{u_\epsilon}{\epsilon^2} \in L^2((0,T)\times\Omega_\epsilon)$. Finally, we also require that $\widehat{p}_\epsilon$ is bounded in $L^2_T(H^1(\Omega)) + \epsilon L^2_T(L^2(\Omega))$. We assume that the bounds given above are uniform in ϵ. We point out that the fact that we can consider a sequence of solutions satisfying the above uniform estimates can be proved using the methods of [109].

Before studying the limit of the sequence $(u_\epsilon, \rho_\epsilon, p_\epsilon)$, we have to prolong it to Ω. Let $\widetilde{u}_\epsilon$, $\widetilde{\rho}_\epsilon$ and $\widehat{p}_\epsilon$ be the extensions of u_ϵ, ρ_ϵ and p_ϵ to the whole domain Ω.

Theorem 6.6 *Under the above assumptions,*

$$\begin{aligned} \widetilde{\rho}_\epsilon \quad &\rightarrow \quad \theta\rho \quad \textit{weakly in} \quad L^r_T(L^\gamma(\Omega)) \cap L^{2\gamma}((0,T)\times\Omega), \\ \widehat{\rho}_\epsilon \quad &\rightarrow \quad \rho \quad \textit{strongly in} \quad L^r_T(L^\gamma(\Omega)) \cap L^{\gamma+1}((0,T)\times\Omega), \\ \frac{\widetilde{u}_\epsilon}{\epsilon^2} \quad &\rightarrow \quad u \quad \textit{weakly in} \quad L^2_T(L^2(\Omega)) \end{aligned}$$

for all $r < \infty$ where $\rho \in L^{2\gamma}((0,T)\times\Omega)$, $\rho^\gamma \in L^2_T(H^1(\Omega))$ and ρ is the *solution of the following system*

$$\begin{cases} \theta\partial_t\rho + \frac{1}{\mu} div \cdot \left[\rho\bar{A}(\rho f + g - \nabla\rho^\gamma)\right] = 0 \\ \\ \rho\bar{A}(\rho f + g - \nabla\rho^\gamma).n = 0 \quad on \quad \partial\Omega \\ \\ \rho(t=0) = \rho_0 \end{cases} \tag{255}$$

and u is given by

$$u = \bar{A}(\rho f + g - \nabla\rho^\gamma) \quad on \quad \{\rho > 0\}. \tag{256}$$

We point out here that even though each one of the terms f, g and $\nabla\rho^\gamma$ does not have necessary a trace on the boundary $\partial\Omega$, the combination of them appearing in (255) has a sense. A formal derivation of the system (255) can be found in [55]. The relation (256) giving u as a function of the pressure is a Darcy law [48, 160].

Remark 6.7 *if $\bar{A} = \alpha I$ (which is the case if for instance $\mathcal{Y}_s$ is a ball) and $f = g = 0$ then we get the following system*

$$\begin{cases} \partial_t\rho - \beta\Delta\rho^{\gamma+1} = 0 \\ \\ \frac{\partial\rho^{\gamma+1}}{\partial n} = 0 \quad on \quad \partial\Omega \\ \\ \rho(t=0) = \rho_0 \end{cases} \tag{257}$$

where $\beta = \frac{\alpha\gamma}{\theta\mu(\gamma+1)}$. This system is the so-called "porous medium" equation.

6.3 Homogenization of the Euler system

We consider an incompressible perfect fluid governed by the Euler equation. We consider the following system of equations

$$\begin{cases} \partial_t u^\epsilon + \epsilon u^\epsilon.\nabla u^\epsilon = -\nabla p^\epsilon + f^\epsilon(x) \\ \text{div }(u^\epsilon) = 0, \\ u^\epsilon.n = 0 \quad \text{on} \quad \partial\Omega_\epsilon \\ u^\epsilon_{|t=0} = u^\epsilon_0 \end{cases} \tag{258}$$

where u^ϵ is the velocity, p^ϵ is the pressure, f^ϵ is an exterior force and n is the outward normal vector to Ω_ϵ. Arguing as in the book of A. Bensoussan, J.-L. Lions and G. Papanicolaou [21] (see also [107]) and the book of E. Sanchez-Palencia [152], we make an asymptotic development using both a microscopic scale and a macroscopic scale. Hence, we can derive a (formal) limit system. Indeed taking u^ϵ of the form $u^\epsilon = u^0(t,x,\frac{x}{\epsilon}) + \epsilon u^1(t,x,\frac{x}{\epsilon}) + ...$, we get formally the following system for $v(t,x,y) = u^0(t,x,y)$

$$\begin{cases} \partial_t v + v.\nabla_y v = -\nabla_y p(x,y) - \nabla_x\, q(x) + f(t,x,y) \\ \\ \mathrm{div}_y\ (v) = 0, \quad \mathrm{div}_x\ \left(\int_{\mathcal{Y}_f} v(x,y)dy\right) = 0, \\ \\ v(x,y).n = 0 \quad \text{on} \quad \Omega \times \partial\mathcal{Y}_s, \\ \\ \left(\int_{\mathcal{Y}_f} v(x,y)dy\right).n = 0 \quad \text{on} \quad \partial\Omega \\ \\ v_{|t=0} = v_0 \end{cases} \tag{259}$$

where $f(t,x,y)$ and $v_0(x,y)$ are the two-scale limits of the sequences f^ϵ and u_0^ϵ and here n is the inward normal vector to $\mathcal{Y}_s$. The notion of two-scale convergence is aimed at a better description of sequences of oscillating functions with a known scale. It was introduced by G. Nguetseng [137, 138] and later extended by G. Allaire [5] where one can find the mathematical setting we use here.

Definition 6.8 *Let u^ϵ be a sequence of functions such that $u^\epsilon \in L^2(\Omega_\epsilon)$ and $||u^\epsilon||_{L^2(\Omega_\epsilon)}$ is bounded uniformly in ϵ. If $v(x,y) \in L^2(\Omega \times \mathcal{Y}_f)$, then we say that u^ϵ two-scale converges to v if and only if $\forall \psi \in C(\Omega \times \mathcal{Y}_f)$, we have*

$$\lim_{\epsilon\to 0} \int_{\Omega_\epsilon} u^\epsilon(x)\psi(x,\frac{x}{\epsilon})\ dx = \int_{\Omega\times\mathcal{Y}_f} v(x,y)\psi(x,y)\ dxdy. \tag{260}$$

Moreover, we say that u^ϵ two-scale converges strongly to v if and only if $v(x,y) \in L^2(\Omega, C(\mathcal{Y}_f))$ and we have

$$\lim_{\epsilon\to 0} ||u^\epsilon(x) - v(x,\frac{x}{\epsilon})||_{L^2(\tilde{\Omega}_\epsilon)} = 0, \tag{261}$$

and

$$\lim_{\epsilon\to 0} ||u^\epsilon(x)||_{L^2(\Omega_\epsilon - \tilde{\Omega}_\epsilon)} = 0, \tag{262}$$

We will state two results. The first one concerns the Cauchy problem for the limit system and the second one concerns the convergence of a sequence of the solutions to (258) toward a solution to (259). We start by defining the following functional spaces

$$\begin{aligned}\mathcal{A} &= \{v(x,y), \quad v \in L^2(\Omega \times \mathcal{Y}_f),\ \mathrm{div}_y(v) = 0,\ \ \mathrm{div}_x(\overline{v}) = 0, \\ &\qquad\qquad v.n = 0 \text{ on } \Omega \times \partial\mathcal{Y}_s,\ \overline{v}.n = 0 \text{ on } \partial\Omega\} \quad (263)\\ \mathcal{A}_\infty &= \{v(x,y), \quad v \in \mathcal{A} \ \text{ and } \mathrm{curl}_y(v) \in L^\infty(\Omega \times \mathcal{Y}_f)\}, \quad (264)\end{aligned}$$

where div_y and div_x denote respectively the divergence in the y and in the x variables, namely $\mathrm{div}_y(v) = \partial_{y_1} v_1 + \partial_{y_2} v_2$ and $\mathrm{div}_x(v) = \partial_{x_1} v_1 + \partial_{x_2} v_2$. Moreover, $\overline{v}$ denotes the integral of v over $\mathcal{Y}_f$, namely $\overline{v}(x) = \int_{\mathcal{Y}_f} v(x,y)\, dy$. Finally, n denotes the exterior normal vector to $\partial\mathcal{Y}_f$ or to $\partial\Omega$.

Now, we give an existence result for the limit system (259)

Theorem 6.9 *Take $v_0 \in \mathcal{A}_\infty$ and $f \in L^1((0,\infty);\mathcal{A}_\infty))$. Then, there exists a global solution to the system (259) such that*

$$v \in C([0,\infty);\mathcal{A}) \cap L^\infty((0,\infty);\mathcal{A}_\infty). \quad (265)$$

This result is similar to the existence result for the incompressible Euler system by V.-I. Yudovich [175]. However, unlike Yudovich solutions, the uniqueness of the solutions constructed in theorem 6.9 is not known.

Now, we focus on the convergence result. We have to assume that u_0^ϵ is bounded in $L^3(\Omega_\epsilon)$, $\mathrm{div}\,(u_0^\epsilon) = 0$, $u_0^\epsilon.n = 0$ on $\partial\Omega_\epsilon$, ϵ curl (u_0^ϵ) is in L^∞ (which implies the existence and uniqueness for the initial system) and that u_0^ϵ two-scale converges strongly to v_0 where $v_0 \in \mathcal{A}_\infty$. Moreover, we assume that f^ϵ is divergence-free, that it is bounded in $L^1((0,\infty);L^3(\Omega_\epsilon))$, that curl f^ϵ is bounded in $L^1((0,\infty);L^\infty(\Omega_\epsilon))$ and that f^ϵ two-scale converges strongly to f, namely

$$\lim_{\epsilon\to 0} ||u_0^\epsilon(x) - v_0(x,\frac{x}{\epsilon})||_{L^2(\Omega_\epsilon)} = 0, \quad (266)$$

$$\lim_{\epsilon\to 0} ||f^\epsilon(t,x) - f(t,x,\frac{x}{\epsilon})||_{L^1((0,\infty);L^2(\Omega_\epsilon))} = 0, \quad (267)$$

where v_0 and f satisfy the hypotheses of theorem 6.9. Here, we only take the two-scale convergence in the x variable, then we have

Theorem 6.10 *Under the above conditions there exists a sequence u^ϵ of solutions to the initial system (258). Moreover, extracting a subsequence if necessary u^ϵ two-scale converges to v where v is a solution to the limit system (259).*

We refer to [116, 128] for the proof.

7 Conclusion

Before giving some concluding remarks we would like to mention some other limit problems which we did not develop in the previous sections. These asymptotic problems are very important and we want to give some references to the interested reader.

7.1 Other limits

7.1.1 The infinite Prandtl number limit

The infinite Prandtl number limit was considered in [170] (see equation (8) for the definition of the Prandtl number). At the limit the so-called infinite Prandtl number convection system is retrieved at the limit. It is a system where he velocity is slaved by the temperature field since velocity diffuses more rapidly than the temperature. The proof is based on an expansion using two time scales.

7.1.2 The zero surface tension limit

The infinite Weber limit was considered in [7]. This is the same as the zero surface tension limit. It was proved in [7] that when surface tension goes to zero the water wave system with surface tension [6] converges to the water wave system without surface tension [172]. This is a singular limit since surface tension has a regularizing effect even though the initial system and the limit system are of the same type.

7.1.3 The quasi-neutral limit

The convergence from the Vlasov-Poisson system towards the incompressible Euler equation in the quasi-neutral limit was considered in [24] and [126].

These two works deal with the zero temperature case, namely the density $f(t, x, v)$ is a delta function in velocity.

A related problem, is the relation between the Euler system and the N vortices problem. This was considered in [121]. We also refer to [120] for an inviscid limit with concentrated vorticity.

For related asymptotic problems in plasma physics, we refer to [131] for the limit from the Klein-Gordon-Zakharov system to the nonlinear Schrödinger equation. We also refer to [130] for the limit from Maxwell-Klein-Gordon and Maxwell-Dirac to Poisson-Schrödinger when the speed of light c goes to infinity.

7.1.4 Thin domains

Fluid equations considered in thin domains give rise to many asymptotic problems (see [91] and [146] and the references therein). Indeed, taking for instance the Navier-Stokes equation in a thin domain $(0, \epsilon) \times \mathbb{T}^2$, we can try to describe the solutions when ϵ goes to zero. To do so, we have to make a change of variable and rescale the domain to a fixed domain $(0, 1) \times \mathbb{T}^2$. This introduces a small parameter ϵ in the equation written in the fixed domain. The small parameter ϵ is the ratio between the vertical length scale and the horizontal one.

7.2 Concluding remarks

As can be seen from the different section of this chapter, asymptotic problem in hydrodynamics is a vast subject by the number of problems one can consider and the number of methods used to treat them. It is an important subject from physical and numerical point of view. Besides, it is the motor behind the development of many new mathematical tools such as (the group method, defect measures, boundary layer theory ...) to handle the several physical phenomenon such as (oscillations, boundary layers ...).

In this review paper, we tried to give an idea about some of the advances made in these singular limits during the last few years. At several places, the author put more emphasis on results he is more aware of.

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