WELL POSEDNESS FOR THE FENE DUMBBELL MODEL OF POLYMERIC FLOWS

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Abstract
We prove local and global well-posedness for the FENE dumbbell model for a very general class of potentials. Indeed, in prior local or global well-posedness results conditions on the parameter b were made. Here we give a proof in the case $b = 2k > 0$. We also prove global existence results if the data is small or if we restrict to the co-rotational model in dimension 2.

1. Introduction

Systems coupling fluids and polymers are of great interest in many branches of applied physics, chemistry and biology. Although a polymer molecule may be a very complicated object, there are simple theories to model it. One of these model is the FENE (Finite Extensible Nonlinear Elastic) dumbbell models. In this model, a polymer is idealized as an "elastic dumbbell" consisting of two "beads" joined by a spring which can be modeled by a vector R (see Bird, Curtis, Amstrong and Hassager [3], Doi and Edwards, [10] and Ottinger [28]). At the level of the polymeric liquid, we get a system coupling the Navier-Stokes equation for the fluid velocity with a Fokker-Planck equation describing the evolution of the polymer density. The coupling comes from and extra stress term in the fluid equation due to the microscopic polymers. This is the micro-macro interaction. There is also a drift term in the Fokker-Planck that depends on the spatial gradient of the velocity. This is a macro-micro term. The coupling satisfies the fact that the free-energy dissipates which is important to get energy estimates.

The system obtained attempt to describe the behavior of this complex mixture of polymers and fluid, and as such, it presents numerous challenges, simultaneously at the level of their derivation, the level of their numerical simulation and that of their mathematical treatment. In this paper we concentrate on the mathematical treatment and more precisely at the well-posedness for the FENE dumbbell model (1).

An approximate closure of the linear Fokker-Planck equation reduces the description to closed viscoelastic equations for the added stresses themselves. This leads to well-known non-Newtonian fluid models such as the Oldroyd B model. that has been studied extensively. In Guillopé and Saut [17] and [18], the existence of local strong solutions was proved. Also, Fernández-Cara, Guillén and Ortega [13], [12] and [14] proved local well posedness in Sobolev spaces. In Chemin and Masmoudi [6] local and global well-posedness in critical Besov spaces was given. For global existence of weak solutions, we refer to Lions and Masmoudi [26]. We also mention Lin, Liu and Zhang [23] where a formulation based on the deformation tensor is used to study the Oldroyd-B model.

At the micro-macro level, there are also several works. Indeed, from mathematical point of view, the model was studied by several authors. In particular Renardy [31] proved the local existence in Sobolev space where the potential $\mathcal{U}$ is given by $\mathcal{U}(R) = (1 - |R|^2)^{1-\sigma}$

for some $\sigma > 1$. W. E, Li and Zhang [11] proved local existence when R is taken in the whole space and under some growth condition on the potential. Also, Jourdain, Lelievre and Le Bris [21] proved local existence in the case $b = 2k > 6$ for a Couette flow by solving a stochastic differential equation (see also [19] for the use of entropy inequality methods to prove exponential convergence to equilibrium). Zhang and Zhang [32] proved local well-posedness for the FENE model when $b > 76$. Moreover, Lin, Liu and Zhang [24] proved global existence near equilibrium under some restrictions on the potential. Global existence of weak solutions was also proved in [27] for the co-rotational model (see also [2]).

We end this introduction by mentioning other micro-macro models. Indeed, a principle based on an energy dissipation balance was proposed in [7], where the regularity of nonlinear Fokker-Planck systems coupled with Stokes equations in 3D was also proved. In particular the Doi model (or Rigid model) was considered in [29] where the linear Fokker-Planck system is coupled with a stationary Stokes equations. The nonlinear Fokker-Planck equation driven by a time averaged Navier-Stokes system in 2D was studied in [8]. Also, the Doi model was considered in [9].

1.1. **The FENE model.** A macromolecule is idealized as an "elastic dumbbell" consisting of two "beads" joined by a spring which can be modeled by a vector R (see [3]). The micro-macro approach consists in writing a coupled multi-scale system of the

$$\begin{cases} \partial_t u + (u\cdot\nabla)u - \nu\Delta u + \nabla p = \mathrm{div}\tau, \quad \mathrm{div} u = 0, \\ \partial_t\psi + u.\nabla\psi = \mathrm{div}_R\Big[-\nabla u\cdot R\psi + \beta\nabla\psi + \nabla\mathcal{U}\psi\Big]. \\ \tau_{ij} = \int_B (R_i\otimes\nabla_j\mathcal{U})\psi(t,x,R)dR \qquad (\nabla\mathcal{U}\psi + \beta\nabla\psi).n = 0 \text{ on } \partial B(0,R_0). \end{cases} \tag{1}$$

In (1), $\psi(t,x,R)$ denotes the distribution function for the internal configuration and $F(R) = \nabla\mathcal{U}$ is the spring force which derives from a potential $\mathcal{U}$. Besides, β is related to the temperature of the system and $\nu > 0$ is the viscosity of the fluid. In the sequel, we will take $\beta = 1$.

Here, R is in a bounded ball $B(0,R_0)$ which means that the extensibility of the polymers is finite. Moreover, $\mathcal{U}(R) = -k\log(1-|R|^2/|R_0|^2)$ for some constant $k > 0$. We have also to add a boundary condition to insure the conservation of ψ, namely $(-\nabla uR\psi + \nabla\mathcal{U}\psi + \beta\nabla\psi).n = 0$ on $\partial B(0,R_0)$. The boundary condition on $\partial B(0,R_0)$ insures the conservation of the polymer density and should be understood in the weak sense, namely for any function $g(R)\in C^1(B)$, we have

$$\partial_t \int_B g\psi dR + u.\nabla_x \int_B g\psi dR = -\int_B \nabla_R g\Big[-\nabla u\cdot R\psi + \beta\nabla\psi + \nabla\mathcal{U}\psi\Big]dR. \tag{2}$$

Notice in particular that it implies that $\psi = 0$ on $\partial B(0,R_0)$ and that if initially $\int\psi(t = 0,x,R)dR = 1$, then for all t and x, we have $\int\psi(t,x,R)dR = 1$. We will see later an other way of understanding this singular boundary condition.

If ∇u is replaced by $W(u)$ (the anti-symmetric part of ∇u, namely $W(u) = \frac{\nabla u - {}^t\nabla u}{2}$) in the second equation of (1), then we get the so-called co-rotational FENE model. The fact of putting $W(u)$ instead of the whole ∇u in (1) allows to get better estimate on ψ. This will only be done in theorem 2.3. Also, we point out that, if R is in the whole space, we get the Hooke model for which $\mathcal{U}(R) = k|R|^2$ and the model reduces to the Oldroyd-B model (see [6] and [26] for some local and global existence results).

When doing numerical simulation on the FENE model, it is usually better to think of the distribution function ψ as the density of a random variable R which solves (see [28])

$$dR + u.\nabla R dt = (\nabla u R - \nabla_R\mathcal{U}(R))dt + \sqrt{2}dW_t \tag{3}$$

where the stochastic process W_t is the standard Brownian motion in $\mathbb{R}^N$ and the additional stress tensor is given by the following expectation $\tau = \mathbb{E}(R_i \otimes \nabla_j \mathcal{U})$. Of course, we may need to add a boundary condition for (3) if R reaches the boundary of B. This is done by requiring that R stays in $\overline{B}$ (see [20]). Using this stochastic formulation has the advantage of replacing the second equation of (2.1) which has $2N+1$ variables by (3). Of course one has to solve (3) several times to get the expectation τ which is the only information needed in the fluid equation. This strategy was used for instance by Keunings [22] (see also [15]) and by Öttinger [28] (see also [16]).

In the sequel, we will only deal with the FENE model and we will take $\beta = 1$ and $R_0 = 1$.

2. Statement of the results

In this paper, we present three different results which all hold for any $k > 0$. The first one deals with local existence in Sobolev spaces. The second one deals with global existence if the data is small or more precisely if the initial state is close to equilibrium. The third result treats the global existence in 2d for the co-rotational FENE model.

2.1. **Local existence.**

The system (1) has to be complemented with an initial data $u(t=0) = u_0$ and $\psi(t=0) = \psi_0$. Before stating our results, let us mention that local well-posedness for (1) was considered by Renardy [31], by Jourdain, Lelievre and Le Bris [21], by Zhang and Zhang [32] for $b = 2k > 76$ and by Lin, Zhang and Zhang [25] for $b > 12$.

We take, $s > \frac{N}{2} + 1$. Notice that (u, ψ) with $u = 0$ and

$$\psi_\infty(R) = \frac{e^{-\mathcal{U}(R)}}{\int_B e^{-\mathcal{U}(R')} dR'} \tag{4}$$

defines a stationary solution of (1).

Theorem 2.1. *Take $u_0 \in H^s(\mathbb{R}^N)$ and $\psi_0 \in H^s(\mathbb{R}^N; L^2(\frac{dR}{\psi_\infty}))$ with $\int \psi_0 dR = 1$ a.e in x. Then, there exists a time T^* and a unique solution (u, ψ) of (1) in $C([0,T^*); H^s) \times C([0,T^*); H^s(\mathbb{R}^N; L^2(\frac{dR}{\psi_\infty})))$. Moreover, $u \in L^2_{loc}([0,T^*); H^{s+1})$ and $\psi \in L^2_{loc}([0,T^*); H^s(\mathbb{R}^N; \mathcal{H}^1))$*

2.2. **Global existence for small data.**

The local existence result of the previous section gives global existence if the data is small or more precisely if it is close to equilibrium $(0, \psi_\infty)$.

Theorem 2.2. *There exists a constant c_0 such that for $u_0 \in H^s(\mathbb{R}^N)$ and $\psi_0 \in H^s(\mathbb{R}^N; L^2(\frac{dR}{\psi_\infty}))$, $\int \psi_0 dR = 1$ a.e in x, if*

$$\nu |u_0|^2_{H^s} + |\psi_0 - \psi_\infty|^2_{H^s(L^2(\frac{dR}{\psi_\infty}))} \leq c_0(\nu min(1,\nu)^2), \tag{5}$$

then the solution constructed in theorem 2.1 is global.

We refer to Lin, Liu and Zhang [24] for a similar result under some restrictive condition on the potential.

2.3. **Global existence for the co-rotational model in 2d.**

In dimension $N = 2$, we also have global existence if we restrict to the so-called co-rotational model, namely, we replace ∇u by $W(u)$ in the second equation of (1)

$$\partial_t \psi + u.\nabla\psi = \mathrm{div}_R\Big[- W(u) \cdot R\psi + \beta \nabla \psi + \nabla \mathcal{U} \psi \Big]. \tag{6}$$

This extends the result of Lin, Zhang and Zhang [25] to the case $b > 0$. We also refer to Constantin and Masmoudi [9] for a similar result concerning the Doi model. The two results mentioned above use losing regularity estimates in the spirit of [6] and [1].

Theorem 2.3. *Take $u_0 \in H^s(\mathbb{R}^2)$ and $\psi_0 \in H^s(\mathbb{R}^2; \mathcal{L}^r \cap L^2(\frac{dR}{\psi_\infty}))$ for some r such that $(r-1)k > 1$ with $\int \psi_0 dR = 1$ a.e in x. Then, the solution constructed in theorem 2.1 for the co-rotational model (the second equation of (1) is replaced by (6)) is global.*

Remark 2.4. *1) The assumption $\int \psi_0 dR = 1$ a.e in x in the 3 previous theorems is not essential and can be replaced by the fact that $\int \psi_0 dR \le C_0$ which follows from the fact that $\psi_0 \in H^s(\mathbb{R}^2; L^2(\frac{dR}{\psi_\infty}))$.*

2) The regularity assumption $s > \frac{N}{2} + 1$ can be weakened to prove existence in some critical spaces as was done in [6]. *This will be done elsewhere.*

The paper is organized as follows. In the next section, we give some preliminaries where we prove three inequalities and study the linearized operator in the R variable. In section 4, we give some a priori estimates for the full model (1) which are needed for the proof of theorems 2.1 and 2.2. In section 5 we prove theorems 2.1 and 2.2 by using a fixed point argument. Section 6 is devoted to the study of the co-rotational model in $2d$.

3. Preliminaries

3.1. **Notations.** We will use the following notations. For $\alpha \in \mathbb{N}^N$, ∂^α will denote α_1 derivatives in x_1, ... and α_N derivatives in x_N. Also for $s \in \mathbb{N}$, ∂^s will denote all the derivatives ∂^α for $|\alpha| \le s$.

$$|u|_s^2 = \sum_{|\alpha| \le s} \int_\Omega |\partial^\alpha u|^2 dx \tag{7}$$

$$|\psi|_s^2 = \sum_{|\alpha| \le s} \int_\Omega \int_B |\partial^\alpha \psi|^2 \frac{dR}{\psi_\infty} dx \tag{8}$$

$$|\psi|_{s,1}^2 = \sum_{|\alpha| \le s} \int_\Omega \int_B \psi_\infty |\partial^\alpha \nabla_R \frac{\psi}{\psi_\infty}|^2 dR dx \tag{9}$$

We will also use the notation $C_T(H^s)$, $L^2_T(H^s)$ to denote $C([0,T]; H^s(\mathbb{R}^N))$, $L^2(0,T; H^s(\mathbb{R}^N))$.

We also recall that $\psi_\infty(R) = e^{-\mathcal{U}} / \int e^{-\mathcal{U}} = (1 - |R|^2)^k$ which behaves like $(1 - |R|)^k$ when R goes to the boundary of B.

We will also denote $\mathcal{H} = L^2(\frac{dR}{\psi_\infty})$ and

$$\mathcal{H}^1 = \left\{ \psi \mid \quad \int \psi_\infty \left| \nabla \frac{\psi}{\psi_\infty} \right|^2 + \frac{\psi^2}{\psi_\infty} \, dR < \infty \right\}.$$

For $r \ge 1$, we denote $\mathcal{L}^r$ and $\mathcal{L}^{r,1}$ the spaces

$$\mathcal{L}^r = \left\{ \psi \mid \quad |\psi|_{\mathcal{L}^r}^r = \int \psi_\infty \left| \frac{\psi}{\psi_\infty} \right|^r dR < \infty \right\}$$

$$\mathcal{L}^{r,1} = \left\{ \psi \in \mathcal{L}^r \mid \quad |\psi|_{\dot{\mathcal{L}}^{r,1}}^r = \int \psi_\infty \left| \nabla \frac{\psi}{\psi_\infty} \right|^r dR < \infty \right\}.$$

Notice that $\mathcal{H} = \mathcal{L}^2 = L^2(\frac{dR}{\psi_\infty})$ and $\mathcal{H}^1 = \mathcal{L}^{2,1}$. Finally, we denote $C_0^\infty(B) = \mathcal{D}(B)$ the set of C^∞ functions on B with compact support.

3.2. **Some inequalities.** One of the main ingredients of the proof is the use the following Hardy type inequality. We denote $x = 1 - |R|$

Proposition 3.1. *For all $\varepsilon > 0$, there exists a C_ε such that*

$$\left(\int_B \frac{|\psi|}{x} dR\right)^2 \le \varepsilon \int_B \psi_\infty \left|\nabla_R \frac{\psi}{\psi_\infty}\right|^2 dR + C_\varepsilon \int_B \frac{|\psi|^2}{\psi_\infty} dR \tag{10}$$

Remark 3.2. *1) In the case $k > 1$, we can take $\varepsilon = 0$ in proposition 3.1 since, we have*

$$\left(\int_B \frac{|\psi|}{1-|R|} dR\right)^2 \le C \int_B \frac{|\psi|^2}{\psi_\infty} dR \tag{11}$$

2) As can be seen from the proof, we only need the radial part of the gradient in (10)

Proof. The proof is a simple consequence of the following $1d$ inequality

$$\left(\int_0^1 \frac{\psi}{x} dx\right)^2 \le \varepsilon \int_0^1 x^k \left|\left(\frac{\psi}{x^k}\right)'\right|^2 dx + C_\varepsilon \int_0^1 \frac{\psi^2}{x^k} dx. \tag{12}$$

To prove (12), we have to distinguish between three cases.

Case $k > 1$: In this case, we can take $\varepsilon = 0$ and we have just to use Cauchy-Schwarz inequality, namely

$$\int_0^1 \frac{\psi}{x} dx \le \left(\int_0^1 \frac{\psi^2}{x^k} dx\right)^{1/2} \left(\int_0^1 x^{k-2} dx\right)^{1/2} \tag{13}$$

and the last integral converges since $k > 1$.

Case $k < 1$: We make the following change of variables $y = x^{1-k}$ hence $dy = (1-k)x^{-k}dx$. We also denote $g(y) = \psi(x)/x^k$. Hence

$$\int_0^1 x^k \left|\left(\frac{\psi}{x^k}\right)'\right|^2 dx = (1-k) \int_0^1 g'(y)^2 dy \tag{14}$$

Moreover, denoting $\alpha = \frac{k}{1-k}$, we get

$$\int_0^1 \frac{\psi(x)}{x} dx = \frac{1}{1-k} \int_0^1 y^\alpha \frac{g(y)}{y} dy \tag{15}$$

$$\int_0^1 \frac{\psi^2(x)}{x^k} dx = \frac{1}{1-k} \int_0^1 y^{2\alpha} g(y)^2 dy \tag{16}$$

Hence, it is enough to prove that for all $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that for all $g \in H^1(0,1)$, we have

$$\left(\int_0^1 y^\alpha \frac{g(y)}{y} dy\right)^2 \le \varepsilon \int_0^1 g'(y)^2 dy + C_\varepsilon \int_0^1 y^{2\alpha} g(y)^2 dy. \tag{17}$$

We prove this by contradiction. Assume that for some $\varepsilon > 0$, (17) does not hold. Hence, there exists a sequence $g_n \in H^1(0,1)$ such that

$$\begin{cases} \int_0^1 y^\alpha \frac{g_n(y)}{y} dy = 1 \\ \int_0^1 y^{2\alpha} g_n(y)^2 dy \to 0 \\ \int_0^1 g_n'(y)^2 dy \quad \text{is bounded.} \end{cases} \tag{18}$$

Extracting a subsequence, we deduce that g_n converges weakly to some g in $H^1(0,1)$. Moreover, from the second relation of (18), we deduce that $y^\alpha g_n$ converges to 0 in L^2 and hence $g = 0$. Besides, by compact embedding, we deduce that g_n converges to 0 in $L^\infty(0,1)$. Since,

$y^{\alpha-1}$ is in L^1, we deduce that $\int_0^1 y^{\alpha-1} g_n(y)dy \to 0$ when n goes to infinity and this yields a contradiction. Hence, there exists a $C_\varepsilon > 0$ such that (17) holds

Case $k=1$: We make the following change of variables $x = e^{-y}$ hence $dx = -e^{-y}dy$. We also denote $g(y) = \psi(x)/x$. Arguing as in the case, $k<1$, we see that (13) is equivalent to the existence of C_ε such that for $g \in \dot{H}^1(\mathbb{R}_+)$, we have

$$\left(\int_0^\infty g(y)e^{-y}dy\right)^2 \leq \varepsilon \int_0^\infty g'(y)^2 dy + C_\varepsilon \int_0^\infty g(y)^2 e^{-2y}dy \tag{19}$$

We prove this by contradiction. Assume that for some $\varepsilon > 0$, (19) does not hold. Hence, there exists a sequence $g_n \in \dot{H}^1(\mathbb{R}_+)$ such that

$$\begin{cases} \int_0^\infty g_n(y)e^{-y}dy = 1 \\ \int_0^\infty g_n(y)^2 e^{-2y}dy \to 0 \\ \int_0^\infty g_n'(y)^2 dy \quad \text{is bounded.} \end{cases} \tag{20}$$

Extracting a subsequence, we deduce that $g_n e^{-y}$ converges to 0 in $L^2(\mathbb{R}_+)$. Hence, g_n converges to 0 in $L^2_{loc}(\mathbb{R}_+)$. Since g_n is bounded in $\dot{H}^1(\mathbb{R}_+)$, we deduce that $g_n(0)$ goes to zero when n goes to infinity. Besides,

$$g_n(y) \leq g_n(0) + y^{1/2}\int_0^y g_n'(y)^2 dy. \tag{21}$$

Hence, we deduce that $\int_0^\infty g_n(y)e^{-y}dy \to 0$ when n goes to infinity and this yields a contradiction. Hence, there exists a $C_\varepsilon > 0$ such that (19) holds. This end the proof of the Proposition □

We point out that in the case, $k>1$, we have a similar change of variable, namely $y = x^{1-k}$ and hence denoting $\alpha = \frac{k}{k-1}$, we see that (12) is equivalent to

$$\left(\int_1^\infty \frac{g(y)}{y^{1+\alpha}}dy\right)^2 \leq \varepsilon \int_0^\infty g'(y)^2 dy + C_\varepsilon \int_0^\infty \frac{g(y)^2}{y^{2\alpha}}dy \tag{22}$$

Remark 3.3. *We called our inequality (10) a Hardy type inequality even though it is of a different nature. We would like here to explain this more and compare (10) to the Hardy inequality. We assume that $\psi \in \mathcal{H}^1$.*

Case $k>1$: First, we focus on the 1d *problem and denote $x = 1-|R|$. Notice that when $k>1$ and we denote $f(x) = \psi(x)/x^{k/2}$, we get that $f(0)=0$ and $f \in H^1(0,1)$. Indeed, if we make the change of variable $y = x^{1-k}$, we get by Cauchy-Schwarz that for each $\varepsilon > 0$, $g(y) \leq \varepsilon\sqrt{y} + C_\varepsilon$. Hence, $f(x) \leq \varepsilon\sqrt{x} + C_\varepsilon x^{k/2}$. Moreover, after integration by parts we get*

$$\int_0^1 x^k \left|\left(\frac{\psi}{x^k}\right)'\right|^2 dx = \int_0^1 f'(x)^2 + \left(\frac{k}{2}\right)^2 \frac{f^2}{x^2} - 2\frac{k}{2}f'\frac{f}{x} \tag{23}$$

$$= \int_0^1 f'(x)^2 + \left(\frac{k^2-2k}{4}\right)\frac{f^2}{x^2} - \frac{k}{2}f^2(1) \tag{24}$$

If $k \geq 2$, we get easily a bound on $\int_0^1 f'(x)^2$ and then on $\int_0^1 \frac{f^2}{x^2}$ by the Hardy inequality. If $1<k<2$, we use Hardy inequality to control the second term of the right hand side and get

$$\frac{k}{2}f^2(1) + \int_0^1 x^k \left|\left(\frac{\psi}{x^k}\right)'\right|^2 dx \geq (k-1)^2 \int_0^1 f'(x)^2 \geq \frac{(k-1)^2}{4}\int_0^1 \frac{f^2}{x^2} \tag{25}$$

Of course, the $f^2(1)$ of the left hand side can be replaced by $\int_0^1 f^2(x)dx$ modulo some constant. Hence, written in the R variable, we get

$$\int_B \frac{\psi^2}{\psi_\infty x^2}dR \leq C|\psi|^2_{\mathcal{H}^1}. \tag{26}$$

<u>*Case $k \le 1$:*</u> *If $k \le 1$, we can not use Hardy inequality and a bound on $\int_0^1 x^k \left|\left(\frac{\psi}{x^k}\right)'\right|^2 dx + f^2(1)$ does not imply a bound on $\int_0^1 f'(x)^2$ and on $\int_0^1 \frac{f^2}{x^2}$ as can be seen by taking $\psi = \psi_\infty$. However, we have a weaker Hardy estimate, namely*

$$\int_0^1 \frac{\psi^2}{x^{2k}x^\beta}dx \le C\int_0^1 x^k \left|\left(\frac{\psi}{x^k}\right)'\right|^2 dx + C\int_0^1 \frac{\psi^2}{x^k}dx \tag{27}$$

or in the original variables

$$\int_B \frac{\psi^2}{\psi_\infty^2 x^\beta}dR \le C\int_B \psi_\infty \left|\left(\nabla(\frac{\psi}{\psi_\infty}\right)\right|^2 dR + C\int_B \frac{\psi^2}{\psi_\infty}dR \le C|\psi|^2_{\mathcal{H}^1} \tag{28}$$

for any $\beta < 1$. This inequality can be easily deduced, in the case $k < 1$, from the following inequality in the y variable

$$\int_0^1 \frac{g^2(y)}{y^\gamma}dy \le C\int_0^1 g'(y)^2 dy + C\int_0^1 y^{2\alpha} g(y)^2 dy \tag{29}$$

and where we denote $\gamma = \frac{\beta-k}{1-k} < 1$. In the case $k = 1$ it can be deduced from

$$\int_0^\infty g(y)^2 e^{-(1-\beta)y}dy \le C\int_0^\infty g'(y)^2 dy + C\int_0^\infty g(y)^2 e^{-2y}dy \tag{30}$$

The inequalities (29) and (30) can be proved by an argument similar to the one in the proof of the proposition. In Remark 3.8, we will prove an improved version of (28) using logarithmic terms.

For the global existence result, we will also need the following Poincare inequality with weight.

Proposition 3.4. *There exists a constant C such that, for all $\tilde\psi \in \mathcal{H}^1$ with $\int_B \tilde\psi = 0$, we have*

$$\int_B \frac{|\tilde\psi|^2}{\psi_\infty} dR \le \int_B \psi_\infty \left|\nabla_R \frac{\tilde\psi}{\psi_\infty}\right|^2 dR \tag{31}$$

Proof. By contradiction. Assume that there exists a sequence, $\tilde\psi_n \in \mathcal{H}^1$, $\int_B \tilde\psi_n = 0$, and

$$\int_B \frac{|\tilde\psi_n|^2}{\psi_\infty} dR = 1, \qquad \int_B \psi_\infty \left|\nabla_R \frac{\tilde\psi_n}{\psi_\infty}\right|^2 dR \to 0. \tag{32}$$

Hence, $\sqrt{\psi_\infty}\nabla_R \frac{\tilde\psi_n}{\psi_\infty}$ goes to 0 in $L^2(B)$ and $\nabla_R \frac{\tilde\psi_n}{\psi_\infty}$ goes to 0 in $L^2_{loc}(B)$. Hence $\frac{\tilde\psi_n}{\psi_\infty}$ goes to some constant c in $L^2_{loc}(B)$ Since, $\tilde\psi_n$ is bounded in $L^2(B)$, we deduce that $\tilde\psi_n$ goes to c in $L^1(B)$ and using that $\int_B \tilde\psi_n = 0$, we deduce that $c = 0$.

From the Hardy inequality (26) if $k > 1$ or (28) if $k \le 1$, we deduce in all cases that

$$\int_B \frac{|\tilde\psi_n|^2}{\psi_\infty x^\beta} dR \le C \tag{33}$$

for some $0 < \beta < 1$. This gives some tightness of the sequence $\frac{|\tilde\psi_n|^2}{\psi_\infty}$. Hence, we deduce from the strong convergence of $\tilde\psi_n$ in $L^2_{loc}(B)$ to 0 that $\frac{\tilde\psi_n}{\sqrt{\psi_\infty}}$ converges to 0 in $L^2(B)$. This gives a contradiction with (32). Hence, (31) holds.

□

For the global existence for the co-rotational model in 2d, we will need the following inequality. If p is such that $pk > 1$ then

Proposition 3.5. *If p is such that $pk > 1$ then*

$$\int_B \frac{|\psi|}{x} dR \leq C \left(\int_B \frac{|\psi|^{p+1}}{\psi_\infty^p} dR \right)^{\frac{1}{p+1}} \tag{34}$$

Proof. The proof is based on Holder inequality. Indeed, using that $x \sim x^{1-kp/(p+1)} \psi_\infty^{p/(p+1)}$, we get

$$\int_B \frac{|\psi|}{x} dR \quad \leq \quad C \int_B \frac{1}{x^{1-kp/(p+1)}} \frac{|\psi|}{\psi_\infty^{p/(p+1)}} dR \tag{35}$$

$$\leq \quad C \left(\int_B \frac{1}{x^{1+1/p-k}} \right)^{\frac{p}{p+1}} \left(\int_B \frac{|\psi|^{p+1}}{\psi_\infty^p} dR \right)^{\frac{1}{p+1}} \tag{36}$$

and the result follows. □

3.3. **The linearized problem in R.** One important ingredient in proving our existence result is the study of the following linear operator in the R variable

$$L\psi = -div(\psi_\infty \nabla \frac{\psi}{\psi_\infty}) \tag{37}$$

on the space $\mathcal{H} = L^2(\frac{dR}{\psi_\infty})$ and with domain

$$D(L) = \left\{ \psi \in \mathcal{H} | \psi_\infty \nabla \frac{\psi}{\psi_\infty} \in \mathcal{H}, \quad div(\psi_\infty \nabla \frac{\psi}{\psi_\infty}) \in \mathcal{H} \quad \text{and}\, \psi_\infty \nabla \frac{\psi}{\psi_\infty}|_{\partial B} = 0 \right\} \tag{38}$$

We also define the Hilbert spaces $\mathcal{H}^1$ and $\mathcal{H}^2$ by

$$\mathcal{H}^1 = \left\{ \psi \,| \quad \int \psi_\infty \left| \nabla \frac{\psi}{\psi_\infty} \right|^2 + \frac{\psi^2}{\psi_\infty} \; dR < \infty \right\} \tag{39}$$

$$\mathcal{H}^2 = \left\{ \psi \in \mathcal{H}^1 \,| \quad \int \left(div(\psi_\infty \nabla \frac{\psi}{\psi_\infty}) \right)^2 \frac{dR}{\psi_\infty} < \infty \right\} \tag{40}$$

The boundary condition $\psi_\infty \nabla \frac{\psi}{\psi_\infty}|_{\partial B} = 0$ should be understood in the weak sense, namely for any $\phi \in C^1(B)$, we have

$$\int_B \phi L\psi \; dR = \int_B \psi_\infty \nabla \frac{\psi}{\psi_\infty} . \nabla \phi \; dR \tag{41}$$

and for any $\phi \in \mathcal{H}^1$, we have

$$\int_B \phi L\psi \frac{dR}{\psi_\infty} = \int_B \psi_\infty \nabla \frac{\psi}{\psi_\infty} . \nabla \frac{\phi}{\psi_\infty} \; dR. \tag{42}$$

Notice that for any $\phi \in C^1(B)$, $\phi\psi_\infty \in \mathcal{H}^1$ and hence (41) follows from (42).

Proposition 3.6. *L is self-adjoint and positive. Moreover, it has a discrete spectrum formed by a sequence (ℓ_n) such that $\ell_n \to \infty$ when $n \to \infty$.*

Proof. Let us prove that L is self-adjoint. First, it is easy to see that $D(L)$ is dense, indeed $C_0^\infty(B) \subset D(L)$ and is dense in $\mathcal{H}$. Next, to see that L is symmetric, we notice that for $\phi, \psi \in D(L)$, we have

$$\int_B \phi L\psi \frac{dR}{\psi_\infty} = \int_B \psi_\infty \nabla \frac{\psi}{\psi_\infty} . \nabla \frac{\phi}{\psi_\infty} dR = \int_B \psi L\phi \frac{dR}{\psi_\infty}. \tag{43}$$

Next, we use Riesz representation theorem (or Lax-Milgram) to deduce that for all $f \in \mathcal{H}$, there exists a unique $\psi \in \mathcal{H}^1$ such that for all $\phi \in \mathcal{H}^1$, we have

$$\int_B \psi_\infty \nabla \frac{\psi}{\psi_\infty} . \nabla \frac{\phi}{\psi_\infty} + \frac{\psi\phi}{\psi_\infty} \; dR = \int_B \frac{f\phi}{\psi_\infty} \; dR. \tag{44}$$

By taking $\phi \in \mathcal{D}(B)$, we deduce that $-div(\psi_\infty \nabla \frac{\psi}{\psi_\infty}) + \psi = f$ and then that (42) holds. Hence, we have

$$\begin{cases} (L+1)\psi = f \\ \psi_\infty \nabla \frac{\psi}{\psi_\infty}|_{\partial B} = 0. \end{cases} \tag{45}$$

This insures that the operators $L+1$ and L are closed. Moreover, -1 is in the resolvent of L which implies that necessary L is self-adjoint (see for instance Chapter X, p137 of [30])

To prove that L has a discrete spectrum, we define the operator K by for $f \in \mathcal{H}$, Kf is the unique solution ψ of the equation $(L+1)\psi = f$ in $D(L)$. Hence, it is easy to see that K is compact and symmetric. Hence, it has a discrete spectrum formed by a decreasing sequence $\lambda_n > 0$ which goes to zero when n goes to infinity. Besides, it has a countable orthonormal basis of eigenvectors w_n. This implies that L has the same basis of eigenvectors with the eigenvalues $\ell_n = 1/\lambda_n - 1$, namely $Lw_n = \ell_n w_n$.

□

We end this subsection by two remarks about the space $\mathcal{H}^1$ and about the boundary condition for L. These two remarks will not be used in the existence proof but they are interesting in themselves.

Remark 3.7. *If $k \geq 1$ then*

$$\overline{C_0^\infty}^{\mathcal{H}^1} = \mathcal{H}^1. \tag{46}$$

We will only give the proof when $k = 1$. The other case is simpler. First, we notice that (28) can be improved, namely

$$\int_B \frac{\psi^2}{x^3 \log(x)^2} dR \leq C|\psi|^2_{\mathcal{H}^1} \tag{47}$$

Indeed, this is a consequence of the following inequality

$$\int_0^\infty \frac{g(y)^2}{y^2} dy \leq C \int_0^\infty g'(y)^2 dy + C \int_0^\infty g(y)^2 e^{-2y} dy. \tag{48}$$

We define the function χ by $\chi(t) = 1$ for $0 \leq t \leq 1$, $\chi(t) = 2 - t$ for $1 \leq t \leq 2$ and $\chi(t) = 0$ for $t \geq 2$. For $\psi \in \mathcal{H}^1$, we take

$$\psi_n(R) = \psi(R)\chi(\frac{-\log(1-|R|)}{n}).$$

It is clear that $\psi_n \in \mathcal{H}^1$. Moreover,

$$\|\psi - \psi_n\|^2_{H^1} \leq C \int_{1-|R| \leq e^{-n}} \frac{\psi^2}{x^3 \log(x)^2} + |\nabla\psi|^2 \quad dR \tag{49}$$

which goes to 0 when n goes to infinity. Now, it is easy to see that ψ_n can be approximated in $\mathcal{H}^1$ by a sequence of $C_0^\infty(B)$. This ends the proof of (46).

It is clear that (46) does not hold when $k < 1$ since ψ_∞ is not in $\overline{C_0^\infty}^{\mathcal{H}^1}$.

Remark 3.8. *We point out that if $k \geq 1$ (which is equivalent to $b \geq 2$), then the boundary condition $\psi_\infty \nabla \frac{\psi}{\psi_\infty}|_{\partial B} = 0$ is a consequence of the fact that $\psi \in \mathcal{H}^2$ and hence $D(L) = \mathcal{H}^2$. For the proof, we use the fact that for all $\psi \in \mathcal{H}^2$, the relation (42) holds when $\phi \in C_0^\infty$. Then, we use that any $\phi \in \mathcal{H}^1$ can be approximated in $\mathcal{H}^1$ by a sequence $\phi_n \in C_0^\infty$. Then, we* *pass to the limit and deduce that (42) holds for all* $\phi \in \mathcal{H}^1$*. This implies that* $\psi_\infty \nabla \frac{\psi}{\psi_\infty}|_{\partial B} = 0$ *and hence,* $D(L) = \mathcal{H}^2$*.*

The fact that the boundary condition $\psi_\infty \nabla \frac{\psi}{\psi_\infty}|_{\partial B} = 0$ *is not needed when* $k \geq 1$ *should be related to a similar property of the stochastic differential equation (3). Indeed, in* [20]*, it is proved that when* $k \geq 1$*, then the process* R_t *defined by (3) does not reach the boundary almost surely. Besides, when* $k < 1$*, the process* R_t *reaches the boundary in finite time almost surely. This explains why we need a boundary condition for the operator* L *when* $k < 1$*. Notice that in this case the inclusion* $D(L) \subset \mathcal{H}^2$ *is strict. Indeed, we can notice that for* $k < 1$*,* $\psi_\infty^{1/k} \in \mathcal{H}^2$ *but does not satisfy the boundary condition and hence it is not in* $D(L)$*.*

3.4. **Solution for the linear equation in R.** Now, we can solve the following linear problem in R.

Proposition 3.9. *Assume that* $A(t) \in C([0,\infty))$ *is a matrix valued function and that* $f \in C([0,\infty);\mathcal{H}^1)$*, then*

$$\left\{ \begin{array}{l} \partial_t \psi = -div_R(A(t)R\psi) - L(\psi) + div_R(f) \\ \psi_\infty \nabla \frac{\psi}{\psi_\infty}{}_{|\partial B} = 0 \end{array} \right. \tag{50}$$

with the initial value $\psi(t=0) = \psi_0(R) \in \mathcal{H}$ *has a unique very weak solution* ψ *in* $C([0,\infty);\mathcal{H})$ *(see the definition below). Moreover,* $\psi \in L^2_{loc}([0,\infty);\mathcal{H}^1)$*.*

Before giving the proof, we have to give a sense to (50). For, $\psi \in C([0,\infty);\mathcal{H}) \cap L^2_{loc}(0,\infty;\mathcal{H}^1)$, we say that ψ is a weak solution of (50) if for all $T > 0$, $\phi \in C^1([0,T];\mathcal{H}^1)$, $\phi(T) = 0$, we have

$$-\int_B \frac{\psi_0 \phi(0)}{\psi_\infty} - \int_0^T \int_B \frac{\psi \partial_t \phi}{\psi_\infty} = \int_0^T \int_B A(t)R\psi \nabla \frac{\phi}{\psi_\infty} - \psi_\infty \nabla \frac{\psi}{\psi_\infty} . \nabla \frac{\phi}{\psi_\infty} - f . \nabla \frac{\phi}{\psi_\infty} \quad dRdt \tag{51}$$

For $\psi \in C([0,\infty);\mathcal{H})$, we say that ψ is a very weak solution of (50) if for all $T > 0$, $\phi \in C^1([0,T];D(L))$, $\phi(T) = 0$, we have

$$-\int_B \frac{\psi_0 \phi(0)}{\psi_\infty} - \int_0^T \int_B \frac{\psi \partial_t \phi}{\psi_\infty} = \int_0^T \int_B A(t)R\psi \nabla \frac{\phi}{\psi_\infty} - \frac{\psi}{\psi_\infty} L(\phi) - f . \nabla \frac{\phi}{\psi_\infty} \quad dRdt \tag{52}$$

Proof. The proof uses a Galerkin approximation based on the eigenfunctions of the operator L. Let us denote V_N the space spanned by the eigenfunctions w_n of L with eigenvalue $\ell_n \leq N$. Let P_N be the orthogonal projection onto V_N. We consider the Galerkin approximation of (50)

$$\left\{ \begin{array}{l} \partial_t \psi_N = -P_N(\mathrm{div}_R(A(t)R\chi_N \psi_N)) - L(\psi_N) + P_N(\mathrm{div}_R(\chi_N f)) \\ \psi_N(t=0) = P_N(\psi_0) \end{array} \right. \tag{53}$$

where $\chi_N(R) \in C^1(B)$ is a cut-off function which is used to insure that $\mathrm{div}_R(A(t)R\chi_N\psi_N), \mathrm{div}_R(\chi_N f) \in \mathcal{H}$. It satisfies, $\chi_N = 1$ on $B(0, 1 - \frac{2}{N})$ and $\chi_N = 0$ for $|R| > 1 - \frac{1}{N}$. The equation (53) is an ODE which can be solved locally in time. Moreover, the solution is global because of the following estimate

$$\partial_t \int_B \frac{\psi_N^2}{2\psi_\infty} = \int_B A(t)R\chi_N\psi_N \nabla \frac{\psi_N}{\psi_\infty} - \psi_\infty \left| \nabla \frac{\psi_N}{\psi_\infty} \right|^2 - \chi_N f . \nabla \frac{\psi_N}{\psi_\infty} \quad dR \tag{54}$$

$$\leq C|A(t)|^2 \int_B \frac{\psi_N^2}{\psi_\infty} - \frac{1}{2} \int_B \psi_\infty \left| \nabla \frac{\psi_N}{\psi_\infty} \right|^2 + C|f|^2_{\mathcal{H}} \tag{55}$$

and

$$\int_B \frac{\psi_N^2}{\psi_\infty}(t) + \frac{1}{2} \int_0^t \int_B \psi_\infty \left| \nabla \frac{\psi_N}{\psi_\infty} \right|^2 \leq \left(\int_B \frac{\psi_0^2}{\psi_\infty} + \int_0^t C|f(s)|^2_{\mathcal{H}} ds \right) \, e^{C \int_0^t |A(s)|^2 ds}. \tag{56}$$

Besides, we have for $T>0$, $\phi \in C^1([0,T];V_N)$, $\phi(T)=0$,

(57)
$$-\int_B \frac{P_N(\psi_0)\phi(0)}{\psi_\infty}-\int_0^T\int_B \frac{\psi_N\partial_t\phi}{\psi_\infty}=\int_0^T\int_B A(t)R\psi_N\chi_N\nabla\frac{\phi}{\psi_\infty}-\psi_\infty\nabla\frac{\psi_N}{\psi_\infty}.\nabla\frac{\phi}{\psi_\infty}-\chi_N f.\nabla\frac{\phi}{\psi_\infty}\,dRdt$$

Extracting a subsequence and passing to the limit when N goes to infinity, we recover a weak solution $\psi \in L^\infty_{loc}([0,\infty);\mathcal{H})\cap L^2_{loc}([0,\infty;\mathcal{H}^1)$ to (50).

To see that $\psi \in C([0,\infty);\mathcal{H})$, we first notice that, $\psi \in C([0,\infty);w-\mathcal{H})$ where $w-\mathcal{H}$ is the space $\mathcal{H}$ equipped with the weak topology. Then

$$\frac{1}{2}\Big||\psi(t)|^2-|\psi(s)|^2_{\mathcal{H}}\Big| = \Big|\int_s^t \partial_t\psi\psi\Big| \tag{58}$$

$$\leq \int_s^t C|A(t)|^2\int_B\frac{\psi^2}{\psi_\infty}+\frac{1}{2}\int_B\psi_\infty\left|\nabla\frac{\psi}{\psi_\infty}\right|^2+C|f|^2_{\mathcal{H}} \tag{59}$$

and hence $|\psi(s)|^2_{\mathcal{H}}$ goes to $|\psi(t)|^2$ when s goes to t. This yields that $\psi \in C([0,\infty);\mathcal{H})$.

To prove uniqueness in $C([0,\infty);\mathcal{H})$, we use the dual problem. Let ψ be a very weak solution of (50) in $C([0,\infty);\mathcal{H})$ with zero initial data and with $f=0$. For $\phi \in C^1([0,T];D(L))$ and $\phi(t=T)=0$, we have

$$-\int_0^T\int_B\frac{\psi\partial_t\phi}{\psi_\infty}ds=\int_0^T\int_B A(t)R\psi\nabla_R\frac{\phi}{\psi_\infty}+div_R(\psi_\infty\nabla_R\frac{\phi}{\psi_\infty})\frac{\psi}{\psi_\infty} \tag{60}$$

For $F \in C([0,T];\mathcal{H})$, let ϕ be the solution of the following backward equation

$$\begin{cases} -\partial_t\phi=\psi_\infty A(t)R.\nabla_R\frac{\phi}{\psi_\infty}-L(\phi)+F \\ \psi_\infty\nabla\frac{\phi}{\psi_\infty}{}_{|\partial B}=0, \qquad \phi(t=T)=0 \end{cases} \tag{61}$$

The solution ϕ can be constructed using the same Galerkin approximation as above. Moreover, due to the fact that $\phi \in L^2_{loc}([0,\infty;\mathcal{H}^1)$, we see that the force term $\psi_\infty A(t)R.\nabla_R\frac{\phi}{\psi_\infty}+F$ is in $L^2_{loc}([0,\infty;\mathcal{H})$ and maximal regularity results insure that $\phi \in L^2_{loc}([0,\infty;D(L))$ and $\partial_t\phi \in L^2_{loc}([0,\infty;\mathcal{H})$. Hence, ϕ can be used as a test function in (52) and yields that $\int_0^T\int_B\psi\frac{F}{\psi_\infty}=0$ and hence $\psi=0$. This ends the proof of the uniqueness and the proof of the proposition. □

Next, we prove a proposition giving the regularity in the x variable, namely

Proposition 3.10. *Given $u \in C([0,\infty);H^s)\cap L^2_{loc}([0,\infty);H^{s+1})$. Then,*

$$\begin{cases} \partial_t\psi+u.\nabla_x\psi=-div_R(\nabla uR\psi)-L(\psi) \\ \psi_\infty\nabla\frac{\psi}{\psi_\infty}|_{\partial B}=0 \end{cases} \tag{62}$$

with the initial value $\psi(t=0)=\psi_0(x,R)\in H^s(\Omega;\mathcal{H})$ has a unique solution ψ in $C([0,\infty);H^s(\Omega;\mathcal{H}))$. Moreover, $\psi \in L^2_{loc}([0,\infty);H^s(\Omega;\mathcal{H}^1))$.

Proof. First, we define the flow associated with u, namely $\Phi(t,x)$ such that

$$\begin{cases} \partial_t\Phi(t,x)=u(t,\Phi(t,x)) \\ \Phi(t=0,x)=x. \end{cases} \tag{63}$$

Making the change of variable $\phi(t,x,R)=\psi(t,\Phi(t,x),R)$, we see that $\psi(t,x,R)$ solves (62) if and only if $\phi(t,x,R)$ solves

$$\begin{cases} \partial_t\phi=-\mathrm{div}_R((\nabla u)(t,\Phi(t,x))R\phi)-L(\phi) \\ \psi_\infty\nabla\frac{\phi}{\psi_\infty}|_{\partial B}=0 \end{cases} \tag{64}$$

Using Proposition 3.9 for each x, we deduce the existence and uniqueness of $\phi(t,x,R)$ in $C([0,\infty);\mathcal{H}))$. Integrating (56) with ψ_N replaced by $\phi(t,x,R)$, we deduce that

$$\int_{\Omega\times B}\frac{\phi^2}{\psi_\infty}(t)+\frac{1}{2}\int_0^t\int_{\Omega\times B}\psi_\infty\left|\nabla\frac{\phi}{\psi_\infty}\right|^2\leq\int_{\Omega\times B}\frac{\psi_0^2}{\psi_\infty}dRdx\ e^{C\int_0^t|\nabla u(s)|^2_{L^\infty}ds}. \tag{65}$$

Hence, $\phi(t,x,R)\in C([0,\infty);L^2(\Omega;\mathcal{H}))\cap L^2_{loc}([0,\infty);L^2(\Omega;\mathcal{H}^1))$

To prove regularity in the x variable, we use difference quotients

$$\phi_h = D_k^h\phi(t,x,R)=\frac{\phi(t,x+he_k,R)-\phi(t,x,R)}{h} \tag{66}$$

$$(\nabla u)_h = D_k^h[\nabla u(t,\Phi(t,x),R)]=\frac{\nabla u(t,\Phi(t,x+he_k))-\nabla u(t,\Phi(t,x))}{h} \tag{67}$$

for $h>0$ and $1\leq k\leq N$. Hence, ϕ_h solves

$$\begin{cases}\partial_t\phi_h=-\mathrm{div}_R((\nabla u)(t,\Phi(t,x))R\phi_h)-\mathrm{div}_R((\nabla u)_hR\phi(t,x+he_k,R))-L(\phi_h)\\ \psi_\infty\nabla\frac{\phi_h}{\psi_\infty}{}_{|\partial B}=0\end{cases} \tag{68}$$

Applying proposition 3.9, we deduce that $\phi_h\in C([0,\infty);L^2(\Omega;\mathcal{H}))\cap L^2_{loc}([0,\infty);L^2(\Omega;\mathcal{H}^1))$. Hence, taking the limit h to zero, we deduce that $\phi\in C([0,\infty);H^1(\Omega;\mathcal{H}))\cap L^2_{loc}([0,\infty);H^1(\Omega;\mathcal{H}^1))$. This gives a similar bound on $\psi(t,x,R)$. Moreover, we can take higher order derivatives and we can argue in a similar manner to prove the regularity of ψ stated in the proposition.

□

4. A priori estimates

From the first equation of (1), we deduce that

$$\partial_t|u|_s^2+\nu|u|^2_{s+1}\leq C|u|_s^3+\frac{C}{\nu}|\tau|_s^2. \tag{69}$$

From the second equation, we get

$$\partial_t\int_B\frac{\psi^2}{\psi_\infty}dR + u.\nabla\int_B\frac{\psi^2}{\psi_\infty}dR+\int_B\psi_\infty\left|\nabla_R\frac{\psi}{\psi_\infty}\right|^2 \tag{70}$$

$$\leq |Du|\left(\int_B\frac{\psi^2}{\psi_\infty}dR\right)^{1/2}\left(\int_B\psi_\infty\left|\nabla_R\frac{\psi}{\psi_\infty}\right|^2\right)^{1/2} \tag{71}$$

$$\leq C|Du|^2\left(\int_B\frac{\psi^2}{\psi_\infty}dR\right)+\frac{1}{2}\left(\int_B\psi_\infty\left|\nabla_R\frac{\psi}{\psi_\infty}\right|^2\right) \tag{72}$$

We define the flow Φ by

$$\begin{cases}\partial_t\Phi(t,x)=u(t,\Phi(t,x))\\ \Phi(0,x)=x\end{cases} \tag{73}$$

Integrating along the flow, we deduce that

$$\sup_x\int_B\frac{\psi^2(t)}{\psi_\infty}dR+\sup_x\int_0^t\int_B\psi_\infty\left|\nabla_R\frac{\psi}{\psi_\infty}\right|^2(s,\Phi(s,x))ds\leq\sup_x\int_B\frac{\psi_0^2}{\psi_\infty}dRe^{C\int_0^t|Du|^2_{L^\infty}} \tag{74}$$

Taking s derivatives in x and taking the L^2 norm in R, we get

$$\partial_t \int_B \frac{(\partial^s\psi)^2}{\psi_\infty} dR + u.\nabla \int_B \frac{(\partial^s\psi)^2}{\psi_\infty} dR \quad + \quad \int_B \psi_\infty \left|\nabla_R \frac{\partial^s \psi}{\psi_\infty}\right|^2 = \tag{75}$$

$$= \quad - \sum_{|\alpha|+|\beta|\leq s} \int_B div_R(D\partial^\alpha u R \partial^\beta \psi) \frac{\partial^{\alpha+\beta}\psi}{\psi_\infty} \tag{76}$$

Integrating by part and using the Cauchy-Schwarz inequality, the right hand side can be controlled by

$$\sum_{|\alpha|+|\beta|\leq s} |D\partial^\alpha u| \left(\int_B \frac{(\partial^\beta\psi)^2}{\psi_\infty}\right)^{1/2} \left(\int_B \left|\nabla_R \frac{\partial^{\alpha+\beta}\psi}{\psi_\infty}\right|^2\right)^{1/2} \tag{77}$$

$$\leq \sum_{|\alpha|+|\beta|\leq s} 2|D\partial^\alpha u|^2 \left(\int_B \frac{(\partial^\beta\psi)^2}{\psi_\infty}\right) + \frac{1}{4}\left(\int_B \psi_\infty \left|\nabla_R \frac{\partial^{\alpha+\beta}\psi}{\psi_\infty}\right|^2\right) \tag{78}$$

Integrating in the x variable, we get

$$\partial_t |\psi|_s^2 + \frac{1}{2}|\psi|_{s,1}^2 \leq C\left(|Du|_{L^\infty}^2 |\psi|_s^2 + |u|_{s+1}^2 \sup_x \int \frac{\psi^2}{\psi_\infty} dR\right) \tag{79}$$

We choose T such that

$$\int_0^T |u|_s^2 + |Du|_{L^\infty}^2 + |u|_s \leq A \tag{80}$$

for some fixed constant A. Hence,

$$\sup_{t,x} \int \frac{\psi^2}{\psi_\infty} dR \leq \sup_x \int \frac{\psi_0^2}{\psi_\infty} dR e^{CA} \tag{81}$$

Besides, integrating (79) in time and applying Gronwall lemma, we get

$$|\psi(t)|_s^2 + \frac{1}{2}\int_0^t |\psi|_{s,1}^2 \leq |\psi_0|_s^2 e^{CA} + Ce^{2CA}\int_0^t |u|_{s+1}^2. \tag{82}$$

Moreover, we have from (69)

$$|u(t)|_s^2 + \nu \int_0^t |u|_{s+1}^2 \leq (|u_0|_s^2 + \int_0^t |\tau|_s^2) e^{C\int_0^t |u|_s} \tag{83}$$

and from (10)

$$\int_0^t |\tau|_s^2 \leq \varepsilon \int_0^t |\psi|_{s,1}^2 + C_\varepsilon \int_0^t |\psi|_s^2 \leq (\varepsilon + C_\varepsilon T)e^{2CA}(C + \int_0^t |u|_{s+1}^2) \tag{84}$$

Hence, if ε and T are chosen small enough, we get

$$|u(t)|_s^2 + \frac{\nu}{2}\int_0^t |u|_{s+1}^2 \leq (|u_0|_s^2 + C)e^{CA}. \tag{85}$$

4.1. **Small data.** Here, we explain the changes we have to make in the small data case. Instead of using ψ, we will use $\tilde{\psi} = \psi - \psi_\infty$.

We have to use inequality (10) and (31) to bound the stress tensor τ. Hence

$$|\tau|_s^2 \leq |\tilde{\psi}|_{s,1}^2 + C|\tilde{\psi}|_s^2 \leq C|\tilde{\psi}|_{s,1}^2 \tag{86}$$

We assume that $\|Du\|_{L^\infty}^2 \leq \frac{\varepsilon}{C}$, that $\sup_x \int \frac{\tilde{\psi}^2}{\psi_\infty} dR \leq \frac{\varepsilon}{C}\nu^2$ and that $|u|_s \leq \frac{\varepsilon\nu}{C}$ for some ε small enough. Hence, (69) and (79) yield

$$\begin{cases} \partial_t|\tilde{\psi}|_s^2 + \frac{1}{2}|\tilde{\psi}|_{s,1}^2 \leq \varepsilon|\tilde{\psi}|_s^2 + \varepsilon\nu^2|u|_{s+1}^2 \\ \partial_t|u|_s^2 + \nu|u|_{s+1}^2 \leq \varepsilon\nu|u|_s^2 + \frac{C}{\nu}|\tilde{\psi}|_{s,1}^2. \end{cases} \tag{87}$$

Multiplying the second equation of (87) by $\frac{\nu}{4C}$ and adding the first one, we get

$$\partial_t(|\tilde{\psi}|_s^2 + \frac{\nu}{4C}|u|_s^2) + \frac{\min(1,\nu)}{8}(|\tilde{\psi}|_s^2 + \frac{\nu}{4C}|u|_s^2) \leq 0 \tag{88}$$

if ε is taken small enough compared to C. Hence, if the initial data satisfies $|\tilde{\psi}_0|_s^2 + \frac{\nu}{4C}|u_0|_s^2 \leq \frac{\varepsilon^2}{4C^3}\nu\min(1,\nu)^2$, we see that the assumptions made before (87) hold and hence $|\tilde{\psi}|_s^2 + \frac{\nu}{4C}|u|_s^2 \leq \frac{\varepsilon^2}{4C^3}\nu\min(1,\nu)^2 e^{-\min(1,\nu)t/8}$.

5. Existence proofs

In this section, we prove theorem 2.1 and 2.2. To prove the existence and uniqueness of a solution in theorem 2.1, we use a fixed point argument. For $T > 0$, we define $X = C_T(H^s) \cap L_T^2(H^{s+1}) \times C_T(H^s(\mathcal{H}))$. We define the operator Φ from X to X by $\Phi((u,\phi)) = (v,\psi)$ where ψ is the unique solution in $C_T(H^s(\mathcal{H})) \cap L_T^2(H^s(\mathcal{H}^1))$ of

$$\begin{cases} \partial_t\psi + u.\nabla_x\psi = -\mathrm{div}_R(\nabla u R\psi) - L(\psi) \\ \psi_\infty \nabla \frac{\psi}{\psi_\infty}|_{\partial B} = 0 \\ \psi(t=0) = \psi_0 \end{cases} \tag{89}$$

Then v is the unique solution of the following linear problem

$$\begin{cases} \partial_t v + u.\nabla v - \nu\Delta v = \mathrm{div}(\tau) \\ \mathrm{div} v = 0 \\ v(t=0) = u_0 \end{cases} \tag{90}$$

where τ is deduced from ψ. Let X_0 be given by

$$X_0 = \{(u,\phi) \in X| \sup_{0\leq t\leq T} |u|_s^2 + \frac{\nu}{2}\int_0^T |u|_{s+1}^2 \leq 9|u_0|_s^2 + 1, \quad \sup_{0\leq t\leq T} |\phi|_s^2 \leq A_1\} \tag{91}$$

where $A_1 = (1 + \frac{Ce(20|u_0|_s^2+2)}{\nu})e|\psi|_0^2$. We assume that T satisfies

$$T \leq \frac{1}{C(9|u_0|_s^2+1)}. \tag{92}$$

If $(u,\phi) \in X_0$, then $(v,\psi) = \Phi(u,\phi)$ satisfies

$$\sup_{t,x} \int \frac{\psi^2}{\psi_\infty} dR \leq \sup_x \int \frac{\psi_0^2}{\psi_\infty} dR\, e^{C\int_0^T |u_0|_s^2} \leq e|\psi_0|_s^2. \tag{93}$$

Moreover,

$$\sup_{0\leq t\leq T} |\psi(t)|_s^2 + \frac{1}{2}\int_0^T |\psi|_{s,1}^2 \quad \leq \quad |\psi_0|_s^2 e + C\, e \sup_{t,x} \int \frac{\psi^2}{\psi_\infty} dR \int_0^t |u|_{s+1}^2 \leq A_1. \tag{94}$$

Hence,

$$\int_0^T |\tau|_s^2 \leq \varepsilon \int_0^t |\psi|_{s,1}^2 + C_\varepsilon \int_0^t |\psi|_s^2 \leq 2\varepsilon A_1 + C_\varepsilon T A_1 \tag{95}$$

$$\sup_{0\leq t\leq T} |v(t)|_s^2 + \frac{\nu}{2}\int_0^T |v|_{s+1}^2 \leq (|u_0|_s^2 + \frac{C}{\nu}A_1(2\varepsilon + C_\varepsilon T))e. \tag{96}$$

We take ε such that $\frac{C}{\nu}A_1 2\varepsilon < 1/8$ and then T such that $\frac{C}{\nu}A_1 C_\varepsilon T < 1/8$. Hence, the right hand side of (96) is bounded by $9|u_0|_s^2 + 1$. Next, we have to prove that Φ is a contraction on X_0. We put the L^2 norm on X_0, namely

$$\|(u,\phi)\|_{X_0} = \sup_{0\leq t\leq T} \|u\|_{L^2}^2 + \frac{\nu}{2}\int_0^T \|u\|_{H^1}^2 + \sup_{0\leq t\leq T} \|\phi\|_{L^2(\mathcal{H})}^2 \tag{97}$$

We want to prove by similar energy estimates that if T is taken even smaller then

$$\|\Phi(u_1,\phi_1) - \Phi(u_2,\phi_2)\|_{X_0} \leq 1/2\|(u_1,\phi_1) - (u_2,\phi_2)\|_{X_0} \tag{98}$$

Indeed, we denote $(v_i,\psi_i) = \Phi(u_i,\phi_i)$ for $i = 1$ or 2 and define $(v,\psi) = (v_2 - v_1, \psi_2 - \psi_1)$. Hence, $(v,\psi)(t=0) = 0$ and

$$\begin{cases} \partial_t\psi + u_2.\nabla_x\psi + (u_2 - u_1).\nabla_x\psi_1 = -\mathrm{div}_R(\nabla u_2 R\psi) - \mathrm{div}_R(\nabla(u_2 - u_1)R\psi_1) - L(\psi) \\ \partial_t v + u_2.\nabla_x v + (u_2 - u_1).\nabla_x v_1 - \nu\Delta v + \nabla p = \nabla(\tau_2 - \tau_1). \end{cases} \tag{99}$$

A simple computation, similar to the H^s estimate, yields that

$$\partial_t(|v|_0^2 + |\psi|_0^2) + |\nabla v|_0^2 \leq C(|v|_0^2 + |\psi|_0^2 + |u_2 - u_1|_0^2 + |\phi_2 - \phi_1|_0^2).$$

Hence, taking T smaller if necessary, we see that (97) holds. This proves that Φ is a contraction and yields the existence an uniqueness of a solution in the space X.

To prove that the solution is actually, unique in $C([0,T^*); H^s) \times C([0,T^*); H^s(\mathbb{R}^N; L^2(\frac{dR}{\psi_\infty})))$, we can use the same computation (99) were $(v_i,\psi_i) = (u_i,\phi_i)$. Without loss of generality, we can assume that the solution (v_2,ψ_2) is the solution given by the fixed point argument in X. Hence, Gronwall lemma implies that $(v,\psi) = (0,0)$ which gives the uniqueness.

5.1. **The small data case.** Now, we turn to the proof of the global existence if the data is small. Using the local existence result of the previous subsection, we get a solution (u,ψ) on a time interval $[0,T^*)$. We would like to prove that we can take $T^* = \infty$, The a priori estimate of subsection 4.1 implies that $(|\tilde{\psi}|_s^2 + \frac{\nu}{4C}|u|_s^2)$ decreases on the time interval $[0,T^*)$. Then using that the existence time T in the previous subsection only depends on $|u_0|_s^2$ and $|\tilde{\psi}_0|_s^2$, we see that we can iterate the argument and prove the global existence. This proves theorem 2.2.

Remark 5.1. *An other way of proving the global existence for small data is to use a fixed point argument on* $[0,\infty)$ *and take advantage of the fact that the data is small to prove that* Φ *is a contraction on some* X_0 *(which is global in time) to be chosen accordingly.*

6. The co-rotational model in 2D

Let us start by explaining the idea of the proof of theorem 2.3. The main difference between the full model (1) and the co-rotational model is that we have here an extra a priori estimate, namely for $r > 1$

$$\partial_t \int_B \psi_\infty \left|\frac{\psi}{\psi_\infty}\right|^r dR + u.\nabla \int_B \psi_\infty \left|\frac{\psi}{\psi_\infty}\right|^r dR = -\frac{4(r-1)}{r}\int_B \psi_\infty \left|\nabla\left(\frac{\psi}{\psi_\infty}\right)^{r/2}\right|^2 dR \tag{100}$$

This yields an L^∞ bound on $\int_B \psi_\infty \left|\frac{\psi}{\psi_\infty}\right|^r dR$. Combining this with (34) when $(r-1)k > 1$, we get and L^∞ bound for the additional stress τ which is uniform in time. In [6], while studying the Oldroyd-B model, the authors proved that a control on the L^∞ norm of τ yields global existence in the 2d case. The ideas of [6] were then used in [25] and [9] in the micro-macro case.

Here, we follow the proof of [9]. For this theorem we use the Littlewood-Paley decomposition.

6.1. **Preliminaries.** We define $\mathcal{C}$ to be the ring of center 0, of small radius 1/2 and great radius 2. There exist two nonnegative radial functions χ and φ belonging respectively to $\mathcal{D}(B(0,1))$ and to $\mathcal{D}(\mathcal{C})$ so that

$$\chi(\xi) + \sum_{q\geq 0} \varphi(2^{-q}\xi) = 1, \tag{101}$$

$$|p-q| \geq 2 \Rightarrow \text{Supp } \varphi(2^{-\mathrm{q}}\cdot) \cap \text{Supp } \varphi(2^{-\mathrm{p}}\cdot) = \emptyset. \tag{102}$$

For instance, one can take $\chi \in \mathcal{D}(B(0,1))$ such that $\chi \equiv 1$ on $B(0,1/2)$ and take

$$\varphi(\xi) = \chi(\xi/2) - \chi(\xi).$$

Then, we are able to define the Littlewood-Paley decomposition. Let us denote by $\mathcal{F}$ the Fourier transform on $\mathbb{R}^d$. Let $h,\ \widetilde{h},\ \Delta_q,\ S_q\ (q \in \mathbb{Z})$ be defined as follows:

$$\begin{aligned} h = \mathcal{F}^{-1}\varphi \quad &\text{and} \quad \widetilde{h} = \mathcal{F}^{-1}\chi, \\ \Delta_q u = \mathcal{F}^{-1}(\varphi(2^{-q}\xi)\mathcal{F}u) &= 2^{qd}\int h(2^q y)u(x-y)dy, \\ S_q u = \mathcal{F}^{-1}(\chi(2^{-q}\xi)\mathcal{F}u) &= 2^{qd}\int \widetilde{h}(2^q y)u(x-y)dy. \end{aligned}$$

We use the para-product decomposition of Bony ([4])

$$uv = T_u v \ + \ T_v u \ + \ R(u,v)$$

where

$$T_u v = \sum_{q\in\mathbb{Z}} S_{q-1}u\Delta_q v \quad \text{and} \quad R(u,v) = \sum_{|q-q'|\leq 1} \Delta_{q'}u\Delta_q v.$$

We define the inhomogeneous and homogeneous Besov spaces by

Definition 6.1. *Let s be a real number, p and r two real numbers greater than 1. Then we define the following norm*

$$\|u\|_{\widetilde{B}^s_{p,r}} \stackrel{def}{=} \|S_0 u\|_{L^p} + \left\|(2^{qs}\|\Delta_q u\|_{L^p})_{q\in\mathbb{N}}\right\|_{\ell^r(\mathbb{N})}$$

and the following semi-norm

$$\|u\|_{B^s_{p,r}} \stackrel{def}{=} \left\|(2^{qs}\|\Delta_q u\|_{L^p})_{q\in\mathbb{Z}}\right\|_{\ell^r(\mathbb{Z})}.$$

Definition 6.2.

- *Let s be a real number, p and r two real numbers greater than 1. We denote by $\widetilde{B}^s_{p,r}$ the space of tempered distributions u such that $\|u\|_{\widetilde{B}^s_{p,r}}$ is finite.*
- *If $s < d/p$ or $s = d/p$ and $r = 1$ we define the homogeneous Besov space $B^s_{p,r}$ as the closure of compactly supported smooth functions for the norm $\|\cdot\|_{B^s_{p,r}}$.*

We refer to [5] for the proof of the following results and for the multiplication law in Besov spaces.

Lemma 6.3.

$$\|\Delta_q u\|_{L^b} \leq 2^{d(\frac{1}{a}-\frac{1}{b})q}\|\Delta_q u\|_{L^a} \quad \text{for } b \geq a \geq 1$$

$$\|e^{t\Delta}\Delta_q u\|_{L^b} \leq C2^{-ct2^{2q}}\|\Delta_q u\|_{L^b}$$

The following corollary is straightforward.

Corollary 6.4. *If $b \geq a \geq 1$, then, we have the following continuous embeddings*

$$B^s_{a,r} \subset B^{s-d\left(\frac{1}{a}-\frac{1}{b}\right)}_{b,r}.$$

Definition 6.5. *Let p be in $[1,\infty]$ and r in $\mathbb{R}$; the space $\widetilde{L}^p_T(C^r)$ is the space of distributions u such that*

$$\|u\|_{\widetilde{L}^p(0,T;C^r)} \stackrel{def}{=} \sup_q 2^{qr}\|\Delta_q u\|_{L^p_T(L^\infty)} < \infty.$$

We will use the following theorem from [6]

Theorem 6.6. *Let v be the solution in $L^2_T(H^1)$ of the two dimensional Navier-Stokes system*

$$(NS_\nu)\begin{cases} \dfrac{\partial v}{\partial t} + v\cdot\nabla v - \nu\Delta v &= -\nabla p + f \\ \mathrm{div} v &= 0 \\ v|_{t=0} &= v_0. \end{cases}$$

with an initial data in L^2 and an external force f in $L^1_T(C^{-1})\cap L^2_T(H^{-1})$; then, for any ε, a T_0 in the interval $]0,T[$ exists such that

$$\|\nabla v\|_{\widetilde{L}^1((T_0,T);C^0)} \leq \varepsilon.$$

6.2. **A deteriorating regularity estimate.** The main part of this subsection is the proof of a deteriorating regularity estimate for transport equations in the spirit of [1] and [6]. After this proof, we will apply this estimate in order to prove Theorem 2.3.

Theorem 6.7. *Let σ and β be two elements of $]0,1[$ such that $\sigma+\beta<1$. A constant C exists that satisfies the following properties. Let T and λ be two positive numbers and v a smooth divergence free vector field so that*

$$\sigma - \lambda\|\nabla v\|_{\widetilde{L}^1_T(C^0)} \geq \beta \quad \text{and} \quad \sigma + \lambda\|\nabla v\|_{\widetilde{L}^1_T(C^0)} \leq 1-\beta. \tag{103}$$

Consider two smooth functions f and v so that f is the solution of

$$\begin{cases} \partial_t f + v\cdot\nabla f = -\mathrm{div}_R(W(v)Rf) + \mathrm{div}_R(\psi_\infty\nabla\frac{f}{\psi_\infty}) \\ \hfill f_{|t=0} = f_0. \end{cases} \tag{104}$$

Then we have, if $\lambda \geq 3C$ and $|f_0|_{L^\infty(\mathcal{L}^r)} \leq C$,

$$M^\sigma_\lambda(f) \leq 3\|f_0\|_{B^\sigma_{p,\infty}(\mathcal{L}^r)} + \frac{3C}{\lambda}M^{\sigma+1}_\lambda(v) \tag{105}$$

where

$$M^\sigma_\lambda(v) \stackrel{def}{=} \sup_{t\in[0,T],q} 2^{q\sigma-\Phi_{q,\lambda}(t)}\|\Delta_q v(t)\|_{L^p} \quad \textit{and} \tag{106}$$

$$M^\sigma_\lambda(f) \stackrel{def}{=} \sup_{t\in[0,T],q} 2^{q\sigma-\Phi_{q,\lambda}(t)}\|\Delta_q f(t)\|_{L^p(\mathcal{L}^r)} \quad \textit{with} \tag{107}$$

$$\Phi_{q,\lambda}(t,t') \stackrel{def}{=} \lambda\int_{t'}^t (\|S_{q-1}\nabla v(t'')\|_{L^\infty}+1)dt'',\ \Phi_{q,\lambda}(t) = \Phi_{q,\lambda}(t,0). \tag{108}$$

Proof. The proof is the same as the proof of theorem 2.1 of [9], the only difference is that we have to replace the space H^{-s} in the R variable by the space $\mathcal{L}^r$ and that we have to split the rest term into 2 parts and use integration by parts to estimate one of them. We give a sketch of the proof here. We will use the notation $f_q \stackrel{\text{def}}{=} \Delta_q f$. Applying the operator Δ_q to the transport equation (104), we get

(109)
$$\begin{cases} \partial_t f_q + S_{q-1}v \cdot \nabla f_q + \operatorname{div}_R(W(S_{q-1}v)Rf) + \operatorname{div}_R(\psi_\infty \nabla \frac{f_q}{\psi_\infty}) + \tilde{R}_q(v,f) + \operatorname{div}_R(R_q(v,f)) = 0 \\ f_{q|t=0} = \Delta_q f_0. \end{cases}$$

where $\tilde{R}_q(v,f) + \operatorname{div}_R(R_q(v,f))$ is a rest term which will be computed later. We denote

$$N_q^r(t,x) = \int_B \psi_\infty \left| \frac{f_q}{\psi_\infty} \right|^r dR = |f_q|^r_{\mathcal{L}^r}. \tag{110}$$

Hence, multiplying (109) by $\left(\frac{f_q}{\psi_\infty}\right)^{r-1}$ and integrating in R, we get

$$\partial_t N_q^r + S_{q-1}v \cdot \nabla N_q^r + \frac{4(r-1)}{r} |f_q|^r_{\dot{\mathcal{L}}^{r,1}} \quad = \quad - \int_B \tilde{R}_q(v,f) \left(\frac{f_q}{\psi_\infty}\right)^{r-1} - R_q(v,f).\nabla \left(\frac{f_q}{\psi_\infty}\right)^{r-1} \tag{111}$$

The right hand side is controlled by

$$C(|f_q|^{r-1}_{\mathcal{L}^r} |\tilde{R}_q|_{\mathcal{L}^r} + |f_q|^{\frac{r}{2}-1}_{\mathcal{L}^r} |f_q|^{\frac{r}{2}}_{\mathcal{L}^{r,1}} |R_q|_{\mathcal{L}^r}) \tag{112}$$

Hence, we get

$$\partial_t N_q + S_{q-1}v \cdot \nabla N_q \leq C(N_q + \|R_q\|_{\mathcal{L}^r} + \|\tilde{R}_q\|_{\mathcal{L}^r}) \tag{113}$$

We also recall from (100) that $|f|_{L^\infty_T(L^\infty(\mathcal{L}^r))} \leq |f_0|_{L^\infty(\mathcal{L}^r)} \leq C$.

To prove theorem 6.7, we have to prove the following lemma to control the rest term and then argue exactly as in [9] to conclude

Lemma 6.8. *$R_q(v,f)$ and $\tilde{R}_q(v,f)$ satisfy*

$$\begin{aligned} &2^{q\sigma - \Phi_{q,\lambda}(t)} (\|R_q(v(t),f(t))\|_{L^p(\mathcal{L}^r)} + \|\tilde{R}_q(v(t),f(t))\|_{L^p(\mathcal{L}^r)}) \leq Ce^{C\lambda \|\nabla v\|_{\tilde{L}^1_T(C^0)}} \\ &\qquad \times \Big(M_\lambda^{\sigma+1}(v) + \Big(1 + \|S_q \nabla v(t)\|_{L^\infty} + \sum_{|q'-q|\leq N} \|\Delta_{q'} \nabla v(t)\|_{L^\infty}\Big) M_\lambda^\sigma(f)\Big). \end{aligned} \tag{114}$$

To prove Lemma 6.8, we have to split $\tilde{R}_q + \operatorname{div}_R(R_q)$ into several terms and analyze each one separately. Here, we will only focus on the term which is not in [9]. Indeed, as in [9], we have

$$
\begin{aligned}
\tilde{R}_q(v,f) &= \sum_{\ell=1}^{3} R_q^\ell(v,f) \quad \text{and} \quad \mathrm{div}_R(R_q(v,f)) = \sum_{\ell=4}^{6} \mathrm{div}_R(R_q^\ell(v,f)) \text{ with}\\
R_q^1(v,f) &= \sum_{j=1}^{d} \Delta_q(T_{\partial_j f} v^j),\\
R_q^2(v,f) &= \sum_{j=1}^{d} [\Delta_q, T_{v^j}\partial_j] f,\\
R_q^3(v,f) &= \sum_{j=1}^{d} \Delta_q \partial_j R(v^j, f) + \Delta_{q-1} v^j \partial_j \Delta_{q+1} f_q - \Delta_{q-2} v^j \partial_j \Delta_{q-1} f_q\\
R_q^4(v,f) &= \sum_{i,j=1}^{d} \Delta_q(T_f W(v)^i_j)\\
R_q^5(v,f) &= \sum_{i,j=1}^{d} [\Delta_q, T_{W(v)^i_j}] f\\
R_q^6(v,f) &= \sum_{i,j=1}^{d} \Big(R(W(v)^i_j, f) + \Delta_{q-1} W(v)^i_j \Delta_{q+1} f_q - \Delta_{q-2} W(v)^j_i \Delta_{q-1} f_q \Big).
\end{aligned}
$$

The first three terms are exactly treated as in [9]. The last three terms come from

$$
\Delta_q(\mathrm{div}_R(W(v)Rf)) = \mathrm{div}_R\Big(\sum_{\ell=4}^{6} R_q^\ell(v,f)\Big) + \mathrm{div}_R(W(S_{q-1}v) f_q).
$$

Here, we only explain the estimate for $R_q^4(v,f)$ and $R_q^5(v,f)$. The estimate for $R_q^6(v,f)$ is the same as $\tilde{R}_q^3(v,f)$. We have

$$
\begin{aligned}
\|R_q^4(v(t), f(t))\|_{L^p(\mathcal{L}^r)} &\leq C \sum_{|q-q'|\leq 2} \|S_{q'-1} f\|_{L^\infty(\mathcal{L}^r)} \|\Delta_{q'} \nabla v(t)\|_{L^p}\\
&\leq C \sum_{|q-q'|\leq 2} \|\Delta_{q'} \nabla v(t)\|_{L^p}
\end{aligned}
$$

and hence (114) holds for R_q^4. For $R_q^5(v,f)$, we have

$$
[\Delta_q, T_{W(v)^i_j}] f = -\sum_{j=1}^{d} \sum_{q'} [S_{q'-1} W(v)^i_j, \Delta_q] \Delta_{q'} f.
$$

The terms of the above sum are equal to 0 except if $|q - q'| \leq 2$. We also recall that $2W(v)^i_j = \partial_j v^i - \partial_i v^j$. By definition of the operators Δ_q, we have

$$
[S_{q'-1}\partial_j v^i, \Delta_q] \Delta_{q'} f(x) = 2^{qd} \int_{\mathbb{R}^d} h(2^q(x-y))(S_{q'-1}\partial_j v^i(x) - S_{q'-1}\partial_j v^i(y)) \Delta_{q'} f(y) dy.
$$

So we infer that

$$
\|R_q^5(v,f)\|_{\mathcal{L}^r} \leq 2^{-q} |\nabla^2 S_{q'-1} v| 2^{qd} \Big(\Big(2^q |\cdot| \times |h(2^q \cdot)|\Big) \star \|\Delta_{q'} f\|_{\mathcal{L}^r} \Big)(x).
$$

Hence,

$$
\|R_q^5(v,f)\|_{L^p(\mathcal{L}^r)} \leq 2^{-q} \|\nabla^2 S_{q'-1} v\|_{L^p} \|\Delta_{q'} f\|_{L^\infty(\mathcal{L}^r)}.
$$

Then, we have,

$$2^{q\sigma-\Phi_{q,\lambda}(t)}\|HR_q^5(v,f)\|_{L^p(\mathcal{L}^r)} \leq C \sum_{\substack{|q-q'|\leq 2\\ q''\leq q'-1}} 2^{(\sigma-1)(q-q'')-\Phi_{q,\lambda}(t)+\Phi_{q'',\lambda}(t)} M_\lambda^{\sigma+1}(v)\|\Delta_{q'}f\|_{L^\infty(\mathcal{L}^r)}.$$

Hence,

$$2^{q\sigma-\Phi_{q,\lambda}(t)}\|HR_q^5(v,f)\|_{L^p(\mathcal{L}^r)} \leq C \sum_{q''\leq q+1} 2^{-\beta(q-q'')} M_\lambda^{\sigma+1}(v)\|f\|_{L^\infty(\mathcal{L}^r)}$$

and the sum is uniformly bounded since $\sigma - 1 + \lambda\|\nabla v\|_{\widetilde{L}^1_T(C^0)} \leq -\beta < 0$. This ends the proof of Lemma 6.8 and of theorem 6.7.

□

6.3. **Proof of theorem 2.3.** The proof follows the same ideas as in Chemin and Masmoudi [6], Lin, Zhang and Zhang [25] and Constantin and Masmoudi [9]. We will only sketch the proof.

First, we notice that theorem 2.1 yields the local existence of a solution with $u \in L^\infty_{loc}([0,T^*);H^s) \cap L^2_{loc}([0,T^*);H^{s+1})$ and $\psi \in L^\infty_{loc}([0,T^*);H^s(\mathcal{H}))$. Moreover, estimating $\partial^s\psi$ in $\mathcal{L}^r$, we deduce that $\psi \in L^\infty_{loc}([0,T^*);H^s(\mathcal{L}^r))$. (see (79) for a similar estimate when $r=2$). Besides, from regularity estimates for the heat equation, we have for all $0 < T_0 < T$, $u \in L^\infty_{loc}((T_0,T^*);H^{s+1-\varepsilon})$.

To prove that we can extend the solution beyond the time T^*. It is enough to prove that $\nabla u \in L^\infty((0,T^*)\times\mathbb{R}^2)$.

By Sobolev type embeddings of Corollary 6.4, we have

$$(u,\psi) \in L^\infty_{loc}\Big([T_0,T[;\widetilde{B}^{s-\varepsilon}_{p,\infty}\times\widetilde{B}^{s-1}_{p,\infty}(\mathcal{L}^r)\Big).$$

This implies that $(u,\psi) \in L^\infty_{loc}(\widetilde{C}^{1+\sigma}\times\widetilde{C}^\sigma(\mathcal{L}^r))$ for any $0<\sigma<1$. We fix such a σ. So we can apply Theorem 6.6 and we can choose T_0 such that, with the notations of Theorem 6.7, we have

$$\|\nabla u\|_{\widetilde{L}^1(T_0,T;C^0)} \leq \frac{min(\sigma-\beta,1-\sigma-\beta)}{3\lambda}.$$

The deteriorating regularity estimate of Theorem 6.7 applied with σ and between T_0 and T tells exactly that ψ satisfies

$$M_\lambda^\sigma(\psi) \leq 3\|\psi\|_{C^\sigma(\mathcal{L}^r)} + \frac{3C}{\lambda}M_\lambda^{\sigma+1}(u). \tag{115}$$

Now, we have to estimate ∇u using that u solves the two dimensional Navier-Stokes equation. Arguing as in [6] and [9] and using that $M_\lambda^\sigma(\tau) \leq M_\lambda^\sigma(\psi)$, we deduce that

$$M_\lambda^{\sigma+1}(u) \leq \|u_0\|_{C^{\sigma+1}} + \frac{3C}{\nu}\|\psi_0\|_{C^\sigma} + \left(\frac{C}{\lambda}+\frac{C}{\lambda\nu}+\frac{C}{\nu^{\frac{3}{4}}}\|u\|_{L^4_{T_0,T]}(H^{\frac{1}{2}})}\right)M_\lambda^{\sigma+1}(u)$$

Now it is enough to choose T_0 such that the quantity

$$\left(\frac{C}{\lambda}+\frac{C}{\lambda\nu}+\frac{C}{\nu^{\frac{3}{4}}}\|u\|_{L^4_{T_0,T]}(H^{\frac{1}{2}})}\right)$$

is small enough. Then as σ is greater than 0, u is such that ∇u belongs to $L^\infty([T_0,T]\times\mathbb{R}^2)$; this concludes the proof of Theorem 2.3.

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