Diffusion and Homogenization approximation for semiconductor Boltzmann-Poisson system

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Abstract. We are concerned with the study of the diffusion and homogenization approximation of the Boltzmann-Poisson system in presence of a spatially oscillating electrostatic potential. By analyzing the relative entropy, we prove uniform energy estimate for well prepared boundary data. An averaging lemma and two scale convergence techniques are used to prove rigorously the convergence of the scaled Boltzmann equation (coupled to Poisson) to a homogenized Drift-Diffusion-Poisson system.

Key words: semiconductors, Boltzmann-Poison system, Drift-Diffusion equation, renormalized solution, diffusion approximation, homogenization. two-scale convergence, averaging lemma.

1 Introduction

In this paper, we study the diffusion limit of the semiconductor Boltzmann-Poisson system (see [24, 27]) in the presence of a spatially oscillating electrostatic potential. This generalizes the study done in [21] where the same problem is treated without the oscillating electrostatic potential. The study of the diffusion and homogenization of the linear Boltzmann system was also done in [7]. In the present paper, the major difficulty is the combination of the nonlinearity with the two scale limit which requires some compactness to pass to the limit. We refer to the previous mentioned papers for the physical background.

We consider the following scaled Boltzmann equation

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon}\left(v \,.\, \nabla_x f^\varepsilon - \nabla_x \Phi_T^\varepsilon \,.\, \nabla_v f^\varepsilon\right) = \frac{Q(f^\varepsilon)}{\varepsilon^2} \tag{1}$$

The position variable x belongs to a bounded and regular domain ω, the velocity is $v \in \mathbb{R}^d$ and the time t is nonnegative. The initial value of the distribution function is

$$f^\varepsilon(0,x,v) = f_0^\varepsilon(x,v) \tag{2}$$

where f_0^ε is a given function which might depend on ε.

The electrostatic potential $\Phi_T^\varepsilon(t,x)$ is given by

$$\Phi_T^\varepsilon(t,x) = \Phi_H\left(x, \frac{x}{\varepsilon}\right) + \Phi_P^\varepsilon(t,x) \tag{3}$$

where $\Phi_H(x,y)$ is a given regular and cell-periodic function with respect to y. For simplicity we assume that the cell period is the unit cube $[0,1]^d$ and Φ_H is time-independent. The additional potential Φ_P^ε is self-consistent, it is obtained by solving the Poisson equation

$$\begin{cases} -\Delta_x \Phi_P^\varepsilon(t,x) = \rho^\varepsilon(t,x) = \displaystyle\int_{\mathbb{R}^d} f^\varepsilon(t,x,v)dv \\ \Phi^\varepsilon_{P\,|\,\partial\omega} = \Phi_b \end{cases} \tag{4}$$

where Φ_b is given on the boundary. We denote $\bar{\Phi}_b$ the harmonic extension of Φ_b in ω.

The incoming boundary conditions are assumed to be well-prepared:

$$f^\varepsilon(t,x,v) = \rho_b(t,x)\; M(v) \exp\left(\Phi_e(x) - \Phi_H\left(x, \frac{x}{\varepsilon}\right)\right) \qquad x \in \partial\omega,\; v.n(x) < 0 \tag{5}$$

where $n(x)$ is the outward normal vector at the point x, $\rho_b(t,x)$ is a boundary data, $\Phi_e(x)$ is the effective potential defined in (10) and M is the normalized Maxwellian with zero mean velocity

$$M(v) = \frac{e^{-|v|^2/2}}{\left(\sqrt{2\pi}\,\right)^d} \tag{6}$$

The collision operator is the low density approximation electron-phonon interaction, given by

$$Q(f) = \int_{\mathbb{R}^d} \sigma(v,v')\,[\,(M(v)\,f(v') - M(v')\,f(v)\,]\,dv' \tag{7}$$

The cross section σ is assumed to be symmetric (micro-reversibility principle) and bounded from above and below

$$\begin{cases} \sigma(v,v') = \sigma(v',v), \quad (v,v') \in \mathbb{R}^{2d} \\ \\ \exists\, \sigma_1,\, \sigma_2 > 0 \quad / \quad 0 < \sigma_1 \leq \sigma \leq \sigma_2 \end{cases} \tag{8}$$

We recall that this operator is the linearization of the Fermi-Dirac model, we refer to [20, 24, 30] for the study of such operator. The properties of this operator can be summarized in the following proposition due to Poupaud [27].

Proposition 1.1 *[27] The collision operator Q is continuous on $L^1(dv)$. It satisfies*

1. *$-Q$ is nonnegative and self-adjoint operator on $L^2(M^{-1}(v)\,dv)$.*
2. *$\mathcal{K}er(Q)$ is spanned by the Maxwellian: $\mathcal{K}er(Q) = \mathbb{R}\,M(v)$.*
3. *Q is invertible on its range:*

$$\mathcal{R}(Q) = \mathcal{K}er(Q)^{\perp} := \left\{ g \in L^2(M^{-1}\,dv), \quad \int_{\mathbb{R}^d} g(v)dv = 0 \right\}$$

4. *Q satisfies an H-theorem which we state here in the following form*

$$\mathcal{H}(f) = \int_{\mathbb{R}^d} Q(f)\log\left(\frac{f}{M}\right) dv \leq -2\sigma_1 \int_{\mathbb{R}^d} \left(\sqrt{f} - \sqrt{\rho\, M}\right)^2 dv \tag{9}$$

where $\rho = \int_{\mathbb{R}^d} f\, dv$.

1.1 Assumptions and Convergence result

Before giving the assumptions we are considering, let us define by $\Omega = \omega \times \mathbb{R}^d$, the phase (position-velocity) space. The incoming (-) and the outgoing (+) parts of the boundary are given by

$$\Gamma^{\pm} = \{(x,v) \in \partial\Omega, \; / \; \pm\; v.n(x) > 0\}$$

We will denote by Φ_e the homogenized effective potential

$$\Phi_e(x) = -\log\left(\int_Y e^{-\Phi_H(x,y)} dy\right). \tag{10}$$

The Maxwellians M_{Φ_H} and $M_{\Phi_T^\varepsilon}$ will denote

$$\begin{cases} M_{\Phi_H}(x,y,v) &= & M(v)\exp\left(\Phi_e(x) - \Phi_H(x,y)\right) \\ M_{\Phi_T^\varepsilon}(t,x,v) &= & M(v)\exp\left(\Phi_e(x) - \Phi_T^\varepsilon(t,x)\right) \end{cases} \tag{11}$$

We define the total mass, the kinetic energy and two distances to the local equilibrium by

$$
\begin{cases}
\mathcal{M}^\varepsilon(t) = \displaystyle\int_\Omega f^\varepsilon(t,x,v)\,dv\,dx, \qquad \mathcal{K}^\varepsilon(t) = \int_\Omega \frac{|v|^2}{2} f^\varepsilon(t,x,v)\,dv\,dx \\
\mathcal{R}^\varepsilon(t) = \displaystyle\int_0^t \int_\Omega \left(\sqrt{f^\varepsilon} - \sqrt{\rho^\varepsilon M}\right)^2 dv\,dx\,ds \\
\mathcal{R}_1^\varepsilon(t) = \displaystyle\int_0^t \int_\Omega \int_{\mathbb{R}^d} \left(\frac{f'}{M'} - \frac{f}{M}\right)\left(\log\frac{f'}{M'} - \log\frac{f}{M}\right) M M'\,dv'\,dv\,dx\,ds
\end{cases}
\tag{12}
$$

The charge and current densities will be denoted by

$$
\rho^\varepsilon(t,x) = \int_{\mathbb{R}^d} f^\varepsilon(t,x,v)\,dv, \qquad j^\varepsilon(t,x) = \frac{1}{\varepsilon}\int_{\mathbb{R}^d} f^\varepsilon(t,x,v) v\,dv \tag{13}
$$

We will also use the notations

$$
\begin{cases}
\widetilde{f^\varepsilon}(t,x,v) = f^\varepsilon(t,x,v)\,e^{\Phi_H(x,x/\varepsilon)}, \\
\widetilde{\rho^\varepsilon}(t,x) = \rho^\varepsilon(t,x)\,e^{\Phi_H(x,x/\varepsilon)}, \\
\widetilde{j^\varepsilon}(t,x) = j^\varepsilon(t,x)\,e^{\Phi_H(x,x/\varepsilon)}
\end{cases}
\tag{14}
$$

Assumptions. The assumptions of this study are the following

(**A1**) The initial data f_0^ε is uniformly bounded in $L^1(\Omega)$ with bounded mass, kinetic energy and entropy: i. e. there exists a constant $C > 0$ such that

$$
\int_\Omega f_0^\varepsilon \left(|\log f_0^\varepsilon| + 1 + \frac{|v|^2}{2}\right) dxdv \leq C \tag{15}
$$

(**A2**) The boundary data ρ_b is bounded from below and above: $\exists\, c_b > 0$ and $C_b > 0$ such that

$$
c_b \leq \rho_b \leq C_b \tag{16}
$$

(**A3**) The potential Φ_H belongs to $L^\infty_{loc}(\mathbb{R}^+;\ W^{2,\infty}(\bar\omega \times Y))$. The boundary data $\Phi_b \geq 0$ and $(\bar\Phi_b, \partial_t \bar\Phi_b) \in L^\infty_{loc}(\mathbb{R}^+;\ W^{1,\infty} \times L^\infty(\bar\omega))$.

(**A4**) The cross-section σ is uniformly bounded and satisfies the detailed balance principle:

$$
0 < \underline{\sigma} \leq \sigma(v,v') = \sigma(v',v) \leq \bar\sigma \tag{17}
$$

Main result. The main result of the paper is the following

Theorem 1.2 *Assume that assumptions (A1), (A2), (A3) and (A4) are satisfied. Let $(f^\varepsilon, \Phi_P^\varepsilon)$ be a renormalized solution in the sense of definition 3.1 of the Boltzmann-Poisson system (1)–(4) and which satisfies in addition the properties of theorem 3.2. Then,*

$$
\begin{array}{llll}
\widetilde{f^\varepsilon} := f^\varepsilon\, e^{\Phi_H(x,x/\varepsilon)} & \to & \rho\, M(v)\, e^{\Phi_e(t,x)} & \text{in } L^1(0,T;\ L^1(\Omega)) \\
\Phi_P^\varepsilon & \to & \Phi_P & \text{in } L^2(0,T;\ W^{1,p}(\omega)), \quad \forall\, 1 \leq p < 2
\end{array}
\tag{18}
$$

In particular ρ^ε converges weakly in $L^1(0,T;\ L^1(\omega))$ towards ρ and (ρ, Φ_P) is the solution of the Drift-Diffusion-Poisson system

$$\begin{cases} \partial_t \rho + \nabla_x \,.\, J(\rho, \Phi_P + \Phi_e) = 0 \\ J(\rho, \Phi_P + \Phi_e) = -\mathbb{D}(x) \left[\nabla_x \rho + \rho \nabla_x (\Phi_P + \Phi_e) \right] \\ -\Delta_x \Phi_P = \rho \\ (\rho, \Phi_P)_{|\partial\omega} = (\rho_b, \Phi_b) \\ \rho(t=0) = \rho_0 = \displaystyle\int_{\mathbb{R}^d} f_0 \, dv \end{cases} \tag{19}$$

The function Φ_e and the matrix $\mathbb{D}$ are given respectively by (10) and (31) *and f_0 is the weak limit of f_0^ε.* □

The outline of the paper is as follows. In Section 2, we recall some useful properties concerning the notion of two-scale limit. Then, we derive formally the homogenized fluid system. Section 3 is devoted to the existence of renormalized solutions to the initial system. Then, in section 4, we recall some uniform estimates for well-prepared incoming data. In Section 5, we prove the compactness of $\widetilde{f}^\varepsilon$ and $\widetilde{\rho}^\varepsilon$ by using an averaging lemma and a Lions-Aubin lemma. This result will be essential to pass to the limit in the equation which will be done in section 6. In the last subsection we will pass to the limit in the boundary conditions and recover the limit system which will end the proof of Theorem 1.2.

2 Formal analysis

Due to the presence of an oscillating potential Φ_H, the formal analysis should be treated using a double scale limit on the spatial variable. As usual, when dealing with two-scale limits, the following notations will be used. We denote by $\mathcal{C}_\#(Y)$ and $\mathcal{C}^\infty_\#(Y)$ respectively, continuous and infinitely differentiable functions defined on Y and extended to $\mathbb{R}^d$ by Y-periodicity. For $p \geq 1$ and an open subset $\omega \subset \mathbb{R}^d$, $L^p(\omega;\ \mathcal{C}_\#(Y))$ is the space of functions of $L^p(\omega)$ with value in $\mathcal{C}_\#(Y)$. The following spaces $\mathcal{D}'(\omega;\ \mathcal{C}^\infty_\#(Y))$, $\mathcal{D}(\omega;\ \mathcal{C}^\infty_\#(Y))$, ... are defined in the same manner. We also use the notation $\mathcal{D}_\# = \mathcal{D}((0,T) \times \Omega;\ \mathcal{C}^\infty_\#(Y))$ to denote the space of test functions which are periodic in Y. For Y-periodic functions of y, the notation $(f)_\varepsilon$, $(\nabla_y f)_\varepsilon$, ... will be used instead of $f(t, x, \frac{x}{\varepsilon}, v)$, $\nabla_y f(t, x, \frac{x}{\varepsilon}, v)$, ... We will also use the notation $\Phi_H^\varepsilon := (\Phi_H)_\varepsilon = \Phi_H(x, \frac{x}{\varepsilon})$.

2.1 Two-scale limit

Let us review some useful properties related to the notion of two-scale convergence. Formally speaking, it consists in introducing an antzas like a two scale Hilbert development

$$f^\varepsilon(t,x,v) = f_0\left(t, x, \frac{x}{\varepsilon}, v\right) + \varepsilon\, f_1\left(t, x, \frac{x}{\varepsilon}, v\right) + \varepsilon^2\, f_2\left(t, x, \frac{x}{\varepsilon}, v\right) + \dots \tag{20}$$

A similar idea consists in replacing the sequence $f^\varepsilon(t,x,v)$ by $\widehat{f^\varepsilon}(t,x,y,v)$. This function $\widehat{f^\varepsilon}$ is periodic in y. The existence of such function is justified by using the Riesz theorem. Let us explain this idea and give some properties of the two scale convergence which we will use throughout this paper.
Let $u_\varepsilon(t,x,v)$ be a bounded sequence in $L^2(]0,T[\times\Omega)$. Then, using the Riesz representation theorem, there exists a unique function $\widehat{u_\varepsilon}$ such that for all $\psi := \psi(t,x,y,v)$ in $\mathcal{D}([0,T]_t \times \Omega_{x,v},\ \mathcal{C}^\infty_\#(Y))$, we have

$$\int_0^T \int_\Omega u_\varepsilon(t,x,v)\psi\left(t,x,\frac{x}{\varepsilon},v\right) = \int_0^T \int_\Omega \int_Y \widehat{u_\varepsilon}(t,x,y,v)\psi(t,x,y,v) \qquad (21)$$

Moreover,

$$\|u_\varepsilon\|_{L^2(]0,T[\times\Omega)} = \|\widehat{u_\varepsilon}\|_{L^2(]0,T[\times\Omega\times Y)}$$

The above isometry identifies u_ε and $\widehat{u_\varepsilon}$ and then gives a rigorous sense to the notion of two sale convergence. Indeed, from each bounded sequence $\widehat{u_\varepsilon}$ of L^2_{loc} there exists a subsequence which converges weakly to a function $\widehat{u}$ of L^2_{loc}. The subsequence is still denoted $\widehat{u_\varepsilon}$. The limit $\widehat{u}$ is called the two-scale limit of $\widehat{u_\varepsilon}$. We remark that there is more information in $\widehat{u}$ then in the weak limit u of the sequence u_ε. We can notice that u is the average of $\widehat{u}$ with respect to the fast variable y. We should remark in this context that even if the two scale limit of u_ε does not depend on y, this does not imply that it converges strongly in L^2. However, we say that u_ε converges two-scale strongly if the sequence $\widehat{u_\varepsilon}$ converges in the L^2 norm. we claim that if u_ε converges two-scale strongly towards a $y-$independent function $u(x)$, then it converges strongly in L^2. Let us summarize some of these properties in the following proposition

Proposition 2.1 *Let ω be an open subset of $\mathbb{R}^d$ and u_ε a sequence of $L^2(\omega)$ that two-scale converges to a limit $\widehat{u} \in L^2(\omega \times Y)$. If*

$$\lim_{\varepsilon\to 0} \|u_\varepsilon\|_{L^2(\omega)} = \|\widehat{u}\|_{L^2(\omega\times Y)}$$

then for all sequence v_ε that two-scale converges to $\widehat{v} \in L^2(\omega \times Y)$, we have $\widehat{u_\varepsilon}\,\widehat{v_\varepsilon}$ converges in $\mathcal{D}'(\omega \times Y)$ towards $\widehat{u}\,\widehat{v}$ and

$$u_\varepsilon\, v_\varepsilon \rightharpoonup \int_Y \widehat{u}\,\widehat{v}\,dy \qquad \in \mathcal{D}'$$

Moreover, if $\widehat{u}$ is continuous with respect to the variable y, we infer that

$$\lim_{\varepsilon\to 0} \left\| u_\varepsilon(x) - \widehat{u}\left(x,\frac{x}{\varepsilon}\right)\right\|_{L^2(\omega)} = 0$$

Let us conclude this review with some further remarks:

1. For any smooth function $\psi(x,y)$ which is $Y-$periodic with respect to y, the sequence $\psi_\varepsilon(x) = \psi(x,\frac{x}{\varepsilon})$ two-scale converges strongly towards the function ψ.

2. Let u_ε be a bounded sequence in $W^{1,2}$ which converges weakly to $u \in W^{1,2}$. Then, u_ε converges to u in L^2. Moreover, there exists a function $u_1(x,y) \in L^2\left(\omega,\ W^{1,2}_{\#}(Y)/\mathbb{R}\right)$ such that $\nabla_x u_\varepsilon$ two-scale converges to $\nabla_x u + \nabla_y u_1$.

3. If u_ε and $\varepsilon\,\nabla_x u_\varepsilon$ are bounded in L^2. Then the two-scale limit $\widehat{u}$ of u_ε belongs to $L^2(\omega,\ W^{1,2}_{\#}(Y))$ and $\varepsilon\,\nabla_x u_\varepsilon$ two-scale converges, up to a subsequence, to $\nabla_y \widehat{u}$.

4. Similar remarks are available in L^p or L^p_{loc} for all $p \in (1, +\infty)$.

2.2 Formal expansion

Let us now perform the formal analysis of the scaled Boltzmann equation in order to get the limit system. We start with the linear equation and consider the equation associated with a given potential $\Phi_H^\varepsilon := \Phi_H(x, \frac{x}{\varepsilon})$. As we remarked above, one can rewrite the Boltzmann equation, with unknown $\widehat{f^\varepsilon}$. It reads as follows

$$\partial_t \widehat{f^\varepsilon} + \frac{1}{\varepsilon}\left(v.\nabla_x \widehat{f^\varepsilon} - \nabla_x \Phi_H.\nabla_v \widehat{f^\varepsilon}\right) + \frac{\mathcal{L}_{\Phi_H}(\widehat{f^\varepsilon})}{\varepsilon^2} = 0 \tag{22}$$

where

$$\mathcal{L}_{\Phi_H} = v.\nabla_y - \nabla_y \Phi_H.\nabla_v - Q \tag{23}$$

We assume that $\widehat{f^\varepsilon}$ satisfies the following Hilbert expansion, as ε goes to zero,

$$\widehat{f^\varepsilon} = \widehat{f_0} + \varepsilon\,\widehat{f_1} + \varepsilon^2 \widehat{f_2} + \dots \tag{24}$$

where the coefficients $\widehat{f_i} \in L^2((0,T) \times \Omega \times Y)$. Inserting this development into the equation (22) and identifying the coefficients with the same power of ε, we get

$$\varepsilon^{-2}: \quad \mathcal{L}_{\Phi_H} \widehat{f_0} \quad = \quad 0 \tag{25}$$

$$\varepsilon^{-1}: \quad -\mathcal{L}_{\Phi_H} \widehat{f_1} \quad = \quad v.\nabla_x \widehat{f_0} - \nabla_x \Phi_H(x,y).\nabla_v \widehat{f_0} \tag{26}$$

$$\varepsilon^{0}\ : \quad -\mathcal{L}_{\Phi_H} \widehat{f_2} \quad = \quad \partial_t \widehat{f_0} + v.\nabla_x \widehat{f_1} - \nabla_x \Phi_H(x,y).\nabla_v \widehat{f_1} \tag{27}$$

It is obvious that the limit system relies on the properties of the null space of the cell operator $\mathcal{L}_{\Phi_H}$, so we should study the first equation. For this, we define the weighted Hilbert space $L^2_{M_{\Phi_H}}$ as

$$L^2_{M_{\Phi_H}} = \left\{ f(y,v)\ /\ f(y,v) \in L^2_{loc}\left(\frac{dydv}{M_{\Phi_H}}\right) \text{ and } f \text{ is Y-periodic with respect to y} \right\}$$

equipped with the inner product

$$\langle f, g \rangle = \int_{\mathbb{R}^d} \int_Y f\, g \frac{dydv}{M_{\Phi_H}}.$$

Notice that the inner product depends on x. For each x, the operator $\mathcal{L}_{\Phi_H}$ acting on this space is unbounded, with domain

$$D(\mathcal{L}_{\Phi_H}) = \{f \in L^2_{M_\Phi} \;/\; v.\nabla_y f - \nabla_y \Phi_H.\nabla_v f \in L^2_{M_{\Phi_H}}\}$$

and it satisfies the following

Proposition 2.2 *[7] The operator $\mathcal{L}_{\Phi_H}$ is maximal monotone on $L^2_{M_{\Phi_H}}$ and satisfies*

1. *$\mathcal{K}er(\mathcal{L}_{\Phi_H}) = \mathbb{R}\, M_{\Phi_H}$,*
2. *$\mathcal{R}(\mathcal{L}_{\Phi_H}) = \left\{ g \in L^2_{M_{\Phi_H}} \;/\; \iint_{Y\times\mathbb{R}^d} g(y,v)dydv = 0 \right\}$.*
3. *For all $g \in \mathcal{R}(\mathcal{L}_{\Phi_H})$, there exists $f \in D(\mathcal{L}_{\Phi_H})$ such that $\mathcal{L}_{\Phi_H} f = g$. This solution is unique under the solvability condition $\iint_{Y\times\mathbb{R}^d} f(y,v)dydv = 0$. We denote it $f = \mathcal{L}^{-1}_{\Phi_H} g$* □

According to (25) and the above proposition, there exists a density $\rho(t,x)$ such that the weak limit $\widehat{f}_0$ of the sequence $\widehat{f^\varepsilon}$ has the form:

$$\widehat{f}_0(t,x,y,v) = \rho(t,x)\, M_{\Phi_H}(x,y,v). \tag{28}$$

We stress, here that $\widehat{f}_0$ solves both the transport and the collision parts separately, namely $v.\nabla_y \widehat{f}_0 - \nabla_y \Phi_H.\nabla_v \widehat{f}_0 = 0$ and $Q\widehat{f}_0 = 0$. Ignoring the part of $\widehat{f}_1$ which is in the kernel of $\mathcal{L}_{\Phi_H}$, a simple computation of the right hand side of (26) leads to

$$\widehat{f}_1(t,x,y,v) = -(\nabla_x \rho + \rho \nabla_x \Phi_e)(t,x).\mathcal{L}^{-1}_{\Phi_H}(v\, M_{\Phi_H})(x,y,v). \tag{29}$$

Denoting by χ the unique solution in $[\mathcal{R}(\mathcal{L}_{\Phi_H}) \cap D(\mathcal{L}_{\Phi_H})]^d$ of

$$\mathcal{L}_{\Phi_H}\chi = v\, M_{\Phi_H}, \tag{30}$$

the diffusion matrix $\mathbb{D}$ is defined by

$$\mathbb{D}_{ij}(x) = \iint_{Y\times\mathbb{R}^d} v_i \otimes \chi_j(x,y,v)dydv. \tag{31}$$

By applying the solvability condition, stated in the above proposition, to (27) we obtain the homogenized Drift-Diffusion model. We recall here that we use the convention $\mathrm{div}(\mathbb{D}F) = \partial_i(\mathbb{D}_{ij}F_j)$.
We remark that, this limit equation is associated with an effective potential Φ_e, collecting some microscopic information induced by the rapidly oscillating potential Φ_H. The Drift-Diffusion model we get is the following

$$\begin{cases} \partial_t \rho + \nabla_x \,.\, J(\rho,\Phi_e) = 0 \\ J(\rho,\Phi_e)(t,x) = -\mathbb{D}(x)[\nabla_x \rho + \rho \nabla_x \Phi_e](t,x). \end{cases} \tag{32}$$

We refer to [7] for the proof of proposition 2.2 and for the convergence ($\varepsilon \to 0$) in the case of linear Boltzmann equation. In this paper, we are dealing with a more general situation since we have a coupling with Poisson and the major difficulty is to get enough compactness to pass to the limit in the nonlinear terms.

3 Existence of solutions

In all the sequel we will the notation Let us now give the definition of solution we are going to deal with.

Definition 3.1 *We say that that* $(f^\varepsilon, \Phi_P^\varepsilon)$ *is a renormalized solution of the Boltzmann-Poisson system if the function* $\widetilde{f^\varepsilon} := e^{\Phi_H(x,x/\varepsilon)} f^\varepsilon$ *satisfies*

1. $\forall\ \beta \in \mathcal{C}^1(\mathbb{R}^+)$, $|\beta(t)| \le C(\sqrt{t}+1)$, *and* $|\beta'(t)| \le C$, $\beta(\widetilde{f^\varepsilon})$ *is a weak solution of*

$$\begin{cases} \varepsilon\, \partial_t\, \beta(\widetilde{f^\varepsilon}) + v\,.\,\nabla_x \beta(\widetilde{f^\varepsilon}) - \nabla_v\,.\,(\nabla_x \Phi_T^\varepsilon\, \beta(\widetilde{f^\varepsilon})) = & \dfrac{Q(\widetilde{f^\varepsilon})}{\varepsilon}\beta'(\widetilde{f^\varepsilon}) + \\ \qquad +v.(\nabla_x \Phi_H)_\varepsilon\, \widetilde{f^\varepsilon}\beta'(\widetilde{f^\varepsilon}) + & \dfrac{1}{\varepsilon}\, v\,.\,(\nabla_y \Phi_H)_\varepsilon\, \widetilde{f^\varepsilon}\beta'(\widetilde{f^\varepsilon}) \\ \beta(\widetilde{f^\varepsilon})_{|\Gamma^-} = \beta(\widetilde{f_b^\varepsilon}) \\ \beta(\widetilde{f^\varepsilon})(t=0) = \beta(\widetilde{f_0^\varepsilon}) \end{cases} \tag{33}$$

2. $\forall\ \lambda > 0$, $\theta_{\varepsilon,\lambda} = \sqrt{f^\varepsilon + \lambda\, M\, e^{-\Phi_H(x,x/\varepsilon)}}$ *satisfies*

$$\varepsilon\, \partial_t\, \theta_{\varepsilon,\lambda} + v\,.\,\nabla_x \theta_{\varepsilon,\lambda} - \nabla_v.[\nabla_x \Phi_T^\varepsilon\, \theta_{\varepsilon,\lambda}] = \frac{Q(\widetilde{f^\varepsilon})}{2\,\varepsilon\, \theta_{\varepsilon,\lambda}} + \frac{\lambda}{2\theta_{\varepsilon,\lambda}}\, vM.\,\nabla_x \Phi_P^\varepsilon e^{-\Phi_H(x,x/\varepsilon)} \tag{34}$$

Theorem 3.2 *The semiconductor Boltzmann-Poisson system (1–4) has a renormalized solution in the sense of definition 3.1 which satisfies in addition*

1. *the continuity equation*
$$\partial_t \rho^\varepsilon + \nabla_x . j^\varepsilon = 0 \tag{35}$$

2. *the entropy inequality*

$$\left[\int_\Omega f^\varepsilon \left(\log f^\varepsilon + \frac{|v|^2}{2} + \Phi_H^\varepsilon + \bar{\Phi}_b\right) + \frac{1}{2}\|\nabla_x(\Phi_P^\varepsilon - \bar{\Phi}_b)\|_{L^2}^2\right]_0^t - \int_0^t \int_\omega \partial_t \bar{\Phi}_b\, \rho^\varepsilon$$
$$\le \frac{1}{\varepsilon^2}\int_0^t \int_\Omega Q(f^\varepsilon) \log\left(\frac{f^\varepsilon}{M}\right) - \frac{1}{\varepsilon}\int_0^t \int_{\Gamma^+\cup\Gamma^-} f^\varepsilon \left(\log f^\varepsilon + \frac{|v|^2}{2} + \Phi_T^\varepsilon\right)(v.n(x)) \tag{36}$$

Proof. We refer to [21] for the proof of a similar theorem and for further details on the concept of renormalized solution. We also refer to [11, 22] and the references therein.

Remark 3.3 *We point out here that we are renormalizing the equation satisfied by* $\widetilde{f^\varepsilon}$ *instead of renormalizing the equation satisfied by* f^ε. *This is actually, not equivalent. Indeed, to get the compactness of* $\widetilde{\rho^\varepsilon}$, *we need the renormalization of* $\widetilde{f^\varepsilon}$ *and to pass to the limit in* j_ε *we need the equation satisfied by* $\theta_{\varepsilon,\lambda}$. *We point out that* *we can also change the definition 3.1 to include the renormalization of any function of the form $\phi_1 f^\varepsilon + \phi_2$ where ϕ_1 and ϕ_2 are smooth functions of x and v, namely we can require that for all $\forall\ \beta \in \mathcal{C}^1(\mathbb{R}^+)$, $|\beta(t)| \le C(\sqrt{t}+1)$, and $|\beta'(t)| \le C$, $\beta(\phi_1 f^\varepsilon + \phi_2)$ satisfies a similar equation to (33). We do not do this here since it is not necessary in the proof.* □

4 Uniform estimates

Let us now show the kind of estimates one can establish for these renormalized solutions. We point out that here we try to generalize energy estimates obtained in [21] to the case we add an oscillating potential. So, we will not give all the details and refer to [21] for the proofs. We remark that due to the incoming velocities, estimates (35) and (36) are not uniform at this stage and we should approximate the entropy production terms coming from the boundary in order to get some uniform bounds from (36). Our uniform estimate is the following

Lemma 4.1 *Assume that (A1)–(A4) are satisfied. Let $(f^\varepsilon, \Phi_P^\varepsilon)$ be a renormalized solution of (1-4) given by theorem 3.2. Then,*

$$\mathcal{M}^\varepsilon(t) + \mathcal{K}^\varepsilon(t) + \|\nabla_x \Phi_P^\varepsilon(t)\|_{L^2} + \frac{\mathcal{R}_1^\varepsilon(t)}{\varepsilon^2} + \int_0^t \|j^\varepsilon(s)\|^2_{L^1(\omega)}\, ds \le C_T \qquad (37)$$

Proof. Let us come back to the inequality (36) and denote the total relative entropy $\mathcal{E}^\varepsilon(t)$ by

$$\mathcal{E}^\varepsilon(t) = \int_\Omega f^\varepsilon(t) \left(\log f^\varepsilon + \frac{|v|^2}{2} + \Phi_H^\varepsilon + \bar{\Phi}_b \right)(t) + \frac{1}{2}\|\nabla_x(\Phi_P^\varepsilon - \bar{\Phi}_b)\|^2_{L^2(\omega)}(t).$$

One can rewrite the boundary fluxes in the following manner

$$\begin{aligned}\int_{\Gamma^+\cup\Gamma^-} f^\varepsilon \left(\log f^\varepsilon + \frac{|v^2|}{2} + \Phi_T^\varepsilon\right)(v.n(x)) &= \int_{\Gamma^+} |v.n(x)| f^\varepsilon(v) \log\left(\frac{f^\varepsilon(v)}{f^\varepsilon(-v)}\right) \\ &+ \int_{\Gamma^+} |v.n(x)|(f^\varepsilon(v) - f^\varepsilon(-v)) \left(\log\left(\frac{\rho_b(t,x)}{(\sqrt{2\pi})^d}\right) + \Phi_b\right) \\ &\ge \int_{\Gamma^+} |v.n(x)|(f^\varepsilon(v) - f^\varepsilon(-v))\,(1 + \mathcal{E}_F(t,x))\end{aligned} \qquad (38)$$

where $\mathcal{E}_F$ is a macroscopic quasi-Fermi level given by

$$\mathcal{E}_F(t,x) = \log\left(\frac{\rho_b(t,x)}{(\sqrt{2\pi})^d}\right) + \Phi_b(t,x). \qquad (39)$$

We point out that $\mathcal{E}^\varepsilon$ belongs to $\mathcal{C}^1([0,T])$ and (36) can be replaced by

$$[\mathcal{E}^\varepsilon]_0^t + \frac{\sigma_1}{2}\mathcal{R}_1^\varepsilon(t) \le \int_\omega \partial_t \bar{\Phi}_b\, \rho^\varepsilon - \frac{1}{\varepsilon}\int_0^t \int_{\Gamma^+} |v.n(x)|(f^\varepsilon(v) - f^\varepsilon(-v))(1 + \mathcal{E}_F(t,x)).$$

The remainder of the proof follows exactly the one in [21]. We replace ρ_b by its harmonic extension, $\bar{\rho}_b$, The extended quasi-Fermi level is denoted by

$$\bar{\mathcal{E}}_F(t,x) := \log\left(\frac{\bar{\rho}_b(t,x)}{(\sqrt{2\pi})^d}\right) + \bar{\Phi}_b(t,x)$$

Recall that for all fixed $\varepsilon > 0$, the continuity equation is defined in the weak sense. We multiply (35) by $(1+\bar{\mathcal{E}}_F)$ and integrate with respect to t and x and then bound $\nabla_x \bar{\mathcal{E}}_F$ and $\partial_t \bar{\mathcal{E}}_F$ and deduce

$$\begin{aligned}\int_0^t \int_\omega \partial_t \bar{\Phi}_b\, \rho^\varepsilon \quad &- \quad \frac{1}{\varepsilon}\int_0^t \int_{\Gamma^+} |v.n(x)|[f^\varepsilon(v) - f^\varepsilon(-v)](1+\mathcal{E}_F) \\ &\leq \quad C_T\left(1 + \int_0^t [\mathcal{M}^\varepsilon(s) + \|j^\varepsilon(s)\|_{L^1}]\, ds\right) + \int_\omega \rho^\varepsilon(t)(1+\bar{\mathcal{E}}_F)(t)\end{aligned}$$

and also,

$$\mathcal{E}^\varepsilon(t) + \frac{\mathcal{R}_1^\varepsilon(t)}{\varepsilon^2} \leq C_T\left(1 + \int_0^t \mathcal{M}^\varepsilon(s)ds + \int_0^t \|j^\varepsilon(s)\|_{L^1} ds\right) + \int_\omega \rho^\varepsilon(t)(1+\bar{\mathcal{E}}_F)(t)$$

We deduce easily from the entropy dissipation (9)

$$\int_0^t \|j^\varepsilon\|_{L^1}(s)ds \leq \frac{1}{2\varepsilon^2}\mathcal{R}_1^\varepsilon(t) + \frac{C_T}{2}\int_0^t (\mathcal{M}^\varepsilon(s) + \mathcal{K}^\varepsilon(s))ds \tag{40}$$

Notice that $\mathcal{E}^\varepsilon$ can be negative, however, we can bound it from below in function of $\mathcal{M}^\varepsilon$ and $\mathcal{K}^\varepsilon$ (see [21]). Hence, we get

$$\mathcal{M}^\varepsilon(t) + \mathcal{K}^\varepsilon(t) + \frac{\mathcal{R}_1^\varepsilon(t)}{2\varepsilon^2} \leq C_T\left(1 + \int_0^t \mathcal{M}^\varepsilon(s)ds + \int_0^t \mathcal{K}^\varepsilon(s)ds\right)$$

Finally, a Gronwall argument implies Estimate (37). □

The proof of the next corollary and propositions can be found in [21]

Corollary 4.2 *The renormalized solution satisfies*

$$\int_\Omega f^\varepsilon(1+|v|^2+|\log f^\varepsilon|) + \int_0^t \int_{\Gamma^+} f^\varepsilon(1+|v|^2+|\log f^\varepsilon|)|v.n(x)| \leq C_T.$$

Moreover, f^ε and its trace $f^\varepsilon_{|\Gamma^+}$ are weakly relatively compact in $L^1((0,T)\times\Omega)$ and $L^1((0,T)\times\Gamma^+, |v.n(x)|dtd\sigma_x dv))$ respectively. □

Proposition 4.3 *The renormalized solution $(f^\varepsilon, \Phi_P^\varepsilon)$ satisfies*

1. *ρ^ε is weakly relatively compact in $L^1((0,T)\times\omega)$.*
2. *$\frac{Q(f^\varepsilon)}{\varepsilon}$ is weakly relatively compact in $L^1((0,T)\times\Omega)$.*
3. *$\nabla\Phi_P^\varepsilon$ is relatively compact in $L^2(0,T;\ L^p(\omega))$ for all $1 \leq p < 2$.* □

We also define

$$r_\varepsilon = \frac{1}{\varepsilon\sqrt{M}}\left(\sqrt{f^\varepsilon} - \sqrt{\rho^\varepsilon M}\right).$$

Using the entropy dissipation bound $\mathcal{R}_1^\varepsilon(t) \leq C\varepsilon^2$ and Young inequality, we deduce as in [21] that (see also [2] and [19])

Proposition 4.4 *r_ε is such that $\varepsilon|r_\varepsilon|^2|v|^2M$ is bounded in $L^1((0,T)\times\Omega)$, $\sqrt{\varepsilon}|r_\varepsilon|^2|v|M$ is bounded in $L^1((0,T)\times\Omega)$ and $|r_\varepsilon|^2M$ is bounded in $L^1((0,T)\times\Omega)$.*

5 Compactness of modified density

Proposition 5.1 *The modified density $\widetilde{\rho^\varepsilon}(t,x) := \rho^\varepsilon(t,x)e^{\Phi_H(x,x/\varepsilon)}$ is relatively compact in $L^1((0,T)\times\omega)$ and there exists $\tilde\rho \in L^1((0,T)\times\omega)$ such that, up to extraction of a subsequence if necessary,*

$$\begin{array}{ll} \widetilde{\rho^\varepsilon} \to \tilde\rho \quad \textit{in } L^1((0,T)\times\omega) & \textit{and a. e,} \\ \widetilde{f^\varepsilon} \to \tilde\rho M \quad \textit{in } L^1((0,T)\times\Omega) & \textit{and a. e.} \end{array} \qquad \square$$

The proof of this proposition is done in two steps. We first prove the compactness of $\widetilde{\rho^\varepsilon}$ with respect to the x variable and then show the compactness in time.

In the sequel, we will use the following

$$\beta(s) = \frac{s}{1+s}, \qquad \beta_\delta(s) = \frac{1}{\delta}\beta(\delta\, s), \qquad \forall s>0.$$

We recall that for all fixed parameter $\delta>0$, we have

1. $\quad 0 \le \beta_\delta(s) \le \min\left(s, 1/\delta\right),$
2. $\quad |\beta_\delta(s)-\beta_\delta(t)| \le \min\left(|s-t|, |\sqrt{s}-\sqrt{t}|/\sqrt{\delta}\right),$
3. $\quad |\beta'_\delta(s)-1| \le 2\delta\, s,$
4. $\quad |s\,\beta'_\delta(s) - t\,\beta'_\delta(t)| \le \min((1+\delta)|t-s|,\ C_\delta\,|\sqrt{s}-\sqrt{t}|).$

First, we remark that we only need to show for all $\delta>0$, the compactness of the charge density associated to $(\beta_\delta(\widetilde{f^\varepsilon}))_\varepsilon$. This is a consequence of the following averaging lemma (see [21] for the proof). We also refer to [14].

Lemma 5.2 *[21] Let h^ε be a bounded sequences of $L^2(0,T;\ L^2(\Omega))$, h_0^ε and h_1^ε are bounded in $L^1(0,T;\ L^1(\Omega))$. Assume that h^ε satisfies*

$$\varepsilon\,\partial_t\, h^\varepsilon + v\,.\,\nabla_x\, h^\varepsilon = h_0^\varepsilon + \nabla_v\,.\,h_1^\varepsilon \tag{41}$$

Then, for all $\psi \in \mathcal{D}(\mathbb{R}^d)$

$$\lim_{z\to 0}\left(\sup_{\varepsilon<1}\left\|\int_{\mathbb{R}^d} h^\varepsilon(t,x+z,v)\psi(v)dv - \int_{\mathbb{R}^d} h^\varepsilon(t,x,v)\psi(v)dv\right\|_{L^1(0,T;\ L^1(\Omega))}\right) = 0 \tag{42}$$

where h^ε is extended by zero for $x\notin\bar\omega$. $\square$

Let δ be a (fixed) nonnegative parameter and let us check that $h^\varepsilon := \beta_\delta(\widetilde{f^\varepsilon})$ satisfies the assumptions of the previous lemma. Indeed, we have according to definition (3.1)

$$\varepsilon\,\partial_t\,\beta_\delta(\widetilde{f^\varepsilon}) + v\,.\,\nabla_x\,\beta_\delta(\widetilde{f^\varepsilon}) = h_0^\varepsilon + \nabla_v\,.\,h_1^\varepsilon \tag{43}$$

where

$$h_0^\varepsilon \;=\; \left[\frac{Q(\widetilde{f^\varepsilon})}{\varepsilon} + v\,.\,(\nabla_x\Phi_H)_\varepsilon\,\widetilde{f^\varepsilon}\right]\beta'_\delta(\widetilde{f^\varepsilon}) + \frac{v\,.\,(\nabla_y\Phi_H)_\varepsilon}{\varepsilon}\left[\widetilde{f^\varepsilon}\,\beta'_\delta(\widetilde{f^\varepsilon}) - \widetilde{\rho^\varepsilon}M\,\beta'_\delta(\widetilde{\rho^\varepsilon}\,M)\right]$$

and

$$h_1^\varepsilon = \Big[(\nabla_x \Phi_P^\varepsilon + (\nabla_x \Phi_H)_\varepsilon)\beta_\delta(\widetilde{f^\varepsilon})\Big] + (\nabla_y \Phi_H)_\varepsilon \left(\frac{\beta_\delta(\widetilde{f^\varepsilon}) - \beta_\delta(\widetilde{\rho^\varepsilon}\, M)}{\varepsilon}\right).$$

Using the energy estimate (37), we see that h^ε, h_0^ε and h_1^ε satisfy the assumptions of the above lemma.

Applying this lemma we deduce the compactness in x of $\int_{\mathbb{R}^d} \beta_\delta(\widetilde{f^\varepsilon})\psi(v)dv$ for all $\psi \in \mathcal{D}(\mathbb{R}^d)$, namely (42) holds with h^ε replaced by $\beta_\delta(f^\varepsilon)$.

Next, using that $(\beta_\delta(f^\varepsilon))_\varepsilon$ is bounded in $L^\infty(0,T;\, L^1((1+|v|^2)dxdv))$, we see that we can take $\psi(v)$ to be constant equal to 1 in (42) and hence we deduce, after also sending δ to 0 and using the equi-integrability of f^ε, that

$$\|\widetilde{\rho^\varepsilon}(t,x+z) - \widetilde{\rho^\varepsilon}(t,x)\|_{L^1((0,T)\times\omega)} \;\to\; 0 \,\text{when}\, z \;\to\; 0 \,\text{uniformly in}\, \varepsilon. \tag{44}$$

To complete the proof of the compactness for the modified density we need some regularity in the time variable. Indeed, the mass conservation property

$$\partial_t \rho^\varepsilon + \nabla_x \,.\, j^\varepsilon = 0$$

and the uniform estimate (37) imply that

$$\rho^\varepsilon \text{ is bounded in } L^2(0,T;\; W^{-1,1}(\omega)). \tag{45}$$

Notice that this does not yield a similar bound for $\widetilde{\rho^\varepsilon}$ (see Remark 5.3) and we need some argument to combine (44) and (45) to get the compactness of $\widetilde{\rho^\varepsilon}$. We denote

$$a^\varepsilon(t,x) = \widetilde{\rho^\varepsilon}(t,x) = \rho_\varepsilon(t,x)e^{\Phi_H(x,\frac{x}{\varepsilon})} \quad \text{and} \quad a_R^\varepsilon(t,x) = \inf(a^\varepsilon(t,x), R).$$

Using (44), we deduce that for all $R > 0$ and for all $1 \le p < \infty$, we have

$$\|a_R^\varepsilon(t,x+z) - a_R^\varepsilon(t,x)\|_{L^p((0,T)\times\Omega)} \;\to\; 0 \,\text{when}\, z \;\to\; 0 \,\text{uniformly in}\, \varepsilon. \tag{46}$$

The sequence a_R^ε is bounded in L^p for $p > 1$, let us denote $a_R(t,x,y,v)$, the two scale limit of a_R^ε when ε goes to zero. We also denote $a(t,x,y)$ the two scale limit of a^ε. From (44) and (46), we deduce easily that a_R and a do not depend on y.

Moreover, since $\rho^\varepsilon = a^\varepsilon e^{-\Phi_H(x,\frac{x}{\varepsilon})}$ and $e^{-\Phi_H(x,\frac{x}{\varepsilon})}$ two-scale converges strongly to $e^{-\Phi_H(x,y)}$, we deduce that ρ^ε two scale converges to $ae^{-\Phi_H(x,y)}$ and converges weakly to $ae^{-\Phi_e(x)}$.

Then, on one hand, using that a_R^ε satisfies (46) and that $\partial_t\rho^\varepsilon$ is bounded in $L^1(0,T;\, W^{-1,1}(\omega))$ and that ρ^ε is equi-integrable, namely ρ_ε converges weakly in L^1 we deduce from Lions-Aubin type lemma (see Lemma 5.1 of Lions [18]), that

$$a_R^\varepsilon\, \rho^\varepsilon \rightharpoonup a_R\, a\, e^{-\Phi_e} \qquad \text{in } \mathcal{D}'((0,T)\times\omega). \tag{47}$$

On the other hand, we have

$$\begin{aligned} a_R^\varepsilon\, \rho^\varepsilon \;&=\; (a_R^\varepsilon)^2\, e^{-\Phi_H(x,x/\varepsilon)} + (a^\varepsilon - a_R^\varepsilon)\, a_R^\varepsilon\, e^{-\Phi_H(x,x/\varepsilon)} \\ &=\; (a_R^\varepsilon)^2\, e^{-\Phi_H(x,x/\varepsilon)} + \delta_R^\varepsilon \end{aligned}$$

where $\delta_R^\varepsilon = (a^\varepsilon - a_R^\varepsilon)\, a_R^\varepsilon\, e^{-\Phi_H(x,x/\varepsilon)} = (a^\varepsilon - a_R^\varepsilon)\, R\, e^{-\Phi_H(x,x/\varepsilon)}$ weakly converges to $\delta_R = (a - a_R)\, R\, e^{-\Phi_e}$ when ε goes to 0.

We denote by c_R the two scale limit of $(a_R^\varepsilon)^2$. It is easy to see from (46) that c_R does not depend on y and that $c_R \geq a_R^2$.

To prove that a_R^ε converges strongly to a_R in L^2, it is enough to prove that $c_R = a_R^2$. To prove this, we use the two scale limit of $a_R^\varepsilon\, \rho^\varepsilon$ which was computed by two different methods :

$$c_R\, e^{-\Phi_e} + (a - a_R)\, R\, e^{-\Phi_e} = a_R\, a\, e^{-\Phi_e} = (a_R)^2\, e^{-\Phi_e} + a_R\, (a - a_R)\, e^{-\Phi_e} \tag{48}$$

Since $a_R \leq R$ and $a - a_R \geq 0$ we deduce that $c_R\, e^{-\Phi_e} \leq (a_R)^2\, e^{-\Phi_e}$. Using that $e^{-\Phi_e} > 0$, we infer that $c_R \leq (a_R)^2$. Hence $c_R = (a_R)^2$ and a_R^ε converges strongly to a_R for all $R > 0$. This yields the strong convergence of $a^\varepsilon = \widetilde{\rho^\varepsilon}$ since it is equi-integrable. This yields the first convergence.

The strong convergence of $\widetilde{f^\varepsilon}$ follows then from the fact that $\mathcal{R}_\varepsilon$ goes to zero when ε goes to zero. This ends the proof of the proposition (5.1). □

Remark 5.3 *Let us explain the reason behind the previous argument. On one hand, due to the fact that the two scale limit of ρ^ε depends on the fast variable y, we have no hope to obtain compactness for ρ^ε in L^1 and then one can not proceed like in [21]. On the other hand, using the continuity equation and multiplying by $e^{\Phi_H(x,x/\varepsilon)}$, we see that the modified densities $(\widetilde{\rho^\varepsilon}, \widetilde{j^\varepsilon})$ is a weak solution of*

$$\partial_t \widetilde{\rho^\varepsilon} + \nabla_x . \widetilde{j^\varepsilon} = \left(\nabla_x \Phi_H + \frac{1}{\varepsilon} \nabla_y \Phi_H \right)_\varepsilon . \widetilde{j^\varepsilon} \qquad \text{in } \mathcal{D}'((0,T) \times \omega)$$

and due to the presence of the singular term $\frac{1}{\varepsilon} \nabla_y \Phi_H(x, x/\varepsilon)$ it is not clear how to obtain directly compactness in time for $\widetilde{\rho^\varepsilon}$. Also, we note that we can not use the $div - curl$ lemma as in [17].

6 Passage to the limit

We would like to pass to the limit in the continuity equation

$$\partial_t \rho^\varepsilon + \nabla_x j^\varepsilon = 0.$$

The question is to identify the limit of the current density. To do so, we begin by summarizing some consequences of Proposition 5.1. First for $\lambda > 0$, we define $\xi_{\varepsilon,\lambda}$

$$\xi_{\varepsilon,\lambda}(t,x,v) = \frac{1}{\varepsilon \sqrt{M}} \left(\sqrt{f^\varepsilon + \lambda\, M\, e^{-\Phi_H(x,x/\varepsilon)}} - \sqrt{(\rho^\varepsilon + \lambda\, e^{-\Phi_H(x,x/\varepsilon)})\, M} \right). \tag{49}$$

Using that $|\xi_{\varepsilon,\lambda}| \leq |r_\varepsilon|$, we see that $\xi_{\varepsilon,\lambda}$ is bounded in $L^2(M dv dx dt)$. We denote by $\hat{\xi}_\lambda$ the two scale limit of $\xi_{\varepsilon,\lambda}$ when ε goes to zero. We notice that ξ_λ is bounded in $L^2((0,T) \times \Omega \times Y, dt dx dy M dv)$. Extracting a subsequence, we denote $\hat{\xi}$ the weak limit of $\hat{\xi}_\lambda$ when λ goes to zero. One can easily prove that the whole sequence $\hat{\xi}_\lambda$ converges strongly to $\hat{\xi}$ in $L^2((0,T) \times \Omega \times Y, dt dx dy M dv)$; But this is not necessary in the sequel.

According to the previous section, the sequences $\widetilde{f^\varepsilon}$, $\widetilde{\rho^\varepsilon}$ converge strongly in L^1 and a. e. In particular,

Proposition 6.1 *There exist a density $\rho \in L^1((0,T)\times\omega)$ and a potential $\Phi_P \in L^\infty(0,T;\ W^{1,2})$ such that, up to extraction of a subsequence, we have*

1. $\|\rho^\varepsilon - \rho e^{\Phi_e(x)-\Phi_H(x,x/\varepsilon)}\|_{L^1((0,T)\times\omega)} \to 0$ *and* ρ^ε *converges weakly to* ρ *in* $L^1_{t,x}$.
2. $\sqrt{f^\varepsilon}$ *and* $\sqrt{\rho^\varepsilon\, M}$ *two scale converge strongly in* $L^2((0,T)\times\Omega)$ *towards* $\sqrt{\rho\, M_{\Phi_H}}$.
3. $-\Delta\Phi_P = \rho\,,\ \Phi_P = \Phi_b$ *on* $\partial\omega$ *and*
$$\nabla_x \Phi_P^\varepsilon \to \nabla_x \Phi_P \quad \text{in } L^2(0,T;\ L^p(\omega)) \quad \text{for } 1 \le p < 2 \,\text{and}\, a.\ e.$$
4. *For all* $\psi \in \mathcal{D}_\#$, *we have*
$$\int \frac{Q(f^\varepsilon)}{2\,\varepsilon\,\theta_{\varepsilon,\lambda}}\psi\, dvdxdt \to \int\int_Y \frac{Q(\hat{\xi}\, M)}{\sqrt{M}}\psi\, dydvdxdt$$
as ε *goes to zero and then* λ *goes to zero.* □

Proof of Proposition 6.1 The first point is a consequence of the L^1-strong convergence of $\widetilde{\rho^\varepsilon}$. Let $\tilde\rho(t,x)$ be the limit of $\widetilde{\rho^\varepsilon}$ and ρ the weak limit of ρ^ε. we have
$$e^{-\Phi_H(x,x/\varepsilon)} \rightharpoonup e^{-\Phi_e(x)} := \int_Y e^{-\Phi(x,y)}dy, \quad \text{in } L^\infty\ w-*.$$
This implies that $\rho^\varepsilon := \widetilde{\rho^\varepsilon}\, e^{-\Phi_H(x,x/\varepsilon)} \rightharpoonup \rho := \tilde\rho\, e^{-\Phi_e}$. Moreover, we have
$$\rho^\varepsilon - \rho\, e^{\Phi_e - \Phi_H(x,x/\varepsilon)} = (\widetilde{\rho^\varepsilon} - \rho\, e^{-\Phi_e})e^{-\Phi_H(x,x/\varepsilon)}$$
and hence
$$\rho^\varepsilon - \rho\, e^{\Phi_e - \Phi_H(x,x/\varepsilon)} \to 0 \qquad \text{in } L^1_{t,x} \text{ and a. e.}$$
The second point is a simple consequence of the strong convergence (in L^2) of $\sqrt{\widetilde{f^\varepsilon}}$ and $\sqrt{\widetilde{\rho^\varepsilon}\, M}$ towards $\sqrt{\rho\, e^{\Phi_e}}$ and the 2-scale convergence of the sequence $e^{-\Phi_H(x,x/\varepsilon)}$ towards $e^{-\Phi_H}$.
The third property is a consequence of Proposition 4.3. To deduce the last point, we remark that (37) implies that $\xi_{\varepsilon,\lambda}$ is bounded in $L^2(0,T;\ L^2(M\, dxdv))$ and
$$\frac{Q(f^\varepsilon)}{2\,\varepsilon\,\theta_{\varepsilon,\lambda}} = \frac{\sqrt{(\widetilde{\rho^\varepsilon}+\lambda)}}{\sqrt{\widetilde{f^\varepsilon}+\lambda\, M}}\, Q\,(\xi_{\varepsilon,\lambda}\, M) + \mathbf{O}(\varepsilon)_{L^1_{loc}(dvdxdt)}$$
We also have, for $\lambda > 0$,
$$\sqrt{\widetilde{f^\varepsilon}+\lambda\, M} \quad \text{and} \quad \sqrt{(\widetilde{\rho^\varepsilon}+\lambda)\, M} \quad \text{converge to} \quad \sqrt{(\tilde\rho+\lambda)\, M} \qquad \text{in } L^2_{loc}(dvdxdt).$$
So, if we take an oscillating test function $\psi \in \mathcal{D}((0,T)\times\Omega;\ \mathcal{C}^\infty_\#(Y))$, we infer, as ε goes to zero and λ goes to zero that
$$\lim_{\lambda\to 0}\lim_{\varepsilon\to 0}\int \frac{Q(f^\varepsilon)\,\psi}{2\,\varepsilon\,\theta_{\varepsilon,\lambda}}dvdxdt = \int\int_y \frac{Q(\hat{\xi}\, M\,)\psi}{\sqrt{M}}dydvdxdt.$$
This ends the proof of the above proposition. □

Let us denote j the weak limit of j_ε when ε goes to zero, then we have the following proposition

Proposition 6.2

$$j^{\varepsilon} \overset{\mathcal{D}'}{\rightharpoonup} j := 2\sqrt{\tilde{\rho}} \int_Y \int_{\mathbb{R}^d} \widehat{\xi}\, v\, M\, e^{-\Phi_H(x,y)/2} dy\, dv \tag{50}$$

Proof of Proposition 6.2 Using that

$$f_{\varepsilon} = \rho_{\varepsilon}\, M + 2\, \varepsilon\, \xi_{\varepsilon,\lambda} \sqrt{\rho_{\varepsilon} + \lambda\, e^{-\Phi_H(x,x/\varepsilon)}}\, M + \varepsilon^2\, \xi_{\varepsilon,\lambda}^2\, M$$

we deduce that

$$j_{\varepsilon} = 2\sqrt{\rho_{\varepsilon} + \lambda e^{-\Phi_H(x,x/\varepsilon)}} \int_{\mathbb{R}^d} \xi_{\varepsilon,\lambda}\, v\, M\, dv + \varepsilon \int_{\mathbb{R}^d} \xi_{\varepsilon,\lambda}^2\, v\, M\, dv \tag{51}$$

Using that $|\xi_{\varepsilon,\lambda}| \leq |r_{\varepsilon}|$, Proposition 4.4 and that $\xi_{\varepsilon,\lambda}$ two scale converges to $\hat{\xi}_{\lambda}$, we can pass to the limit in (51) and get

$$j = 2 \int_Y \sqrt{(\tilde{\rho} + \lambda) e^{-\Phi_H}} \left[\int_{\mathbb{R}^d} \hat{\xi}_{\lambda}\, v\, M\, dv \right] dy \tag{52}$$

Sending λ to zero, we deduce that (50) holds.

Now, we have to compute $\hat{\xi}$ to get the expression of j.

Proposition 6.3

$$j^{\varepsilon} \overset{\mathcal{D}'}{\rightharpoonup} j(t,x) := -2\sqrt{\rho}\, \mathbb{D}(x). \left(\nabla_x \sqrt{\rho} + \frac{1}{2} \nabla_x (\Phi_e + \Phi_P) \sqrt{\rho} \right) \tag{53}$$

where $\mathbb{D}$ is the diffusion matrix defined by (31).

Proof of Proposition 6.3 Using that $\theta_{\varepsilon,\lambda}$ is a weak solution of (34), taking a test function $\phi \in \mathcal{D}((0,T) \times \Omega)$ and using the relation

$$\theta_{\varepsilon,\lambda} = \varepsilon \sqrt{M}\, \xi_{\varepsilon,\lambda} + \sqrt{(\widetilde{\rho^{\varepsilon}} + \lambda)\, M\, e^{-\Phi_H^{\varepsilon}}}$$

we get

$$\begin{aligned} &\int_0^T \int_{\Omega} \left(\varepsilon\, \xi_{\varepsilon,\lambda} \sqrt{M} + \sqrt{(\widetilde{\rho^{\varepsilon}} + \lambda)\, M\, e^{-\Phi_H(x,x/\varepsilon)}} \right) [\varepsilon\, \partial_t \phi + v\, .\, \nabla_x \phi - \nabla_x (\Phi_T^{\varepsilon})\, .\nabla_v \phi] dx\, dv\, dt \\ &+ \int_0^T \int_{\Omega} \left[\phi\, \frac{Q(f^{\varepsilon})}{2\, \varepsilon\, \theta_{\varepsilon,\lambda}} + \frac{\lambda\, v.\, \nabla_x \Phi_P^{\varepsilon}\, \phi\, M\, e^{-\Phi_H(x,x/\varepsilon)}}{2\, \theta_{\varepsilon,\lambda}} \right] dx\, dv\, dt = 0 \end{aligned} \tag{54}$$

Taking ϕ of the form $\phi = \psi(t, x, \frac{x}{\varepsilon}, v) = \psi$ where $\psi \in \mathcal{D}_{\#}$, we see that in (54), we have two singular terms. We can rewrite them in the following way

$$\begin{aligned} &\frac{1}{\varepsilon} \int_0^T \int_{\Omega} \sqrt{(\widetilde{\rho^{\varepsilon}} + \lambda)} \sqrt{M\, e^{-\Phi_H}}\, [\, v.\nabla_y \psi - \nabla_y \Phi_H.\, \nabla_v \psi]\, dv\, dx\, dt \\ &= \frac{1}{\varepsilon} \int_0^T \int_{\Omega} \sqrt{(\widetilde{\rho^{\varepsilon}} + \lambda)} \sqrt{M\, e^{-\Phi_H}} \left[v.\nabla_y \psi + \nabla_y \Phi_H \cdot \frac{v}{2} \psi \right] \\ &= \frac{1}{\varepsilon} \int_0^T \int_{\omega} \sqrt{(\widetilde{\rho^{\varepsilon}} + \lambda)} \nabla_y \cdot \left[\int_{\mathbb{R}_v^d} \sqrt{M\, e^{-\Phi_H}}\, v\, \psi\, dv \right] dx\, dt \end{aligned}$$

Hence, we get

$$\begin{aligned}
&\int_0^T \int_\Omega \left(\varepsilon\, \xi_{\varepsilon,\lambda}\sqrt{M} + \sqrt{(\widetilde{\rho^\varepsilon}+\lambda)\, M\, e^{-\Phi_H}}\right)[\varepsilon\, \partial_t\psi + v\,.\,\nabla_x\psi - \nabla_x(\Phi_H+\Phi_P^\varepsilon)\,.\nabla_v\psi] dv\, dx\, dt \\
&+\int_0^T \int_\Omega \xi_{\varepsilon,\lambda}\sqrt{M}\,[\, v.\nabla_y\psi - \nabla_y\Phi_H\, \nabla_v\psi]\, dv\, dx\, dt+ \\
&+\frac{1}{\varepsilon}\int_0^T \int_\omega \sqrt{(\widetilde{\rho^\varepsilon}+\lambda)}\nabla_y \cdot \left[\int_{\mathbb{R}^d} \sqrt{M\, e^{-\Phi_H}}\; v\,\psi dv\right] dx\, dt \\
&+\int_0^T \int_\Omega \left[\psi\, \frac{Q(f^\varepsilon)}{2\,\varepsilon\,\theta_{\varepsilon,\lambda}} + \frac{\lambda\, v.\,\nabla_x\Phi_P^\varepsilon\,\psi\, M\, e^{-\Phi_H}}{2\theta_{\varepsilon,\lambda}}\right] dx\, dv\, dt = 0
\end{aligned} \tag{55}$$

To pass to the limit, we have to cancel the singular term. We define the following subset of $\mathcal{D}_\#$

$$\mathcal{G}_\# := \left\{\psi \in \mathcal{D}_\# \;/\; \nabla_y\,.\left(\int_{\mathbb{R}^d} v\,\psi\, dv\right) = 0\right\}$$

Its orthogonal is given by the following lemma

Lemma 6.4 *[17]. Let $T\in\mathcal{D}'_\#$. Then, T belongs to $\mathcal{G}_\#^\perp$ if and only if there exists $S\in\mathcal{D}'_\#$ such that $\nabla_v S = 0$ and $T = v.\nabla_y S$* □

Now, we take ψ such that $\sqrt{M\, e^{-\Phi_H}}\;\psi \in \mathcal{G}_\#$. Sending ε to zero and then λ to zero in (55), we get

$$\begin{aligned}
&\int \left[\widehat{\xi}\, M\, e^{-\Phi_H/2}\,[\, v.\nabla_y - \nabla_y\Phi_H.\,\nabla_v]\,(\frac{\psi}{\sqrt{M\, e^{-\Phi_H}}}) + \frac{\psi}{\sqrt{M\, e^{-\Phi_H}}}\; Q(\widehat{\xi}\, M\, e^{-\Phi_H/2})\right] dvdxdydt \\
&+\int \sqrt{\rho\, M_{\Phi_H}}\,[v\,.\,\nabla_x\psi - \nabla_x(\Phi_H+\Phi_P)\,.\nabla_v\psi]\, dv\, dx\, dy\, dt = 0
\end{aligned}$$

where we have used that $[v.\nabla_y - \nabla_y\Phi_H.\,\nabla_v]\sqrt{M\, e^{-\Phi_H}} = 0$. Integrating by parts, we get

$$\int \psi\sqrt{M\, e^{-\Phi_H}}\left[\frac{\mathcal{L}_{\Phi_H}(\widehat{\xi}\, M\, e^{-\Phi_H/2})}{M\, e^{-\Phi_H}} + \frac{[v\,.\,\nabla_x - \nabla_x(\Phi_H+\Phi_P)\,.\nabla_v]\,(\sqrt{\rho\, M_{\Phi_H}})}{\sqrt{M\, e^{-\Phi_H}}}\right] dvdxdydt = 0$$

This is equivalent to the existence of $S\in\mathcal{D}'_\#$ such that $\nabla_v S = 0$ and

$$\frac{\mathcal{L}_{\Phi_H}(\widehat{\xi}\, M\, e^{-\Phi_H/2})}{M\, e^{-\Phi_H}} + \frac{[v\,.\,\nabla_x - \nabla_x(\Phi_H+\Phi_P)\,.\nabla_v]\,(\sqrt{\rho\, M_{\Phi_H}})}{\sqrt{M\, e^{-\Phi_H}}} \;=\; v\,.\,\nabla_y S$$

where $\mathcal{L}_{\Phi_H}$ is the homogenized cell operator given in (23). After simple computations, we get

$$\mathcal{L}_{\Phi_H}(\widehat{\xi}\, M\, e^{-\Phi_H/2}) \;=\; (M\, e^{-\Phi_H} S) - v\left[\nabla_x\sqrt{\rho} + \tfrac{1}{2}\,\nabla_x(\Phi_e+\Phi_P)\,\sqrt{\rho}\right] M e^{-\Phi_H} e^{\Phi_e/2} \tag{56}$$

Let us go back to the expression of the current density computed in Proposition 6.2. We denote by $\tilde\chi$ the unique solution in $[\mathcal{R}(\mathcal{L}_{\Phi_H}) \cap D(\mathcal{L}_{\Phi_H})]^d$ of

$$\mathcal{L}^*{}_{\Phi_H}\tilde\chi = v\, M_{\Phi_H}, \tag{57}$$

Using (56), we get

$$\begin{aligned} j &= 2\sqrt{\rho\, e^{\Phi_e}} \int_Y \int_{\mathbb{R}^d} (\widehat{\xi}\, M\, e^{-\Phi_H/2})\, \frac{\mathcal{L}^*_{\Phi_H}(\tilde\chi)}{M_{\Phi_H}} \\ &= 2\sqrt{\rho\, e^{\Phi_e}} \int_Y \int_{\mathbb{R}^d} M e^{-\Phi_H} \left[S - v\left(\nabla_x \sqrt{\rho} + \frac{1}{2} \nabla_x(\Phi_e + \Phi_P)\sqrt{\rho}\right) e^{\Phi_e/2}\right] \frac{\tilde\chi}{M_{\Phi_H}} \\ &= -2\sqrt{\rho}\left[\int_Y \int_{\mathbb{R}^d} \tilde\chi \otimes v\right] \cdot \left(\nabla_x\sqrt{\rho} + \frac{1}{2}\nabla_x(\Phi_e + \Phi_P)\sqrt{\rho}\right) \end{aligned}$$

and we conclude by using that $\int_Y \int_{\mathbb{R}^d} \tilde\chi_i v_j dydv = \int_Y \int_{\mathbb{R}^d} \chi_j v_i dydv$.

6.1 Recovering the limit system

In this last subsection, we would like to explain how we can pass to the limit in the boundary condition and how we can rewrite the current j. The arguments are exactly the same as in [21] and we will not detail them.

Using (43) and arguing as in section 6 of [21], we can deduce that $\tilde\rho = \rho_b e^{\Phi_b}$ on $\partial\omega$ and hence $\rho = \rho_b$ on $\partial\omega$.

Finally, we would like to deduce that $\rho \in L^2(0,T;\ L^2(\omega))$, $\sqrt{\rho} \in L^2(0,T;\ H^1(\omega))$ and $\nabla\phi\sqrt{\rho} \in L^2(0,T;\ L^2(\omega))$. This can be done by applying lemma 7.1 of [21]. We have only to check the hypothesis, namely the fact that $\nabla_x\sqrt{\rho} + \frac{1}{2}\nabla_x\phi\sqrt{\rho} = G \in L^2(0,T;\ L^2(\omega))$. This is a consequence of (56) and the fact that $\hat\xi$ is in $L^2((0,T)\times\Omega\times Y)$.

This ends the proof of theorem 1.2. □

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