Stability of oscillating boundary layers in rotating fluids

Stabilité de couches limites oscillantes dans les fluides tournant

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Abstract

We prove the linear and non-linear stability of oscillating Ekman boundary layers for rotating fluids in the so-called ill-prepared case under a spectral hypothesis. Here, we deal with the case where the viscosity and the Rossby number are both equal to ε. This study generalizes the study of [22] where a smallness condition was imposed and the study of [25] where the well-prepared case was treated.

Résumé

On prouve la stabilité linéaire et non-linéaire de couches limites oscillantes de type Ekman pour les fluides tournant dans le cas de données mal préparées sous une hypothèse spectrale. On s'intéresse au cas où la viscosité et le nombre de Rossby sont du même ordre ε. Cette étude généralise celle de [22] où une condition de petitesse était imposée et celle de [25] où les données bien préparées étaient traitées.

1 Introduction

We consider the following system describing the evolution of a rotating fluid in a rectangular domain

$$\left\{ \begin{array}{l} \partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon + \frac{e\times u^\varepsilon}{\varepsilon} + \frac{\nabla p}{\varepsilon} - \varepsilon \Delta u^\varepsilon = 0, \\ u^\varepsilon(t=0) = u^{\varepsilon,0} \qquad \nabla \cdot u^\varepsilon = 0 \end{array} \right. \tag{1}$$

for $x = (y,z) \in \Omega = \mathbb{T}^2_a \times (0,1)$ with the Dirichlet boundary condition

$$u^\varepsilon_{/\partial\Omega} = 0 \tag{2}$$

and the initial condition

$$u^\varepsilon_{/t=0} = u^{\varepsilon,0}. \tag{3}$$

Here $\mathbb{T}^2_a$ is the periodic torus with periods a_1 and a_2, namely, $\mathbb{T}^2_a = \mathbb{R}^2/(\frac{1}{a_1}\mathbb{Z} \times \frac{1}{a_2}\mathbb{Z})$ and a_1, $a_2 > 0$. Moreover, $e = e_3$ is the vertical unit vector and $\frac{e\times u^\varepsilon}{\varepsilon}$ is the Coriolis force.

This system describes the motion of a rotating fluid as the Ekman and Rossby numbers go to zero (see Pedlovsky [24], and Greenspan [13]). It can model the dynamics of the ocean or the

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atmosphere far from the equator or a rotating fluid in a container. Note that, here, we take the horizontal viscosity and the vertical viscosity to be equal. We point out that in many previous works the horizontal viscosity was supposed constant whereas the vertical viscosity ν goes to 0 (see for instance [16]) or in some other cases, the vertical viscosity was supposed much smaller than the horizontal viscosity. This anisotropy has the advantage of making the boundary layers more stable.

In this paper, we look at the case where the vertical and the horizontal viscosities are equal. We study the convergence of solutions to (1) towards a solution of the limit system (9) defined below once the time oscillations are filtered out.

We recall that this system and related ones were studied by several authors. In the "well-prepared" case in domains with boundary, like Ω, we refer to Colin, Fabrie [4], Grenier, Masmoudi [16], Masmoudi [21]. For general initial data, and for the periodic case, we refer to Grenier [14], Embid and Majda [8], Babin, Mahalov and Nicolaenko [2, 1], Gallagher [11] or in particular cases where there is no boundary layer, or where the boundary layer can be eliminated by symmetry (Beale and Bourgeois [3]). These results rely on the introduction of a group to filter the oscillations in time, a method which was previously used by Schochet [29] to investigate related problems in the torus concerning the compressible-incompressible limit.

In [22], the "group method" was extended to the case of domains with boundary, by solving a superposition of an infinite number of boundary layers. These layers create an extra term in the limit equation. In [22], the stability of these boundary layers was proved in the case where the horizontal viscosity goes to zero slower than the Rossby number (or in the small data case). In this paper, we would like to give a spectral assumption (which we think is optimal) and which yields the stability of such boundary layers.

In the well prepared case, a similar spectral assumption was used to prove the stability of the boundary layer [25]. This spectral assumption is optimal since the instability of the boundary layer was proved in [6] if the spectral assumption does not hold.

In the following sections, we recall the main properties of the approximate solution of (1) constructed in [22], in particular, we recall the properties of the limit system, of the boundary layers and the assumptions on the torus which are needed. Next, we shall give our main assumption on the spectral stability of the boundary layers and state our main result.

1.1 Properties of the approximate solution

To state our main result, we first recall the main properties of the approximate solution u^{app} of (1) constructed in [22]. In particular, u^{app} describes the formal limit of (1) and the boundary layers. The details of the construction will be recalled later. The approximate solution is under the form

$$u^{app} = u^{int}(\frac{t}{\varepsilon}, t, x) + u^b(\frac{t}{\varepsilon}, \frac{z}{\varepsilon}, \frac{1-z}{\varepsilon}, t, y) + u^r, \quad x = (y, z) \in \mathbb{T}_a^2 \times (0, 1) \tag{4}$$

where the remainder term u^r satisfies $u^r = \mathcal{O}(\varepsilon)$ (a precise statement will be given later). The interior term u^{int} can be expressed as

$$u^{int}(\tau, t, x) = \mathcal{L}(\tau) w^{int}(t, x)$$

where $\mathcal{L}(\tau) = e^{\tau L}$, $Lu = -\mathbb{P}(e \times u)$ and $\mathbb{P}$ is the Leray projector on divergence-free vector fields with zero normal component in Ω. We denote $\mathbb{Z}_a^3 = \frac{2\pi}{a_1}\mathbb{Z} \times \frac{2\pi}{a_2}\mathbb{Z} \times \frac{2\pi}{2}\mathbb{Z}$, $\mathbb{Z}_a^2 = \frac{2\pi}{a_1}\mathbb{Z} \times \frac{2\pi}{a_2}\mathbb{Z}$ and we denote elements of $\mathbb{Z}_a^3$ by $\bar{k} = (k, k_3) \in \mathbb{Z}_a^3$ with $k \in \mathbb{Z}_a^2 = \frac{2\pi}{a_1}\mathbb{Z} \times \frac{2\pi}{a_2}\mathbb{Z}$. We have an expansion

$$w^{int}(t, x) = \sum_{\bar{k} \in \mathbb{Z}_a^3} b(t, \bar{k}) e^{ik \cdot y} M^{\bar{k}}(z), \tag{5}$$

so that

$$u^{int}(\tau, t, x) = \mathcal{L}(\tau)(w^{int}(t, x)) = \sum_{\bar{k} \in \mathbb{Z}_a^3} b(t, \bar{k}) e^{ik \cdot y} M^{\bar{k}}(z) e^{i\lambda(\bar{k})\tau} \tag{6}$$

and w^{int} solves the limit system (9). Note that $N^{\bar{k}} = e^{ik\cdot y} M^{\bar{k}}$ is an eigenvector of L. We assume that the initial data is chosen such that $b(0, (0, k_3)) = 0$ for every k_3 i.e. we exclude initial values with modes which depend only on z. We shall also assume that the torus is non resonant in the sense of [2] to insure that the condition $b(t, (0, k_3)) = 0$ for every k_3 remains true for positive times (see below for a precise definition).

We can express the dominant boundary layer term u^b as

$$u^b(\tau, Z, Z', t, y) = u^{b,0}(\tau, Z, t, y) + u^{b,1}(\tau, Z', t, y)$$

where

$$u^{b,\sigma}(\tau, Z, t, y) = -\frac{1}{2} \sum_{\bar{k}} b(t, \bar{k}) e^{ik\cdot y + i\lambda(\bar{k})\tau} (-1)^{\sigma k_3} \Big(h^{\bar{k},+} e^{-\frac{1+i}{\sqrt{2}}\eta^{\bar{k},+} Z} + h^{\bar{k},-} e^{-\frac{1-i}{\sqrt{2}}\eta^{\bar{k},-} Z} \Big), \quad \sigma = 0,\, 1.$$

with

$$\eta^{\bar{k},\pm} = \sqrt{1 \pm \lambda(\bar{k})}, \quad h^{\bar{k},\pm} = M^{\bar{k}}(0) \mp ie \times M^{\bar{k}}(0).$$

Note that since terms under the form $(0, k_3)$ are excluded in the above sum, we have $\eta^{\bar{k},\pm} > 0$ and hence, we have a superposition of terms which are small far from the boundary. Nevertheless, the rate of decay, $\eta^{\bar{k},\pm}$, goes to zero when $\frac{k_3}{|k|}$ tends to ± 1.

In view of the above definition of the boundary layers, we introduce the operators

$$\mathcal{B}^\sigma(\tau, Z) q = -\frac{1}{2} \sum_{\bar{k}} q_{\bar{k}} e^{i\lambda(\bar{k})\tau} (-1)^{\sigma k_3} \Big(h^{\bar{k},+} e^{-\frac{1+i}{\sqrt{2}}\eta^{\bar{k},+} Z} + h^{\bar{k},-} e^{-\frac{1-i}{\sqrt{2}}\eta^{\bar{k},-} Z} \Big)$$

for any sequence $q = (q_{\bar{k}})_{\bar{k} \in \mathbb{Z}_a^3}$ so that if q is taken under the form $q_{\bar{k}} = b(t, \bar{k}) e^{ik\cdot y}$, we have $\mathcal{B}^\sigma(\tau, y, Z) q = u^{b,\sigma}(\tau, Z, t, y)$.

In a similar way, we also define

$$\mathcal{L}^\sigma(\tau) q = \sum_{\bar{k}} q_{\bar{k}} M^{\bar{k}}(\sigma) e^{i\lambda(\bar{k})\tau}, \quad \sigma = 0,\, 1.$$

Again, note that if q is taken such that $q_{\bar{k}} = b(t, \bar{k}) e^{ik\cdot y}$ then, we have $\mathcal{L}^\sigma(\tau) q = w(t, y, \sigma)$.

We shall allways assume that the initial data is sufficiently smooth and vanishes at a sufficient order at $z = 0$, $z = 1$ in order that $b(t, \bar{k})$ decay to zero sufficiently fast. In particular, we assume that

$$\|w^0\|^2_{V^s_{sym}} = \sum_{\bar{k} \in \mathbb{Z}_a^3} |b(0, \bar{k})|^2 |\bar{k}|^{2s} < \infty \quad \text{for some } s \text{ big enough.} \tag{7}$$

This yields that $w(t) \in V^s_{sym}$ for $0 < t < T^*$ where T^* is the life span of a smooth solution of the limit system (9). Hence, by using that $s > \frac{3}{2} + 2$, we have since

$$\Big(\frac{1}{\eta^{\bar{k},\pm}}\Big)^2 \leq \frac{|\bar{k}|}{|\bar{k}| - |k_3|} = \frac{2|\bar{k}|^2}{k_1^2 + k_2^2} \leq 2|\bar{k}|^2$$

that

$$\sum_{\bar{k}} |b(t,\bar{k})| \Big(1 + \Big(\frac{1}{\eta^{\bar{k},+}}\Big)^2 + \Big(\frac{1}{\eta^{\bar{k},-}}\Big)^2\Big) < \infty.$$

to finally obtain the important property

$$\sup_y \int_0^{+\infty} \Big|\partial_Z \mathcal{B}^\sigma(\tau, Z)(w(t,y,\sigma))\Big| (1 + |Z| + |Z^2|)\, dZ < +\infty, \quad \sigma = 0, 1 \tag{8}$$

which insures that the boundary layers are sufficiently localized in the vicinity of the boundary.

1.2 The limit system

We denote $w^{int} = \sum_{\bar{k}\in\mathbb{Z}^3_a} b(t,\bar{k}) N^{\bar{k}}$ the solution in $L^\infty(0,T^*;V^s_{sym})$ of the following system

$$\begin{cases} \partial_t w^{int} + \overline{Q}(w^{int}, w^{int}) + \overline{S}(w^{int}) = -\nabla p \text{ in } \quad \Omega, \\ \nabla \cdot w^{int} = 0 \text{ in } \quad \Omega, \\ w^{int} \cdot n = \pm w_3 = 0 \quad \text{ on } \quad \partial\Omega, \\ w^{int}(t=0) = w^0. \end{cases} \tag{9}$$

where T^* is the time of existence of the smooth solution w^{int} of (9) and $\overline{Q}(w^{int}, w^{int})$, $\overline{S}(w^{int})$ are respectively a bilinear and a linear operators of w^{int}.

The bilinear operator is given by

$$\overline{Q}(w^{int}, w^{int}) = \sum_{\substack{\bar{l},\bar{m},\bar{k} \\ \bar{k}\in\mathcal{A}(\bar{l},\bar{m}) \\ \lambda(\bar{l})+\lambda(\bar{m})=\lambda(\bar{k})}} b(t,\bar{l}) b(t,\bar{m}) \alpha_{\bar{l}\bar{m}\bar{k}} N^{\bar{k}}(X). \tag{10}$$

The numbers $\alpha_{\bar{l}\bar{m}\bar{k}}$ are constants and the set $\mathcal{A}(\bar{l},\bar{m}) = \{\bar{l} + \bar{m},\ S\bar{l} + \bar{m},\ \bar{l} + S\bar{m},\ S\bar{l} + S\bar{m}\}$ with the notation

$$S(\bar{l}_1, \bar{l}_2, \bar{l}_3) = (\bar{l}_1, \bar{l}_2, -l_3)$$

is the set of possible resonances.

The linear operator is defined by

$$\overline{S}(w^{int}) = \sum_{\bar{k}} (D(\bar{k}) + iI(\bar{k})) b(t,\bar{k}) N^{\bar{k}}(X)$$

where

$$D(\bar{k}) = \sqrt{2}\left\{(1 - \lambda(\bar{k})^2)^{\frac{1}{2}}\right\}, \quad I(\bar{k}) = \sqrt{2}\left\{\lambda(\bar{k})(1 - \lambda(\bar{k})^2)^{\frac{1}{2}}\right\}.$$

Note that $\overline{S}(w^{int})$ is a damping term, since $D(\bar{k}) \geq 0$, that depends on the frequencies $\lambda(\bar{k})$.

1.3 Non resonance assumption on the torus

In the case of a non resonant torus (see [2] for the definition), the quadratic term $\overline{Q}(w,w)$ only includes trivial resonances, namely the resonances only take place with the $2d$ non oscillating geostrophic part :

$$\{(\bar{k},\bar{l},\bar{m}) \mid \bar{k} \in \mathcal{A}(\bar{l},\bar{m})\} \subset \{(\bar{k},\bar{l},\bar{m}) \mid k_3 m_3 l_3 = 0\} \tag{11}$$

which yields in particular the global existence of strong solutions to the limit system. We also know in that case that for $\bar{k} \in \mathcal{A}(\bar{l},\bar{m})$, and $\lambda(\bar{l}) + \lambda(\bar{m}) - \lambda(\bar{k}) \neq 0$

$$\frac{1}{\lambda(\bar{l}) + \lambda(\bar{m}) - \lambda(\bar{k})} \leq C(|\bar{l}|^d + |\bar{m}|^d) \tag{12}$$

for some $d > 4$. We recall that for almost all choices of a_1 and a_2, the torus $\mathbb{T}^3_a$ is non resonant (see [2]).

Besides, if at $t = 0$, we have

$$\int_{x,y} w^0 \; dxdy = 0$$

we see that this holds for any t. Indeed, in the non resonant case there are only trivial resonances, namely with the slow modes (the geostrophic modes) $(k_1, k_2, 0)$. Notice then that the modes $(k_1, k_2, 0)$, and $(-k_1, -k_2, k_3)$ do not create a resonance with $(0, 0, k_3)$, since $(k_1, k_2) \neq (0, 0)$, and then $|\lambda(-k_1, -k_2, k_3)| < 1$. Hence we get that for all t the modes such that $\lambda(\bar{k}) = \pm 1$ are absent. This is a crucial fact in our analysis since the boundary layers for the modes $\lambda(\bar{k}) = \pm 1$ behave like the boundary layers in the vanishing viscosity limit of the Navier-Stokes equation whithout the fast rotation (these layers are of Prandtl-type). For such case, the stability of the boundary layer is known if the horizontal viscosity is much bigger than the vertical one (see [21]), in other cases instability is more expected [15] except in dimension 1 [26] or for analytic data [28].

1.4 Stability assumption on the boundary layer profiles

The main difficulty in the convergence proof is to get an estimate for (1) linearized about the approximate solution u^{app} : we study

$$\partial_t v + u^{app} \cdot \nabla v + v \cdot \nabla u^{app} - \varepsilon \Delta v + \nabla p + \frac{e \times v}{\varepsilon} = 0, \quad x \in \mathbb{T}^2_a \times (0,1) \tag{13}$$

with the boundary condition (2) and the initial condition $v_{/t=0} = v_0(x)$. We would like to prove an estimate like

$$||v(T)||^2 \leq e^{\gamma T} ||v_0||^2$$

for some norm $||\cdot||$ with $\gamma > 0$ independent of ε. Even in the well-prepared case such an estimate is not always true, it depends on a spectral stability property of the boundary layer profiles. If the boundary layer profiles are spectrally stable, this estimate can be proven as well as nonlinear stability, [25]. Whereas if they are unstable, we can only get an estimate with γ of the order of ε^{-1} and in this case nonlinear instability can be proven [6]. The spectral stability depends on the amplitude of the boundary layer, numerically, one can prove that boundary layers with too large amplitude are unstable [20]. The aim of the next subsection is to formulate a stability assumption on the boundary layer profiles which generalize the spectral stability assumption of the well-prepared case formulated in [25].

We start by freezing the slow variables $t = t^0$ and $y = y^0$ in the coefficients of the approximate solution. Let us set $q = q(t^0, y^0) = (b(t^0, \bar{k})e^{ik\cdot y^0})_{\bar{k}\in\mathbb{Z}_a^3}$. We want to study the stability property of the equations

$$\partial_t v + \Big(\mathcal{L}^\sigma(\frac{t}{\varepsilon})q + \mathcal{B}^\sigma(\frac{t}{\varepsilon}, \frac{z}{\varepsilon})q\Big) \cdot \nabla v + v \cdot \nabla\Big(\mathcal{L}^\sigma(\frac{t}{\varepsilon})q + \mathcal{B}^\sigma(\frac{t}{\varepsilon}, \frac{z}{\varepsilon})q\Big) + \frac{\nabla p}{\varepsilon} + \frac{e \times v}{\varepsilon} - \varepsilon\Delta v = 0.$$

Let us define for each sequence $q = (q_{\bar{k}})_{\bar{k}\in\mathbb{Z}_a^3}$ the oscillating boundary layer profile $V(\tau, Z, q)$ as

$$V^\sigma(\tau, Z, q) = \mathcal{L}^\sigma(\tau)q + \mathcal{B}^\sigma(\tau, Z)q,$$

we can take the Fourier transform in y and set $Z = z/\varepsilon$ and $\tau = t/\varepsilon$ to get the family of one-dimensional problems

$$\begin{aligned}
&\partial_\tau w + V^\sigma(\tau, Z, q) \cdot \begin{pmatrix} i\varepsilon k \\ \partial_Z \end{pmatrix} w + w \cdot \begin{pmatrix} i\varepsilon k \\ \partial_Z \end{pmatrix} V^\sigma(\tau, Z, q) \\
&\quad + \begin{pmatrix} i\varepsilon k \\ \partial_Z \end{pmatrix} p + e \times w + \varepsilon^2|k|^2 w - \partial_{ZZ} w = 0, \\
&i\varepsilon k \cdot w_h + \partial_Z w_3 = 0.
\end{aligned}$$

which is now set for $Z \in (0, +\infty)$ with the boundary condition

$$w(\tau, \varepsilon k, 0) = 0. \tag{14}$$

We recall here that we use the notation $\bar{k} = (k, k_3)$ where $k \in \mathbb{Z}_a^2$. Finally, we can set $\xi = \varepsilon k$ and use for $\xi \neq 0$ the Leray projection $\mathbb{P}_+(\xi)$ in the half-space which is recalled in section 19 to rewrite the equation as

$$\partial_\tau w = \mathbb{P}_+(\xi)\mathbb{L}_+^\sigma(\xi, q)w, \quad i\xi \cdot w_h + \partial_Z w_3 = 0 \tag{15}$$

where $\mathbb{L}_+^\sigma(\tau, \xi, q)$ is defined as

$$\mathbb{L}_+^\sigma(\tau, \xi, q)w = -V^\sigma(\tau, Z, q) \cdot \begin{pmatrix} i\xi \\ \partial_Z \end{pmatrix} w - w \cdot \begin{pmatrix} i\xi \\ \partial_Z \end{pmatrix} V^\sigma(\tau, Z, q) - e \times w - |\xi|^2 w + \partial_{ZZ} w \tag{16}$$

The non-autonomous operator $\mathbb{P}_+\mathbb{L}_+^\sigma(\tau, \xi, q)$ generates a strongly continuous family of evolution operators in the sense of [19], Chapter 7, $\mathbb{S}_+^\sigma(\tau, \tau', \xi, q)$ on $H_\xi = \{w \in L^2(0, +\infty), i\xi\cdot w_h + \partial_Z w_3 = 0\}$. As usual, the main property of $\mathbb{S}_+^\sigma(\tau, \tau', \xi, q)$ is that $\tau \mapsto \mathbb{S}_+^\sigma(\tau, \tau', \xi, q)w_0$ is the unique solution of (15) for $\tau > \tau'$ with value w_0 for $\tau = \tau'$.

Let us fix $s_0 > 1$ such that (8) holds when we replace $w(t, y, \sigma)$ by $\mathcal{L}^\sigma(\tau)q$. A set $\mathcal{K} \subset V_{sym}^{s_0}(\Omega)$ or more precisely in $\{q \mid \sum_{\bar{k}\in\mathbb{Z}_a^3} |\bar{k}|^{2s_0}|q_{\bar{k}}|^2 < \infty\}$ (using the identification between the function and its Fourier coefficients) will be called a **uniform stability set** if for every r, R, $0 < r < R$, there exists $C(r, R)$ and $\alpha(r, R) > 0$ such that

$$\forall v \in H_\xi, \quad |\mathbb{S}_+^\sigma(\tau, \tau', \xi, q)v|_{L^2(\mathbb{R}_+)} \leq Ce^{-\alpha(\tau-\tau')}|v|_{L^2(\mathbb{R}_+)}, \quad \forall\tau \geq \tau' \geq 0, \sigma = 0, 1 \tag{17}$$

for every $q \in \mathcal{K}$ and ξ such that $r \leq |\xi| \leq R$.

We point out that there exists uniform stability sets. Indeed, a vicinity of zero is a uniform stability set since we can prove as in [22, 5] that all weak amplitude boundary layers are stable when (8) is matched.

Let $w^{int}(t) = \sum_{\bar{k}\in\mathbb{Z}_a^3} b(t, \bar{k})N^{\bar{k}}$ be the solution of (9). Our main stability assumption is:

- **(H)** We assume that the set $\mathcal{K} = \{(q_k(t,y) := b(t,\bar{k})e^{ik\cdot y})_{\bar{k}\in\mathbb{Z}_a^3},\ t\in[0,T],\ y\in\mathbb{T}_a^2\}$, is a uniform stability set.

Remark 1 *Note that we can define more generally the stability set $\mathcal{K}_0 \subset V^{s_0}_{sym}(\Omega)$ as the set of q having the property that for every $0<r<R$, there exists $C(r,R,q)>0$ and $\alpha(r,R,q)>0$ such that (17) holds. By continuity of $\mathbb{S}_+$ with respect to q and compactness, we easily get that every compact set $\mathcal{K}\subset\mathcal{K}_0$ is a uniform stability set. In particular, this yields that for every bounded set $B\subset V^s_{sym}(\Omega)$ with $s>s_0$, we have that if B is included in $\mathcal{K}_0$, then B is a uniform stability set. Consequently, the assumption (H) is matched if $\mathcal{K}\subset\mathcal{K}_0$ (where $\mathcal{K}$ is defined in the statement of (H) and $\mathcal{K}$ is bounded in $V^s_{sym}(\Omega)$ for $s>s_0$.*

Since the operator $\mathbb{L}_+^\sigma$ has only quasi-periodic coefficients there is no easy characterization of the assumption (H), for example in term of the spectrum of $\mathbb{L}_+$. When the initial data is prepared so that the coefficients of $\mathbb{L}_+^\sigma$ are periodic in time, we can use Floquet theory to replace in the assumption (H) the decay estimate (17) by an assumption on the spectrum of $\mathbb{S}_+^\sigma(T,0,\xi,q)$ where T is the period: if the spectrum $\sigma(\mathbb{S}_+^\sigma(T,0,\xi,q))$ is contained in the open unit disk $\mathbb{D}$, then the estimate (17) is verified. Finally, we note that (H) is the natural generalization of the assumption used in the well-prepared case. Indeed, in the well prepared case, since $\mathbb{L}_+^\sigma$ does not depend on τ the stability assumption was formulated in term of the spectrum of $\mathbb{P}_+(\xi)\mathbb{L}_+^\sigma(\xi,q)$: it was basically assumed in [25] that the spectrum of $\mathbb{P}_+(\xi)\mathbb{L}_+^\sigma(\xi,q)$ is contained in $\{Re\,\lambda<0\}$. By the standard theory of analytic semi-groups, it is easy to prove that this assumption on the spectrum implies the estimate (17). This is proven in [12] in a close setting.

1.5 Notations

We denote by $||\cdot||$ the norm of $L^2(\mathbb{T}_a^2\times(0,1))$ and by $(\cdot,\cdot)$ the associated scalar product. We also define the weighted higher order norms :

$$||v||^2_{1,\varepsilon} = ||v||^2 + \varepsilon^2||\nabla v||^2,$$
$$||v||^2_{2,\varepsilon} = ||v||^2 + \varepsilon^2||\nabla v||^2 + \varepsilon^4||\nabla^2 v||^2.$$

We will also use some anisotropic norms, namely

$$||v||^2_m = \sum_{\alpha\in\mathbb{Z}^3,\,|\alpha|\leq m} ||\overline{Z}^\alpha v||^2 \tag{18}$$

where $\overline{Z}_1=\partial_{y_1}$, $\overline{Z}_2=\partial_{y_2}$, and $\overline{Z}_3=\varepsilon z(1-z)\partial_z$ and denote by H^m_{anis} the Hilbert space defined by this norm.

1.6 Main result

Our main result is :

Theorem 2 *We consider a torus $\mathbb{T}_a^3$ non-resonant in the sense of [2] and $w^0\in V^s_{sym}$ for s sufficiently large such that in the expansion*

$$w^0(x) = \sum_{\bar{k}\in\mathbb{Z}_a^3} b^0(\bar{k})e^{ik\cdot y}M^{\bar{k}}(z)$$

we have $b^0(\bar{k}) = 0$ *if* $\bar{k} = (0, 0, k_3)$. *Moreover, with the notation* $q^0(y) = (q^0_{\bar{k}}(y))_{\bar{k}} = (b^0(\bar{k})e^{ik\cdot y})_{\bar{k}}$, *we assume that*

$$\left\{ \begin{array}{l} \|u^{\varepsilon,0} - w^0 - \mathcal{B}^0(0, \frac{z}{\varepsilon})(q^0(y)) - \mathcal{B}^1(0, \frac{1-z}{\varepsilon})(q^0(y))\|_m \leq c\varepsilon^\alpha \\ \varepsilon\|\nabla(u^{\varepsilon,0} - w^0 - \mathcal{B}^0(0, \frac{z}{\varepsilon})(q^0(y)) - \mathcal{B}^1(0, \frac{1-z}{\varepsilon})(q^0(y)))\|_m \leq c\varepsilon^\alpha \end{array} \right. \tag{19}$$

for some $m \geq 2$, $3/4 < \alpha \leq 1$ *and some constant* $c > 0$. *Let* $w^{int}(t)$ *be the solution of* (9), *we assume that* **(H)** *holds. Then, there exists* ε_0 *such that for all* $0 < \varepsilon < \varepsilon_0$, *the system (1) has a unique weak solution* $u^\varepsilon \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1_0(\Omega))$ *with initial value* $u^{\varepsilon,0}$. *Moreover,*

$$\|u^\varepsilon - \mathcal{L}(\frac{t}{\varepsilon})w^{int} - \mathcal{B}^0(\frac{t}{\varepsilon}, \frac{z}{\varepsilon})(q(t, y)) - \mathcal{B}^1(\frac{t}{\varepsilon}\frac{1-z}{\varepsilon})(q(t, y))\|_{L^\infty(0,T;L^2(\Omega))} \leq C_T\varepsilon^\alpha \tag{20}$$

In addition, if $\alpha = 3/4$ *in (19), there exists a time* T_0 *which depends on* w^0 *and on c (but not on* ε*) such that (20) holds on* $(0, T_0)$

Let us give a few remarks about this theorem.

Remark 3 *1) First, we note that when* $3/4 < \alpha \leq 1$, *the uniform time of existence and convergence* T *is only limited by the stability assumption* $(b(t, \bar{k})e^{ik\cdot y})_{\bar{k}\in\mathbb{Z}^3_a} \in \mathcal{K}_0$, *in particular, it may be arbitrary large even if the data is large. This is due to the regularity of the limit system which is better than the regularity of the* 3*D Navier-Stokes in the non-resonant case (see [2]).*

2) We also point out that the assumption that the torus is non-resonant is used to ensure that $b(t, \bar{k}) = 0$ *for* $\bar{k} = (0, 0, k_3)$ *and that the result holds if we know that* $b(t, \bar{k}) = 0$ *for* $\bar{k} = (0, 0, k_3)$ *as well as an estimate of the type (12). Indeed, for the modes* $\bar{k} = (0, 0, k_3)$, *the boundary layer we get is of Prandtl type (the rotation does not play any part). it was handled in [22] only with the crucial assumption that the vertical viscosity over the horizontal one also goes to zero.*

3) We shall see in the proof that the result is actually more precise. A sufficient condition on the regularity of w^0 *is* $s > d+5$ *where* d *is given in (12). The convergence will take place in a space with horizontal regularity* m *where* $2 \leq m \leq s - d - 3$. *Moreover, we notice that the error estimate, namely* $C_T\varepsilon^\alpha$ *sees the boundary layer and hence the boundary layer cannot be removed from the estimate (20). This is stronger than the estimates in [16, 22]. However, our result requires the use of an initial data which depends on* ε *and which is sufficiently close to the approximate solution at* $t = 0$. *The convergence is stated in* $L^\infty(0, T; L^2(\Omega))$ *but as will be seen from the proof in section 5.1, we need to prove estimates in a stronger space, namely* Y_m. *In particular, the solution we construct is a strong solution which yields the uniqueness of the weak solutions by the classical strong-weak uniqueness argument.*

4) Finally, we note that the proof of Theorem 2 is completely different from the proof of the stability in the well-prepared case in [25] and the proof of the small data case in the ill-prepared case [22].

The paper is organized as follows: in the next section, we give some details about the construction of the approximate solution, then, in the next sections, we prove the linear stability of this approximate solution, finally section 5 is devoted to the proof of Theorem 2. The Appendix A is devoted to the definition and the proof of some simple properties of semi-classical operator-valued pseudo-differential calculus. These properties are crucial in the study of the linear stability. Finally, the Appendix B gathers some usefull properties of the Leray projection in the strip and in the half-space.

2 Construction of an approximate solution

2.1 Some definitions and notations

We will use the notations of [22]. Let us denote by V^0 the subspace of $L^2(\Omega)^3$ consisting of divergence-free vectors (div u = 0), and tangent to $\partial\Omega$ ($u_3(z=0) = u_3(z=h) = 0$)

$$V^0 = \{u \in L^2(\Omega)^3, \nabla \cdot u = 0, u_3(z=0) = u_3(z=h) = 0\},$$

we also define, for $m > 0$, V^m the space

$$V^m = H^m(\Omega)^3 \cap V^0.$$

where $H^m(\Omega)$ is the classical Sobolev space $W^{2,m}(\Omega)$

Let $\mathbb{T}_a^3 = \mathbb{T}_a^2 \times]-1,1[= \mathbb{T}_a^2 \times \mathbb{R}/2\mathbb{Z}$ be the torus of periods respectively a_1, a_2 and 2, and E the linear operator from $L^2(\Omega)^3$ into $L^2(\mathbb{T}_a^3)^3$ defined by $Eu(z) = u(z)$ for $0 < z < h$, and for $-h < z < 0$

$$Eu(z) = S(u(-z)), \tag{21}$$

where S is the linear operator from $\mathbb{R}^3$ into $\mathbb{R}^3$, defined by

$$SX_1 = X_1 \quad SX_2 = X_2 \quad SX_3 = -X_3 \tag{22}$$

for all $X \in \mathbb{R}^3$, which corresponds to a symmetry with respect to the plan $X_3 = 0$.

We also introduce $V'^m = \{u \in H^m(\mathbb{T}_a^3)^3, \nabla \cdot u = 0\}$, Hence $E'E\ u = Id_{V^0}$ where E' is the restriction from $L^2(\mathbb{T}^3)^3$ onto $L^2(\Omega)^3$. In the sequel we will work in the space V_{sym}^m,

$$V_{sym}^m = E'(V'^m \cap E(V^m)),$$

which consists of vectors $u \in V^m$ that satisfy extra boundary conditions on the vertical derivatives.

We also introduce a norm on V_{sym}^m

$$|u|_{V_{sym}^m}^2 = \frac{1}{2}|Eu|_{V'^m}^2,$$

which is conserved by the group $\mathcal{L}$, namely $|\mathcal{L}(\tau)u|_{V_{sym}^m} = |u|_{V_{sym}^m}$, for every $\tau \in \mathbb{R}$. We will, also use the following notations (see the appendix 2.2 of [22] for a precise construction)

- $X = (y_1, y_2, z) = (y, z)$
- $N^{\bar{k}}(X) = M^{\bar{k}}(z)e^{ik.y}$ is an eigenvector of L associated to the eigenvalue $i\lambda(\bar{k}) = i\ k_3/|k|$
- For $w \in L^\infty(0, T, V_{sym}^s)$, we set

$$w(t, X) = \sum_{\bar{k} \in \mathbb{Z}_a^3} b(t, \bar{k})N^{\bar{k}}(X), \tag{23}$$

then we have for all t, $|w(t)|_{V_{sym}^s}^2 = 2|\Omega| \sum_{\bar{k}} |b(t, \bar{k})|^2 |\bar{k}|^{2s}$.

2.2 Study of the group

We study here the group $\mathcal{L}$, in particular, we give the expression of the eigenvectors N^k. Using the construction in the appendix of [22], we get for $\bar{k} = (k, k_3) \in \mathbb{Z}_a^3$ and $k \neq (0,0)$ that $N^{\bar{k}}(X) = M^{\bar{k}}(z)e^{ik.y}$ where

$$M^{\bar{k}}(z) = \begin{pmatrix} 2\cos(k_3\, z)n_1(\bar{k}) \\ 2\cos(k_3\, z)n_2(\bar{k}) \\ 2i\sin(k_3\, z)n_3(\bar{k}) \end{pmatrix}. \tag{24}$$

and

$$n_1(\bar{k}) = \frac{n_3(\bar{k})}{1-\lambda(\bar{k})^2}\left(i\frac{k_2}{|\bar{k}|} - \frac{k_1}{|\bar{k}|}\lambda(\bar{k})\right),$$

$$n_2(\bar{k}) = \frac{n_3(\bar{k})}{1-\lambda(\bar{k})^2}\left(-i\frac{k_1}{|\bar{k}|} - \frac{k_2}{|\bar{k}|}\lambda(\bar{k})\right).$$

$$n_3(\bar{k}) = \sqrt{\frac{1-\lambda(\bar{k})^2}{2}}.$$

Notice then that $N^{\bar{k}}(X) \in V^s_{sym}$, and that we have $LN^{\bar{k}} = \lambda(\bar{k})N^{\bar{k}}$.

2.3 Approximate solution

Here, we construct the approximate solution $U^{app} = (u^{app}, p^{app})$. The aim is that U^{app} satisfies (1) up to a small error and the boundary condition (2) exactly.

To guess a good choice for U^{app}, we expand the solution in the following form $U^0 + \varepsilon U^1 + \varepsilon^2 U^2 + ...$ where

$$U^0 = \underline{U}^0(\tau, t, x) + \tilde{U}^0(\tau, t, y, Z) + \breve{U}^0(\tau, t, y, Z'). \tag{25}$$

where we recall that $\tau = t/\varepsilon$, $Z = z/\varepsilon$ and $Z' = (1-z)/\varepsilon$. Even though, it is not clear whether we can push this expansion to all order, this will allow us to guess the first terms. Hence, arguing as in [22], we get

$$\underline{u}^0(\tau, t) = \mathcal{L}(\tau)\underline{u}^0(\tau = 0, t) = \mathcal{L}(\tau)w^{int}(t)$$

which is exactly the term u^{int} in (4).

We notice that $\underline{u}^0$ does not vanish on $\partial\Omega$, we only have $(\underline{u}^0 \cdot n)_{/\partial\Omega}$, where n is the normal to the boundary. This requires the introduction of a boundary layer. For the boundary layer, we only construct $\tilde{U}^0$, near $z = 0$, since the construction of $\breve{U}^0$ near $z = 1$ is similar. We recall that the modes $(0, k_3)$ are excluded due to the assumption on the initial data and on the resonances of the torus. So, we only deal with the case $-1 < \lambda(\bar{k}) < 1$. Using the construction in section 4 of [22], we get $\tilde{u}^0 = \sum_{\bar{k}\in\mathbb{Z}_a^3} R^{\bar{k}}$ where

$$\begin{aligned} R^{\bar{k}}(z) \;&=\; -\frac{1}{2}b(t,k)\exp\left[i\left((k.y) + \frac{\lambda(k)t}{\varepsilon}\right)\right] \times \\ &\left[h^{\bar{k},+}\exp(-\frac{(1+i)}{\sqrt{2}}\eta^{\bar{k},+}\frac{z}{\varepsilon}) + h^{\bar{k},-}\exp(-\frac{(1-i)}{\sqrt{2}}\eta^{\bar{k},-}\frac{z}{\varepsilon})\right] \end{aligned} \tag{26}$$

In a similar way, we get that $\breve{u}^0 = \sum_{\bar{k}\in\mathbb{Z}_a^3} T^{\bar{k}}$ where T^k has the same formula as R^k. Hence, $u^b = \tilde{u}^0 + \breve{u}^0$.

It remains to construct the remainder term u^r. Of course, a good guess for u^r is to take εU^1. However, we would like to get an approximate solution u^{app} which is divergence-free and which vanishes on the boundary. The rest term is under the form

$$u^r = \sum_{\bar{k}} (R_3^{\bar{k}} + T_3^{\bar{k}}) + \sum_{\bar{k}} r^{\bar{k}} + \varepsilon \mathcal{Y} + \mathcal{R}^3$$

and hence consists of four terms. We refer to [22] for the details on the construction of these four terms. The main property of this correction term is that for all $|\alpha| + |\beta| \leq s - d - 5$,

$$||(\varphi(z)\partial_z)^\beta \partial_y^\alpha u^r||_{L^\infty} \leq C_\alpha \varepsilon, \quad ||\partial_z(\varphi(z)\partial_z)^\beta \partial_y^\alpha u^r||_{L^\infty} \leq C_\alpha. \tag{27}$$

where $\varphi(z)$ is a smooth bounded function which is equivalent to z and $1 - z$ in the vicinity of 0 and 1 respectively.

- $R_3^{\bar{k}}$ and $T_3^{\bar{k}}$ which are introduced to insure that $R^{\bar{k}}$ and $T^{\bar{k}}$ satisfy the divergence free condition. However, $R_3^{\bar{k}}$ creates a trace at $z = 0$ and $T_3^{\bar{k}}$ creates a trace at $z = 1$ of order ε.
- $r^{\bar{k}}$ which is used to cancel the traces $R_3^{\bar{k}}(z = 0)$ and $T_3^{\bar{k}}(z = 1)$. Since, we have to take $r^{\bar{k}}$ which is divergence-free, we have to construct $r_1^{\bar{k}}$ and $r_2^{\bar{k}}$ which have a trace at $z = 0$ and $z = 1$ of order ε. This is actually easier to handle than the trace on the third component we started with. Besides, the term $\frac{1}{\varepsilon} e \times r^{\bar{k}}$ is responsible for the Ekman damping in the limit equation.
- $\varepsilon\mathcal{Y}$ is introduced to cancel the non resonant oscillating terms which do not yield a contribution in the limit equation. More precisely, we have

  $$\mathcal{Y}(\tau, t) = -\mathcal{L}(\tau) \int_0^\tau \left[\mathcal{L}(-\tau')Q(w, w) - \overline{Q}(w, w)\right](t) d\tau'.$$

  We also point out that due to the non resonance assumption, we know that $\|\mathcal{Y}(\tau, t)\|_{H^{s-d-1}} \leq C$ for a constant which does not depend on τ or $t < T$.
- $\mathcal{R}^3$ which takes into account the boundary condition of r, $\mathcal{Y}$. This was constructed in section 4.3 of [22].

3 Linear Stability

We study the linearized system about the approximate solution:

$$\mathcal{T}^\varepsilon(v, p) = F, \quad \nabla \cdot v = 0, \quad x \in \mathbb{T}_a^2 \times (0, 1) \tag{28}$$

where

$$\mathcal{T}^\varepsilon(v, p) = \partial_t v + u^{app} \cdot \nabla v + v \cdot \nabla u^{app} + \nabla p \ + \frac{e \times v}{\varepsilon} - \varepsilon \Delta v$$

with initial data

$$v(0, x) = v_0(x) \tag{29}$$

and the boundary condition (2). Let us define v^{HF} by $\mathcal{F}_y v^{HF} = \mathbf{1}_{\varepsilon|k|\geq r} \mathcal{F}_y v$, for some $r > 0$ and $\kappa^s(\varepsilon D_y)v$ by $\mathcal{F}_y(\kappa^s v) = \kappa^s(\varepsilon\xi)\mathcal{F}_y v(\xi)$ with $\kappa^s(\xi)$ a smooth function which vanishes for $|\xi| \geq 2r$. the main result of this section is

Theorem 4 *We assume that* **(H)** *holds. Then, there exists $\gamma_0 > 0$, $\gamma > 0$ such that for every $\varepsilon > 0$, $T > 0$, with $\varepsilon e^{\gamma_0 T} \leq 1$, we have*

$$||v(T)||^2 + \varepsilon^2 ||\nabla v^{HF}(T)||^2 + \int_0^T \varepsilon ||\nabla v||^2 \tag{30}$$
$$\lesssim e^{\gamma t}\Big(||v_0||^2 + ||v_0^{HF}||^2_{1,\varepsilon} + \int_0^T \Big| (\kappa^s F, \kappa^s v) \Big| + \varepsilon \int_0^T ||F||^2 \Big)$$

Throughout the paper, $\lesssim$ stands for $\leq C$ where $C > 0$ is independent of $\varepsilon \in (0,1)$, T if $\varepsilon e^{\gamma_0 T} \leq 1$. Note that by using the Cauchy Schwarz inequality and the Gronwall inequality, we can get from (30) the estimate

$$||v(T)||^2 + \varepsilon^2 ||\nabla v^{HF}(T)||^2 + \int_0^T \Big(\varepsilon ||\nabla v||^2 + \varepsilon^{-1} ||v^{HF}||^2_{1,\varepsilon} \Big)$$
$$\lesssim e^{\tilde{\gamma} t}\Big(||v_0||^2 + ||v_0^{HF}||^2_{1,\varepsilon} + \int_0^T (1+\varepsilon) ||F||^2 \Big)$$

for some $\tilde{\gamma} > \gamma$ which gives an estimate of v with respect to the source term F and the initial data only. Nevertheless, in order to handle the nonlinear stability, it is important to keep the term

$$\int_0^T \Big| (\kappa^s F, \kappa^s v) \Big|$$

in the right-hand side since it will allow to use the structure of the nonlinear term of the Navier-Stokes equation and hence to get some better estimates.

The aim of the remaining part of the section is to prove Theorem 4. Note that for the moment we have a control of the $L^\infty(0,T,H^1)$ norm only for the high frequency part of v. We will derive an estimate for all the frequencies in paragraph 3.6.

3.1 Proof of Theorem 4

We start with a localization in frequency of the equation similar to the well prepared case [25]. We will deal with large, medium and small frequencies in different ways. For a smooth bounded function κ, we apply the Fourier multiplier $\kappa(\varepsilon D_y)$ to the equation 13. We get

$$\mathcal{T}^\varepsilon(\kappa v, \kappa p) = \kappa F + \mathcal{C} \tag{31}$$

where the commutator $\mathcal{C}$ is defined as

$$\mathcal{C} = -[\kappa, u^{app} \cdot \nabla] v - [\kappa, D u^{app}] v$$

By using the same argument as in [25], [27], we have the estimate

$$||\mathcal{C}||^2 \lesssim \varepsilon^2 ||\nabla v||^2 + ||v||^2. \tag{32}$$

Note that these commutator estimates are actually proven in a more general case in Lemma 16.

We first deal with the case where $\kappa = \kappa^L$ is supported in $\varepsilon|k| \geq R$. We have

Proposition 5 *There exists $R > 0$ sufficiently large such that we have for every $\varepsilon \in (0,1)$ and every $T > 0$:*

$$||\kappa^L v(T)||^2_{1,\varepsilon} + \varepsilon^{-1} \int_0^T ||\kappa^L v||^2_{1,\varepsilon} \lesssim \int_0^T \Big(||v||^2_{1,\varepsilon} + \varepsilon ||F||^2 \Big) + ||\kappa^L v_0||^2_{1,\varepsilon}.$$

Note that for the proof of this lemma we do not need to use **(H)**.

Proof

We use the same argument as in the well-prepared case treated in [25]. Using that u^{app} is divergence free, the standard energy estimate for (31) gives

$$\begin{aligned} ||\kappa^L v(T)||^2 + \int_0^T \varepsilon ||\nabla \kappa^L v||^2 &\lesssim ||\kappa^L v_0||^2 + \int_0^T \Big(\varepsilon^{-1} ||\kappa^L v||^2 + ||F||\, ||\kappa^L v|| + ||\mathcal{C}||^2 \Big) \\ &\lesssim ||\kappa^L v_0||^2 + \int_0^T \Big(\varepsilon^{-1} ||\kappa^L v||^2 + \varepsilon ||F||^2 + ||\mathcal{C}||^2 \Big). \end{aligned}$$

We notice that

$$||\nabla \kappa^L v||^2 \geq R^2 \varepsilon^{-1} ||\kappa^L v||^2$$

so that for R sufficiently large, the singular term $\varepsilon^{-1}||\kappa^L v||^2$ in the right hand side of (33) can be absorbed by the left hand side. By using also (32), this yields

$$||\kappa^L v(T)||^2 + \varepsilon^{-1} \int_0^T ||\kappa^L v||_{1,\varepsilon}^2 \lesssim \int_0^T \Big(||v||_{1,\varepsilon}^2 + \varepsilon ||F||^2 \Big) + ||\kappa^L v_0||^2. \tag{33}$$

To conclude, it suffices to estimate $\varepsilon^2 ||\nabla \kappa^L v(T)||^2$. This is an easy consequence of the following lemma :

Lemma 6 *Consider u a solution of*

$$\partial_t u - \varepsilon \Delta u + \frac{e \times u}{\varepsilon} + \nabla p = H, \quad \nabla \cdot u = 0, \quad x \in \Omega \tag{34}$$

with the initial condition $u(0,x) = u_0(x)$ and the boundary condition (2). Then, we have the estimate

$$\varepsilon^2 ||\nabla u(T)||^2 + \int_0^T \Big(\varepsilon^3 ||\nabla^2 u||^2 + \varepsilon ||\partial_t u||^2 \Big) \lesssim \varepsilon^2 ||\nabla u_0||^2 + \int_0^T \Big(\varepsilon^{-1} ||u||^2 + \varepsilon ||H||^2 \Big). \tag{35}$$

We first end the proof of Proposition 5 by using Lemma 6. We can use Lemma 6 with $u = \kappa^L v$ and

$$H = -u^{app} \cdot \nabla \kappa^L v - (\kappa^L v) \cdot \nabla u^{app} + \kappa^L F + \mathcal{C}.$$

This yields

$$\varepsilon^2 ||\nabla \kappa^L v||^2 \lesssim \varepsilon^2 ||\nabla \kappa^L v_0||^2 + \int_0^T \Big(\varepsilon^{-1} ||\kappa^L v||^2 + \varepsilon ||\nabla \kappa^L v||^2 + \varepsilon ||F||^2 + ||v||_{1,\varepsilon}^2 \Big).$$

To conclude, it suffices to add (33) times a sufficiently large constant and the last estimate.

We now turn to the Proof of Lemma 6 :

Proof of Lemma 6

We take the scalar product of (34) by $\partial_t u$, since $\partial_t u$ is divergence free and verifies the boundary condition (2), we have

$$(\nabla p, \partial_t u) = 0, \quad (-\Delta u, \partial_t u) = \frac{d}{dt}\Big(\frac{1}{2}||\nabla u||^2\Big)$$

and hence we get

$$\varepsilon||\nabla u(T)||^2 + \int_0^T ||\partial_t u||^2 \lesssim \varepsilon||\nabla u_0||^2 + \int_0^T \Big(||H|| + \varepsilon^{-1}||u||\Big)||\partial_t u||.$$

By using the Young inequality, we find after multiplication by ε

$$\varepsilon^2||\nabla u(T)||^2 + \int_0^T \varepsilon||\partial_t u||^2 \lesssim \varepsilon^2||\nabla u_0||^2 + \int_0^T \Big(\varepsilon||H||^2 + \varepsilon^{-1}||u||^2\Big). \tag{36}$$

Next, we use the classical regularity result for the Stokes equation [10]. We consider (34) as

$$-\varepsilon\Delta u + \nabla p = H - \frac{e \times u}{\varepsilon} - \partial_t u, \quad \nabla \cdot u = 0$$

and we find that

$$\varepsilon^2||\nabla^2 u||^2 \lesssim ||H||^2 + \varepsilon^{-2}||u||^2 + ||\partial_t u||^2.$$

To end the proof, it suffices to integrate in time, to multiply by ε the last estimate and to use (36). This ends the proof of Lemma 6.

We now consider R as fixed. The next step is to consider the case where $\kappa = \kappa^s$ is supported in $\varepsilon|k| \le r$.

Proposition 7 *There exists $r > 0$ sufficiently small such that we have for every $\varepsilon \in (0,1)$ and every $T > 0$*

$$||\kappa^s v(T)||^2 + \int_0^T \varepsilon||\kappa^s \nabla v||^2 \lesssim \int_0^T \Big(||v||^2_{1,\varepsilon} + |(\kappa^s F, \kappa^s v)|\Big) + ||\kappa^s v_0||^2.$$

Again, note that the assumption **(H)** is not used.

Proof

We again use a direct energy estimate as in [25]. Since $(e \times \kappa^s v, \kappa^s v) = 0$, we get

$$||\kappa^s v(T)||^2 + \int_0^T \varepsilon||\kappa^s \nabla v||^2 \lesssim \int_0^T \Big(||\mathcal{C}||^2 + ||\kappa^s v||^2 + |(\kappa^s F, \kappa^s v)| + S^\varepsilon\Big) + ||\kappa^s v_0||^2 \tag{37}$$

where the singular term S^ε is defined by

$$S^\varepsilon = \varepsilon^{-1}\int_\Omega |\kappa^s v_3|\,\Big(|\partial_Z u^b| + |\partial_{Z'} u^b|\Big)\,|\kappa^s v|\,dx.$$

Again, we use (32) to estimate $\mathcal{C}$ and hence it remains to study S^ε. To estimate this term, we shall use the inequality

$$|f(t,x)|^2 \leq \Big(\int_0^z |\partial_z f|\Big)^2 \leq z\int_0^z |\partial_z f|^2 \leq z\int_0^1 |\partial_z f|^2 \quad \text{if} \quad f_{/z=0}=0. \tag{38}$$

Note that since $\partial_z \kappa^s v_3 = -\partial_1\kappa^s v_1 - \partial_2\kappa^s v_2$ because of the incompressiblity condition, we can use (38) twice to get

$$|\kappa^s v_3|^2 \leq z\int_0^z |\partial_z\kappa^s v_3|^2 \lesssim z\int_0^z |\nabla_h\kappa^s v_h|^2 \lesssim z^3\int_0^1 |\partial_z\nabla_h\kappa^s v_h|^2,$$
$$|\kappa^s v|^2 \lesssim z\int_0^1 |\partial_z\kappa^s v|^2.$$

This yields

$$\begin{aligned}
&\varepsilon^{-1}\int_\Omega |\kappa^s v_3|\,|\partial_Z u^{b,0}|\,|\kappa^s v|\,dx \\
&\lesssim \varepsilon^2||\partial_z\nabla_h\kappa^s v||\,||\partial_z\kappa^s v|| \sup_y \int_0^{+\infty} Z^2|\partial_Z\mathcal{B}^0(\tau,Z)w^{int}(t,y,0)|\,dZ \\
&\lesssim \varepsilon r||\nabla\kappa^s v||^2.
\end{aligned}$$

To get the last inequality, we have used that $\varepsilon||\nabla_h\kappa^s f|| \leq r||\kappa^s f||$ by definition of κ^s and our regularity assumption which gives

$$\sup_y \int_0^{+\infty} Z^2|\partial_Z\mathcal{B}^0(\tau,Z)w^{int}(t,y,0)|\,dZ < +\infty$$

following (8). The same argument in the vicinity of the boundary $z=1$ shows that

$$\varepsilon^{-1}\int_\Omega |\kappa^s v_3|\,|\partial_{Z'} u^{b,1}|\,|\kappa^s v|\,dx \lesssim \varepsilon r||\nabla\kappa^s v||^2.$$

Consequently, we can choose r sufficiently small to absorb the singular term S^ε in the left hand side of (37). This ends the proof of Proposition 7. □

Finally, it remains the most difficult case where $\kappa(\varepsilon k) = \kappa^l$ is supported in $r/2 \leq \varepsilon|k| \leq 2R$. Note that r and R are now fixed. We have the following estimate :

Proposition 8 *Under the assumptions of Theorem 4, we have for ε and T such that $\varepsilon e^{\gamma_0 T} \leq 1$*

$$||\kappa^l v(T)||_{1,\varepsilon}^2 + \int_0^T \varepsilon^{-1}||\kappa^l v||_{1,\varepsilon}^2 \lesssim e^{\gamma_0 T}||\kappa^l v_0||_{1,\varepsilon}^2 + \int_0^T \Big(\varepsilon e^{\gamma_0 T}||F||^2 + ||v||_{1,\varepsilon}^2\Big).$$

The assumption $(b(t,\bar{k})e^{ik\cdot y})_{\bar{k}\in\mathbb{Z}_a^3} \in \mathcal{K}$ is crucial in the proof of Proposition 8.

3.2 Proof of Proposition 8

In this section, due to the oscillations in the boundary layers, we use an approach completely different from the one of the well-prepared case used in [25]. The proof of this Proposition is the most technical part and we split it into various steps. First, we rewrite (31) by using the Leray projection $\mathbb{P}(D_y)$ which is recalled in section B.2. Let us set $v^l = \kappa^l v$, we get the equation

$$\partial_t v^l = \mathbb{P}(D_y)\overline{\kappa}^l(\varepsilon D_y)\mathbb{L}^\varepsilon v^l + \mathbb{P}(D_y)\overline{\kappa}^l\Big(\kappa^l F + \kappa^l \mathcal{C}\Big), \quad \nabla \cdot v^l = 0, \quad x \in \Omega, \tag{39}$$

where $\overline{\kappa}^l$ is compactly supported with a support slightly bigger than κ^l and takes the value 1 on the support of κ^l in order that $\overline{\kappa}^l\kappa = \kappa$ and $\mathbb{L}^\varepsilon$ is defined by

$$\mathbb{L}^\varepsilon v = \varepsilon\Delta v - u^{app}\cdot\nabla v - v\cdot\nabla u^{app} - \frac{e\times v}{\varepsilon}.$$

Next, we shall estimate differently v^l in the interior of the domain and in the vicinity of the boundary. We decompose v^l as

$$v^l = \chi^b(\frac{z}{\delta})v^l + \chi^{int}(z)v^l + \chi^b(\frac{1-z}{\delta})v^l \tag{40}$$

where χ^b is compactly supported in $[0,2]$, and χ^{int} is compactly supported in $[\delta, 1-\delta]$. Note that χ^{int} depends on δ though we forget this dependence in the notation. Multiplying (39) by χ for χ one of the truncation functions χ^{int}, $\chi^{b,0} = \chi^b(z/\delta)$, $\chi^{b,1} = \chi^b(1-z/\delta)$, we get

$$\partial_t(\chi v^l) = \mathbb{P}(D_y)\overline{\kappa}^l(\varepsilon D_y)\mathbb{L}^\varepsilon(\chi v^l) + H \tag{41}$$

$$\nabla\cdot(\chi v^l) = -\partial_z\chi\, v_3^l \tag{42}$$

where

$$H = \chi\mathbb{P}(D_y)\overline{\kappa}^l\Big(\kappa^l F + \kappa^l\mathcal{C}\Big) + \mathcal{C}^1 + \mathcal{C}^2 \tag{43}$$

with the new commutators $\mathcal{C}^1$ and $\mathcal{C}^2$ defined by

$$\mathcal{C}^1 = \Big[\chi, \mathbb{P}(D_y)\overline{\kappa}^l(\varepsilon D_y)\Big]\mathbb{L}^\varepsilon v^l, \tag{44}$$

$$\mathcal{C}^2 = \mathbb{P}(D_y)\overline{\kappa}^l(\varepsilon D_y)\Big[\chi, \mathbb{L}^\varepsilon\Big]v^l. \tag{45}$$

Thanks to (155) in Lemma 19, we get that

$$||\mathcal{C}^1|| \lesssim ||v^l|| + \varepsilon||\nabla v^l|| + \varepsilon^2||\Delta v^l|| \lesssim ||v^l||_{2,\varepsilon}. \tag{46}$$

Note that in the following $\lesssim$ stands for $\leq C$ and that C may depend on δ. Besides, the explicit computation of $\mathcal{C}^2$ and a new use of the commutator estimates (see the appendix of [25] or Lemma 16) gives that

$$||\mathcal{C}^2|| \lesssim ||v^l|| + \varepsilon||\nabla v^l|| \lesssim ||v^l||_{1,\varepsilon}. \tag{47}$$

Finally, note that if we choose $\overline{\overline{\kappa}}^l$ such that the support of $\overline{\overline{\kappa}}^l$ is again slightly larger than the one of $\overline{\kappa}^l$ then we have

$$H = \overline{\overline{\kappa}}^l(\varepsilon D_y)H. \tag{48}$$

3.2.1 Interior estimates

We start with the case $\chi = \chi^{int}$. The estimate of $\chi^{int} v^l$ can be obtained by a direct energy estimate. We shall first establish the estimate :

$$||\chi^{int} v^l(T)||^2 + \varepsilon^{-1} \int_0^T ||\chi^{int} v^l||_{1,\varepsilon}^2 \lesssim ||\chi^{int} v_0^l||^2 + \int_0^T \Big(\varepsilon ||v^l||_{2,\varepsilon}^2 + \varepsilon ||F||^2 + ||v||_{1,\varepsilon}^2 \Big). \tag{49}$$

The only difficulty is that we have to deal with the fact that $\chi^{int} v^l$ is not divergence free. When we take the scalar product of (41) by $\chi^{int} v^l$, we can write

$$\begin{aligned}&\Big(\mathbb{P}(D_y)\overline{\kappa}^l(\varepsilon D_y)\mathbb{L}^\varepsilon(\chi^{int} v^l), \chi^{int} v^l\Big) = \Big(\mathbb{L}^\varepsilon(\chi^{int} v^l), \mathbb{P}(D_y)\overline{\kappa}^l(\varepsilon D_y)(\chi^{int} v^l)\Big)\\ &= \Big(\mathbb{L}^\varepsilon(\chi^{int} v^l), \chi^{int} v^l\Big) + \Big(\mathbb{L}^\varepsilon(\chi^{int} v^l), [\mathbb{P}(D_y)\overline{\kappa}^l(\varepsilon D_y), \chi^{int}] v^l\Big)\end{aligned}$$

since v^l is divergence free. Hence by using again the commutator estimate (155) of Lemma 19, we get

$$\begin{aligned}&\Big(\mathbb{P}(D_y)\overline{\kappa}^l(\varepsilon D_y)\mathbb{L}^\varepsilon(\chi^{int} v^l), \chi^{int} v^l\Big) =\\ &\Big(\mathbb{L}^\varepsilon(\chi^{int} v^l), \chi^{int} v^l\Big) + \mathcal{O}(1)\Big(||v^l|| + \varepsilon ||\nabla v^l|| + \varepsilon^2 ||\Delta v^l||\Big)||\chi^{int} v^l||.\end{aligned}$$

where $\mathcal{O}(1)$ is bounded by a constant which is independent of ε. The first term in the above equality can be handled by standard integration by parts as previously. This yields

$$\begin{aligned}&||\chi^{int} v^l(T)||^2 + \int_0^T \varepsilon ||\nabla(\chi^{int} v^l)||^2 \\ &\lesssim ||\chi^{int} v_0^l||^2 + \int_0^T \Big((||v^l||_{2,\varepsilon} + ||H||)\, ||\chi^{int} v^l|| + S^\varepsilon \Big)\end{aligned} \tag{50}$$

where the singular term S^ε is given by

$$S^\varepsilon = \varepsilon^{-1} \Big| \Big(\chi^{int} \partial_Z u^b v^l, \chi^{int} v^l\Big) \Big| + \varepsilon^{-1} \Big| \Big(\chi^{int} \partial_{Z'} u^b v^l, \chi^{int} v^l\Big) \Big| := S_1^\varepsilon + S_2^\varepsilon.$$

By using the localization of the support of χ^{int} we have

$$\begin{aligned}|S_1^\varepsilon| &\lesssim \sum_{\mathbf{k},\pm} |b(t,\mathbf{k})||M^{\mathbf{k}}(0)|\varepsilon^{-1} \exp\Big(-\frac{\eta^{\mathbf{k},\pm}\delta}{\sqrt{2}\,\varepsilon}\Big)||\chi^{int} v^l||^2 \\ &\lesssim \sum_{\mathbf{k},\pm} \frac{|b(t,\mathbf{k})|M^{\mathbf{k}}(0)}{\eta^{\mathbf{k},\pm}} ||\chi^{int} v^l||^2 \lesssim ||\chi^{int} v^l||^2\end{aligned} \tag{51}$$

and hence this term is well controlled. The estimate of S_2^ε is similar. To conclude, we finally notice that since the Fourier transform of v^l is localized in $\varepsilon|k| \geq r$, we can write

$$\begin{aligned}||\nabla(\chi^{int} v^l)||^2 &= ||\nabla(\chi^{int} \overline{\kappa}^l(\varepsilon\partial_y) v^l||^2 = ||\overline{\kappa}\nabla(\chi^{int} v^l)||^2 \\ &\geq \varepsilon^{-2}||\chi^{int} v^l||^2\end{aligned}$$

and hence, we deduce (49) from (50) by using the Young inequality which gives for every $\eta > 0$

$$(||v^l||_{2,\varepsilon} + ||H||)\, ||\chi^{int} v^l|| \leq C(\eta)\varepsilon(||H||^2 + ||v^l||_{2,\varepsilon}^2) + \eta\,\varepsilon^{-1}||\chi^{int} v^l||^2$$

and the estimates (32), (46), (47).

3.2.2 Estimates near the boundary

We now explain how to estimate $\chi^b(z/\delta)v^l$ which will be denoted as $\chi^b v^l$ in the following in order to simplify the notations. The estimate of $\chi^{b,1}v^l$ which can be obtained in a similar way will not be detailled. We shall establish

$$\begin{aligned}&||\chi^b v^l(T)||^2 + \int_0^T \Big(\varepsilon^{-1}||\chi^b v^l||^2 + \varepsilon||\nabla(\chi^b v^l)||^2\Big) \\ &\lesssim e^{\gamma_0 T}||v^l(0)||^2_{1,\varepsilon} + \varepsilon e^{\gamma_0 T}\int_0^T ||F||^2 \\ &+\varepsilon\, e^{\gamma_0 T}\int_0^T \Big(||v||^2 + \varepsilon^2||\nabla v||^2 + \varepsilon^4||\Delta v^l||^2 + \varepsilon^2||\partial_t v^l||^2\Big).\end{aligned} \tag{52}$$

We study (41), (42) with $\chi = \chi^{b,0}$. Note that since $\chi^b v^l$ is compactly supported in $\mathbb{T}^2_a \times [0, 2\delta]$, we can use Lemma 20 and replace the Leray projection $\mathbb{P}(D_y)$ by the Leray projection in the half-space $\mathbb{P}_+(D_y)$ modulo a small remainder term. This means that we can study in $\mathbb{T}^2_a \times \mathbb{R}_+$ the equation

$$\partial_t(\chi^b v^l) = \mathbb{P}_+(D_y)\overline{\kappa}(\varepsilon D_y)\mathbb{L}^\varepsilon(\chi^b v^l) + H + E^0, \quad \nabla\cdot(\chi^b v^l) = -\partial_z\chi\, v^l \tag{53}$$

where

$$E^0 = \overline{\chi}^b\Big(\mathbb{P}(D_y) - \mathbb{P}_+(D_y)\Big)\overline{\kappa}^l(\varepsilon D_y)\mathbb{L}^\varepsilon(\chi^b v^l)$$

where $\overline{\chi}^b$ is a smooth function with a support slightly bigger than χ^b. Thanks to Lemma 20, we have

$$||E^0|| \lesssim ||v^l|| + \varepsilon||\nabla v^l|| + \varepsilon^2||\Delta v^l|| \lesssim ||v^l||_{2,\varepsilon}. \tag{54}$$

Again, note that E^0 verifies

$$E^0 = \overline{\overline{\kappa}}^l(\varepsilon D_y)E^0. \tag{55}$$

We add to (53) the only boundary condition

$$\chi^b v^l(t, y, 0) = 0. \tag{56}$$

Since $\chi^b v^l$ is not divergence free, we first lift the divergence to recover a problem with a divergence free constraint. We choose in a classical way d such that

$$\nabla\cdot d = -\partial_z\chi^b\, v_3^l, \quad d(t, y, 0) = 0 \tag{57}$$

and also in such a way that

$$\overline{\kappa}^l(\varepsilon D_y)d = d. \tag{58}$$

This is possible (see [10]) since

$$\int \partial_z\chi^b\, v_3^l = -\int \chi^b\partial_z v_3^l = \int_z \chi^b(z)\int_y \nabla_y\cdot v_h^l = 0.$$

Moreover, we can have

$$||d||_{H^{s+1}} \lesssim ||v^l||_{H^s}, \quad s \geq 0. \tag{59}$$

Note that since d is chosen with the property (58), we can use that $r \lesssim \varepsilon|k|$ on the support of $\overline{\kappa}^l$ and (59) to get that

$$\varepsilon^{-1}||d|| \lesssim ||v^l||. \tag{60}$$

Moreover, by taking the time derivative of (57), we also get that

$$\varepsilon^{-1}||\partial_t d|| \lesssim ||\partial_t v^l||. \tag{61}$$

Now, let us set $w = \chi^b v^l - d$, we deduce from (53), (57) that w solves

$$\partial_t w = \mathbb{P}_+(D_y)\overline{\kappa}(\varepsilon D_y)\mathbb{L}^{b,\varepsilon} w + H + E^0 + E^1 + E^2, \tag{62}$$

$$\nabla \cdot w = 0 \tag{63}$$

for $x = (y, z) \in \mathbb{T}_a^2 \times \mathbb{R}_+$, with the boundary condition

$$w(t, y, 0) = 0 \tag{64}$$

where $\mathbb{L}^{b,\varepsilon}$ is the operator

$$\begin{aligned}
\mathbb{L}^{b,\varepsilon} w \quad = \quad & \varepsilon \Delta w - \Big(u^{int}(\frac{t}{\varepsilon}, t, y, 0) + u^{b,0}(\frac{t}{\varepsilon}, \frac{z}{\varepsilon}, t, y) + \overline{\chi}(\frac{z}{\delta})(u^{int}(\frac{t}{\varepsilon}, t, y, z) - u^{int}(\frac{t}{\varepsilon}, t, y, 0))\Big) \cdot \nabla w \\
& - w \cdot \nabla \Big(u^{int}(\frac{t}{\varepsilon}, t, y, 0) + u^{b,0}(\frac{t}{\varepsilon}, \frac{z}{\varepsilon}, t, y) + \overline{\chi}(\frac{z}{\delta})(u^{int}(\frac{t}{\varepsilon}, t, y, z) - u^{int}(\frac{t}{\varepsilon}, t, y, 0))\Big) \\
& - \frac{e \times w}{\varepsilon}.
\end{aligned}$$

In this operator, we have introduced $\overline{\chi}$ which is again a smooth compactly supported function such that $\overline{\chi}\chi^b = \chi^b$. Moreover, we recall that the notation $u^{b,0}$ refers to the main boundary layer in the vicinity of $z = 0$. The main interest of the introduction of $\overline{\chi}$ is that the term

$$w^\delta = \overline{\chi}(\frac{z}{\delta})\Big(u^{int}(\frac{t}{\varepsilon}, t, y, z) - u^{int}(\frac{t}{\varepsilon}, t, y, 0)\Big)$$

verifies

$$||w^\delta||_{L^\infty} \lesssim \delta, \quad ||\nabla w^\delta||_{L^\infty} \lesssim 1. \tag{65}$$

In the above estimates, $\lesssim$ is independent of δ for $0 < \delta \leq 1$. In the right-hand side of (62), E^1 is the error term coming from d, i.e

$$E^1 = \partial_t d - \mathbb{P}_+(D_y)\overline{\kappa}(\varepsilon D_y)\mathbb{L}^{b,\varepsilon} d.$$

Hence we have

$$||E^1|| \lesssim ||\partial_t d|| + ||\nabla d|| + ||d|| + \varepsilon||\Delta d|| + \varepsilon^{-1}||d||$$

and hence, thanks to (59), (60), (61), we get

$$||E^1|| \lesssim \varepsilon||\partial_t v^l|| + ||v^l|| + \varepsilon||\nabla v^l|| \lesssim ||\partial_t v^l|| + ||v^l||_{1,\varepsilon}. \tag{66}$$

The other error term E^2 is defined as

$$\begin{aligned}
E^2 = -\mathbb{P}_+(\partial_y)\kappa(\varepsilon\partial_y)\Big(u^{b,1}(\frac{t}{\varepsilon}, \frac{1-z}{\varepsilon}, y, 0) + u^r\Big) \cdot \nabla(\chi^b v^l) \\
- \chi^b v^l \cdot \nabla\Big(u^{b,1}(\frac{t}{\varepsilon}, \frac{1-z}{\varepsilon}, t, y) + u^r\Big)\Big)
\end{aligned}$$

so that by using (27) and the same trick as in (51) with the regularity assumption on the coefficients, we get

$$||E^2|| \lesssim \varepsilon||\nabla v^l|| + ||v^l|| \lesssim ||v^l||_{1,\varepsilon}. \tag{67}$$

Finally, note that E^1 and E^2 verify

$$\overline{\overline{\kappa}}^l E^1 = E^1, \quad \overline{\overline{\kappa}}^l E^2 = E^2. \tag{68}$$

To estimate the solution of (62), (63), we can use the following general principle:

Lemma 9 *Consider a linear equation of parabolic type in a domain* Ω

$$\partial_t w = A^\varepsilon w + F \tag{69}$$

with the boundary condition $w_{/\partial\Omega} = 0$ *and the initial condition* $w(0,x) = w_0(x)$. *Consider two weighted norms* N_T^ε *and* $||\cdot||_{T,\varepsilon}$. *Assume that there exists an approximate solver* G^{app} *such that if we define*

$$w^{app}(t) = G^{app}\mathcal{F}, \quad \mathcal{F} = (F, w_0)$$

then w^{app} *satisfies the boundary condition and the initial condition and moreover, there exists* $C_{T,\varepsilon} > 0$ *such that*

$$N_T^\varepsilon(w^{app}) \leq C_{T,\varepsilon}||\mathcal{F}||_{T,\varepsilon} \tag{70}$$

and if we define the rest operator R^{app} *as*

$$R^{app}\mathcal{F} = \partial_t w^{app} - A^\varepsilon w^{app} - F,$$

then, there exists $C^1_{T,\varepsilon} > 0$ *such that*

$$||(R^{app}\mathcal{F}, 0)||_{T,\varepsilon} \leq C^1_{T,\varepsilon}||\mathcal{F}||_{T,\varepsilon}. \tag{71}$$

Moreover, for every $\varepsilon > 0$ *and* $T > 0$ *such that*

$$C^1_{T,\varepsilon} < 1, \tag{72}$$

there exists $C > 0$ *(*$C = \sum_{k\geq 0}(C^1_{T,\varepsilon})^k$*) such that the exact solution of* (69) *satisfies*

$$N_T^\varepsilon(w) \leq CC_{T,\varepsilon}||\mathcal{F}||_{T,\varepsilon}. \tag{73}$$

The proof of this Lemma which only relies on a simple iteration scheme is postponed to the end of the section.

We shall first explain how we can use Lemma 9 to estimate the solution of (62), (63). In other words, we need to find an approximate solver G^{app}. A similar idea was used in [17], nevertheless, here our approximate solver will be completely different. We define the dilation operators

$$M_\varepsilon f(Z) = \sqrt{\varepsilon} f(\varepsilon Z), \; \mathcal{M}_\varepsilon f(\tau, Z) = \varepsilon f(\varepsilon\tau, \varepsilon Z).$$

Note that M_ε is an isometry on $L^2(\mathbb{R}_+)$ and $\mathcal{M}_\varepsilon$ from $L^2([0,T]\times\mathbb{R}_+)$ to $L^2([0,T/\varepsilon]\times\mathbb{R}_+)$. We notice that thanks to (65), we can rewrite the operator as

$$\begin{aligned}
&\mathbb{L}^{b,\varepsilon} w = \frac{1}{\varepsilon} M_\varepsilon^{-1}\mathbb{L}^b\Big(q(t,y), \varepsilon D_y\Big) M_\varepsilon w + \mathcal{O}(1)\Big(\delta|\nabla w| + |w|\Big) \\
&\mathbb{P}_+(D_y) = M_\varepsilon^{-1}\mathbb{P}_+(\varepsilon D_y) M_\varepsilon
\end{aligned}$$

where $q(t,y) = (q_{\bar{k}}(t,y))$ with

$$q_{\bar{k}}(t,y) = b(t,\bar{k})e^{ik\cdot y}.$$

In the above equality, $\mathcal{O}(1)$ is bounded by a number independent of δ if $\varepsilon/\delta \leq 1$. The rescaled operator $\mathbb{L}^b$ is defined by

$$\begin{aligned}\mathbb{L}^b(q,\xi)w &= (\partial_{ZZ} - |\xi|^2)w - \Big(\mathcal{L}^0 q + \mathcal{B}^0(\tau,Z)q\Big)\cdot \begin{pmatrix} i\xi \\ \partial_z \end{pmatrix} w \\ &\quad - w\cdot \begin{pmatrix} i\xi \\ \partial_z \end{pmatrix}\Big(\mathcal{L}^0 q + \mathcal{B}^0(\tau,Z)q\Big) - e\times w.\end{aligned}$$

Next, we use a frozen time approximation, we rewrite (62) as

$$\frac{1}{\varepsilon}\mathcal{M}_\varepsilon^{-1}\mathbb{P}_+(\varepsilon D_y)\overline{\kappa}(\varepsilon D_y)\mathcal{T}(q(0,y),\varepsilon D_y)\mathcal{M}_\varepsilon w + \mathcal{R}^\varepsilon w = H^1 \tag{74}$$

where the symbol $\mathcal{T}(q,\xi)$ is the differential operator acting only on the τ and Z variable defined by

$$\mathcal{T}(q,\xi)w = \partial_\tau w - \mathbb{L}_+^0(q,\xi)w,$$

where $\mathbb{L}_+^0$ is defined in (16) and $\mathcal{R}^\varepsilon w$ is defined by

$$\begin{aligned}\mathcal{R}^\varepsilon w &= \frac{1}{\varepsilon}\mathcal{M}_\varepsilon^{-1}\mathbb{P}_+(\varepsilon D_y)\overline{\kappa}(\varepsilon D_y)\Big(\mathcal{T}(q(0,y),\varepsilon D_y) - \mathcal{T}(q(t,y),\varepsilon D_y)\Big)\mathcal{M}_\varepsilon w \\ &\quad + \mathbb{P}_+(D_y)\mathcal{O}(1)\Big(\delta|\nabla w| + |w|\Big)\end{aligned}$$

and hence satisfies the estimate

$$\int_0^T ||\mathcal{R}^\varepsilon w||^2 \lesssim (T^2+\delta^2)\int_0^T ||\nabla w||^2 + \varepsilon^{-2}(\delta^2+T^2)\int_0^T ||w||^2. \tag{75}$$

The source term H^1 in the right-hand side of (74) is defined by

$$H^1 = H + E^0 + E^1 + E^2. \tag{76}$$

Thanks to our assumption (H), it is natural to define our approximate solver G^{app} and our approximate solution w^{app} as

$$w^{app} = G^{app}\mathcal{F} = \mathcal{M}_\varepsilon^{-1} Op_g \overline{\overline{\kappa}}^l(\varepsilon D_y)\mathcal{M}_\varepsilon \mathcal{D}_\varepsilon \mathcal{F}, \tag{77}$$

where

$$\mathcal{F} = \begin{pmatrix} H^1 \\ w_0 \end{pmatrix}, \quad \mathcal{D}_\varepsilon\mathcal{F} = (\varepsilon H^1, \varepsilon^{\frac{1}{2}} w_0),$$

and with a slight abuse of notation, we define $\mathcal{M}_\varepsilon\mathcal{F}$ as

$$\mathcal{M}_\varepsilon\mathcal{F} = (\mathcal{M}_\varepsilon H^1, M_\varepsilon w_0).$$

To define the operator-valued symbol g, we first define for data $\mathcal{F}(\tau,Z) = (F(\tau,Z), u_0(Z))$, the operator $G(q,\xi)$ (acting on functions depending on τ and Z) such that $G(q,\xi)\mathcal{F}(\tau,Z) := u(\tau,Z)$ is the solution of

$$\partial_\tau u = \mathbb{P}_+(\xi)\mathbb{L}_+^0(\xi)u + F, \quad i\xi\cdot u_h + \partial_Z u_3 = 0 \tag{78}$$

such that $u(0,Z)=u_0(Z)$, $u(\tau,0)=0$.

By using the operator $G(q,\xi)$, we can define a symbol g such that

$$g(y,\xi)=G(q(0,y),\xi) \tag{79}$$

and then a semi-classical operator-valued pseudo-differential operator Op_g as in the Appendix A. In particular for $\mathcal{F}(\tau,y,Z)$, we have the definition

$$Op_g\mathcal{F}=\sum_k e^{iky}G(q(0,y),\varepsilon k)\hat{\mathcal{F}}(\tau,k,Z). \tag{80}$$

With a slight abuse of notation, we shall sometimes use the notation $G(q(0,y),\varepsilon D_y)$ in place of Op_g. Note that the operator $\mathcal{M}_\varepsilon$ acts only on the τ and Z variables. Consequently, we can write

$$G^{app}\mathcal{F}=Op_{g^{app}},\quad g^{app}(y,\xi)=\mathcal{M}_\varepsilon^{-1}g(y,\xi)\mathcal{M}_\varepsilon\mathcal{D}_\varepsilon \tag{81}$$

and hence G^{app} is itself an operator valued semi-classical pseudo-differential operator with symbol g^{app}.

Note that because of (48), (55), (68), since we want to solve (74) only for data such that

$$\overline{\overline{\kappa}}^l(\varepsilon D_y)H^1=H^1,\quad \overline{\overline{\kappa}}^l w_0=w_0, \tag{82}$$

the introduction of $\overline{\overline{\kappa}}^l$ in the definition (77) is justified. To prove that $w^{app}=G^{app}\mathcal{F}$ is a good approximate solution, we want to use the semi-classical pseudo-differential calculus of section A seeing the symbol $g(y,\xi)$ as an operator on $L^2([0,T]\times\mathbb{R}_+)$ or $L^\infty([0,T],L^2(\mathbb{R}_+))$.

The aim of the following lemma is to study the dependence of $G(q,\xi)$ in q. In particular, we prove that it is smooth in q and we estimate the derivatives. This will imply that g is smooth in y and hence we will be able to use the Lemma 16 and 17 of the Appendix A.

We introduce the following notations :

$$|v|_{\Theta,2}=||v||_{L^2((0,\Theta)\times\mathbb{R}_+)},\quad |v|_{\Theta,\infty}=||v||_{L^\infty((0,\Theta),L^2(\mathbb{R}_+))},$$
$$||v||_{\Theta,2}=||v||_{L^2((0,\Theta)\times\mathbb{T}_a^2\times\mathbb{R}_+)},\quad ||v||_{\Theta,\infty}=||v||_{L^\infty((0,\Theta),L^2(\mathbb{T}_a^2\times\mathbb{R}_+))}.$$

Lemma 10 *Thanks to (H), for every ξ in the support of $\overline{\overline{\kappa}}^l$, for every q in $\mathcal{K}$ and for every m, there exists $\alpha_m>0$ and $C_m>0$ such that for every $\Theta>0$, for every $\mathcal{F}=(F(t,\cdot),u_0(\cdot))$, $F(t,\cdot)$, $u_0(\cdot)\in H_\xi$*

$$|D_q^mG(q,\xi)\mathcal{F}|_{\Theta,\infty}+|D_q^mG(q,\xi)\mathcal{F}|_{\Theta,2}+|\partial_ZD_q^mG(q,\xi)\mathcal{F}|_{\Theta,2}\le C_m\Big(|F|_{\Theta,2}+|u_0|_{L^2(\mathbb{R}_+)}\Big) \tag{83}$$

$$|\partial_ZD_q^mG(q,\xi)\mathcal{F}|_{\Theta,\infty}+|\partial_\tau D_q^mG(q,\xi)\mathcal{F}|_{\Theta,2}+|\partial_{ZZ}D_q^mG(q,\xi)\mathcal{F}|_{\Theta,2}\le C_m\Big(|F|_{\Theta,2}+|u_0|_{H^1(\mathbb{R}_+)}\Big). \tag{84}$$

We also postpone the proof of this Lemma to the end of the section.

Thanks to Lemma 10 and Lemma 15, we get that

$$\begin{aligned}
&||w^{app}||_{T,2} = ||G^{app}\mathcal{F}||_{T,2} \\
&= \Big|\Big| Op_g\, \overline{\overline{\kappa}}^{l}(\varepsilon\partial_y)\mathcal{M}_\varepsilon\mathcal{D}_\varepsilon\mathcal{F}(\tau', Z)\, d\tau'\Big)\Big|\Big|_{T/\varepsilon,2} \\
&\lesssim \varepsilon||\mathcal{M}^\varepsilon H^1||_{T/\varepsilon,2} + \sqrt{\varepsilon}||M_\varepsilon w_0|| \\
&\lesssim \varepsilon||H^1||_{T,2} + \sqrt{\varepsilon}||w_0||.
\end{aligned}$$

By the same method, we get

$$\begin{aligned}
&||w^{app}||_{T,\infty} = ||G^{app}\mathcal{F}||_{T,\infty} \\
&= \varepsilon^{-\frac{1}{2}}\Big|\Big| Op_g\overline{\overline{\kappa}}^{l}(\varepsilon\partial_y)\mathcal{M}_\varepsilon\mathcal{D}_\varepsilon\mathcal{F}(\tau', Z)\, d\tau'\Big)\Big|\Big|_{T/\varepsilon,\infty} \\
&\lesssim \varepsilon^{\frac{1}{2}}||\mathcal{M}^\varepsilon H^1||_{T/\varepsilon,2} + ||M_\varepsilon w_0|| \\
&\lesssim \varepsilon^{\frac{1}{2}}||H^1||_{T,2} + ||w_0||.
\end{aligned}$$

In a similar way, since

$$\partial_t\mathcal{M}_\varepsilon^{-1} = \varepsilon^{-1}\mathcal{M}_\varepsilon^{-1}\partial_\tau, \quad \partial_z\mathcal{M}_\varepsilon^{-1} = \varepsilon^{-1}\mathcal{M}_\varepsilon^{-1}\partial_Z, \quad \partial_Z\mathcal{M}_\varepsilon = \varepsilon\mathcal{M}_\varepsilon\partial_z,$$

we find

$$\begin{aligned}
||\partial_z(G^{app}\mathcal{F})||_{T,2} &\lesssim ||H^1||_{T,2} + \varepsilon^{-\frac{1}{2}}||w_0||, \\
||\partial_t(G^{app}\mathcal{F})||_{T,2} &\lesssim ||H^1||_{T,2} + \varepsilon^{-\frac{1}{2}}||w_0|| + \varepsilon^{\frac{1}{2}}||\nabla w_0|| \\
||\partial_{zz}(G^{app}\mathcal{F})||_{T,2} &\lesssim \varepsilon^{-1}||H^1||_{T,2} + \varepsilon^{-\frac{3}{2}}||w_0|| + \varepsilon^{-\frac{1}{2}}||\nabla w_0||, \\
||\partial_z(G^{app}\mathcal{F})||_{T,\infty} &\lesssim \varepsilon^{-\frac{1}{2}}||H^1||_{T,2} + \varepsilon^{-1}||w_0|| + ||\nabla w_0||.
\end{aligned}$$

Consequently, if we define the weighted norm $N_T^\varepsilon(u)$ by

$$N_T^\varepsilon(u) = \varepsilon^{-1}||u||_{T,2} + ||\nabla u||_{T,2} + ||\partial_t u||_{T,2} + \varepsilon||\nabla^2 u||_{T,2} + \varepsilon^{-\frac{1}{2}}||u||_{T,\infty} + \varepsilon^{\frac{1}{2}}||\nabla u||_{T,\infty}$$

Note that the norm $N_T^\varepsilon(w)$ involves ∇w in our definition. Nevertheless, since we use only this norm for functions whose Fourier transform in y is supported in $r \le \varepsilon|\xi| \le R$, the terms involving ∇ in the norm actually gives a usefull, non redondant piece of information for ∂_z only.

We also define the weighted norm $||\mathcal{F}||_{T,\varepsilon}$ as

$$||\mathcal{F}||_{T,\varepsilon} = ||H^1||_{T,2} + \varepsilon^{-\frac{1}{2}}||w_0|| + \varepsilon^{\frac{1}{2}}||\nabla w_0||$$

we have actually proven that

$$N_T^\varepsilon(G^{app}\mathcal{F}) \lesssim ||\mathcal{F}||_{T,\varepsilon}. \tag{85}$$

Moreover, by using again Lemma 10 and Lemma 15, we get by the same method that

$$N_T^\varepsilon\Big(Op_{D_y^\alpha g^{app}}\mathcal{F}\Big) \lesssim C_\alpha||\mathcal{F}||_{T,\varepsilon}. \tag{86}$$

We can now check that w^{app} is a suitable approximate solutions. To have clear notations in the following computation, we use the notation

$$Op_{\mathcal{T}} = \mathcal{T}(q(0,y), \varepsilon D_y).$$

To check that G^{app} is a good approximate solution, we write

$$\begin{aligned}
&\frac{1}{\varepsilon}\mathcal{M}_\varepsilon^{-1}\mathbb{P}_+(\varepsilon D_y)\overline{\kappa}^l(\varepsilon D_y)Op_{\mathcal{T}}\mathcal{M}_\varepsilon Op_{g^{app}}\mathcal{F}+\mathcal{R}^\varepsilon Op_{g^{app}}\mathcal{F}\\
&=\frac{1}{\varepsilon}\mathcal{M}_\varepsilon^{-1}\mathbb{P}_+(\varepsilon D_y)K^l(\varepsilon D_y)Op_{\mathcal{T}}\mathcal{M}_\varepsilon Op_{g^{app}}\mathcal{F}+\mathcal{R}^\varepsilon Op_{g^{app}}\mathcal{F}+\mathcal{C}^r
\end{aligned}$$

where $K^l(\xi)$ is a smooth compactly supported function with a support slightly bigger that the one of $\overline{\overline{\kappa}}$ and such that

$$K^l\overline{\overline{\kappa}}=\overline{\overline{\kappa}}. \tag{87}$$

The commutator $\mathcal{C}^r$ is defined by

$$\mathcal{C}^r=\frac{1}{\varepsilon}\mathcal{M}_\varepsilon^{-1}\mathbb{P}_+(\varepsilon D_y)K^l[\overline{\kappa},Op_{\mathcal{T}}]Op_{g^{app}}\mathcal{F}$$

and hence is very similar to $\mathcal{C}$. In particular, thanks to (85), we have

$$||\mathcal{C}^r||_{T,2}\lesssim\varepsilon||\mathcal{F}||_{T,\varepsilon}. \tag{88}$$

Next, we write

$$\begin{aligned}
&\frac{1}{\varepsilon}\mathcal{M}_\varepsilon^{-1}\mathbb{P}_+(\varepsilon D_y)K^l(\varepsilon D_y)Op_{\mathcal{T}}\mathcal{M}_\varepsilon Op_{g^{app}}\mathcal{F}+\mathcal{R}^\varepsilon Op_{g^{app}}\mathcal{F}+\mathcal{C}^r\\
&=\frac{1}{\varepsilon}\mathcal{M}_\varepsilon^{-1}Op_{\mathbb{P}_+K^l}Op_{\mathcal{T}\mathcal{M}_\varepsilon g^{app}}\mathcal{F}+\mathcal{R}^\varepsilon Op_{g^{app}}\mathcal{F}+\mathcal{C}^r+\mathcal{R}^1\mathcal{F}\\
&=\frac{1}{\varepsilon}\mathcal{M}_\varepsilon^{-1}Op_{\mathbb{P}_+K^l\mathcal{T}g}\mathcal{M}_\varepsilon\mathcal{D}_\varepsilon\mathcal{F}+\mathcal{R}^\varepsilon Op_{g^{app}}\mathcal{F}+\mathcal{C}^r+\mathcal{R}^1\mathcal{F}+\mathcal{R}^2\mathcal{F}.
\end{aligned}$$

Since by definition, the symbol g, is chosen such that

$$\mathbb{P}_+(\xi)K^l(\xi)\mathcal{T}(q(0,y),\xi)g(y,\xi)=K^l(\xi)Id,$$

we get thanks to (82) and (87) that

$$\begin{aligned}
&\frac{1}{\varepsilon}\mathcal{M}_\varepsilon^{-1}\mathbb{P}_+(\varepsilon D_y)\overline{\kappa}(\varepsilon D_y)Op_{\mathcal{T}}\mathcal{M}_\varepsilon Op_{g^{app}}\mathcal{F}+\mathcal{R}^\varepsilon Op_{g^{app}}\mathcal{F}\\
&=K^l(\varepsilon D_y)H^1+\mathcal{R}^\varepsilon Op_{g^{app}}\mathcal{F}+\mathcal{C}^r+\mathcal{R}^1\mathcal{F}+\mathcal{R}^2\mathcal{F}\\
&=H^1+\mathcal{C}^r+\mathcal{R}^1\mathcal{F}+\mathcal{R}^2\mathcal{F}.
\end{aligned}$$

The remainder $\mathcal{R}^1\mathcal{F}$ is defined by

$$\mathcal{R}^1\mathcal{F}=\frac{1}{\varepsilon}\mathcal{M}_\varepsilon^{-1}\mathbb{P}_+(\varepsilon D_y)\overline{\kappa}(\varepsilon D_y)Op_{r^1}\mathcal{F}$$

with the symbol r^1 given by

$$r^1(y,\xi)=\Big(V(t/\varepsilon,q(t,y),Z)\cdot\varepsilon D_y+\varepsilon^2D_y^2\Big)g^{app}(y,\xi).$$

Consequently, thanks to (86), we have

$$||\mathcal{R}^1\mathcal{F}||_{T,2}\lesssim\varepsilon||\mathcal{F}||_{T,\varepsilon}. \tag{89}$$

In a similar way, thanks to Lemma 16, and (86), we have for $m > 7$

$$||\mathcal{R}^2\mathcal{F}||_{T,2} \lesssim \varepsilon \sum_{|\alpha|\leq 2} N^\varepsilon\Big(Op_{D_y^\alpha g^{app}}\mathcal{F}\Big) \lesssim \varepsilon||\mathcal{F}||_{T,\varepsilon}. \tag{90}$$

Finally, thanks to (75), we also have

$$||\mathcal{R}^\varepsilon Op_{g^{app}}\mathcal{F}||_{T,2} \lesssim (\delta + T)||\mathcal{F}||_{T,\varepsilon}. \tag{91}$$

To use the result of Lemma 9, we set

$$R^{app}\mathcal{F} = \mathcal{R}^\varepsilon G^{app}(y, \varepsilon D_y)H^1 + \mathcal{C}^r + \mathcal{R}^1\mathcal{F} + \mathcal{R}^2\mathcal{F}$$

and we see thanks to (88), (89), (90), (91) that

$$||R^{app}\mathcal{F}||_{T,2} \lesssim (\delta + T + \varepsilon)||\mathcal{F}||_{T,\varepsilon}. \tag{92}$$

Consequently, thanks to Lemma 9, for ε and δ sufficiently small, there exists T^0 and $C^0 > 0$ such that

$$N^\varepsilon_{T^0}(w) \leq C^0||\mathcal{F}||_{T^0,\varepsilon}.$$

To get an estimate on a longer interval of time, we can reiterate the process as long as $q(t, y) \in \mathcal{K}$. Indeed, since $q(T^0, y) \in \mathcal{K}$, then we can use the same method as previously for $T \leq T^0$, we rewrite the analogous of (74) but we replace $\mathcal{T}(w^{int,0}(0, y), \varepsilon D_y)$ by $\mathcal{T}(w^{int,0}(T^0, y), \varepsilon D_y)$. The same argument as previously allows to get an estimate on $[T^0, 2T^0]$. The iteration of the argument finally allows to get for some $C > 0$, $\gamma_0 > 0$ independent of T that

$$N^\varepsilon_T(w) \leq Ce^{\gamma_0 T}||\mathcal{F}||_{T,\varepsilon} \tag{93}$$

for every $T > 0$ such that $w^{int}(t, y, 0) \in \mathcal{K}$, for $t \in [0, T]$.

Finally, we can multiply (93) by $\varepsilon^{\frac{1}{2}}$, use that $w = \chi^b v^l - d$ and the estimates (43), (46), (47), (66), (67), (59), (60) to get in particular that

$$\begin{aligned} &||\chi^b v^l(T)||^2 + \int_0^T \Big(\varepsilon^{-1}||\chi^b v^l||^2 + \varepsilon||\nabla(\chi^b v^l)||^2\Big) \\ &\lesssim e^{\gamma_0 T}||v^l_0||^2_{1,\varepsilon} + \varepsilon e^{\gamma_0 T}\int_0^T ||F||^2 \\ &+\varepsilon\, e^{\gamma_0 T}\int_0^T \Big(||v||^2 + \varepsilon^2||\nabla v||^2 + \varepsilon^4||\Delta v^l||^2 + \varepsilon^2||\partial_t v^l||^2\Big). \end{aligned} \tag{94}$$

This ends the proof of (52).

We can now end the proof of Proposition 8. At this point, we shall restrict ε and T such that $\varepsilon e^{\gamma_0 T} \lesssim 1$, this will allow to absorb the terms in the right hand side still depending on v^l which involve higher order derivatives by another estimates. At first, we can use the decomposition

(40) and (52), (49) (we recall that we get an estimate near the upper boundary $z = 1$ completely analogous to (52)), to get

$$||v^l(T)||^2 + \int_0^T \Big(\varepsilon^{-1}||v^l||^2 + \varepsilon||\nabla v^l||^2\Big) \tag{95}$$

$$\lesssim e^{\gamma_0 T}||v^l(0)||_{1,\varepsilon}^2 + \varepsilon e^{\gamma_0 T}\int_0^T ||F||^2$$

$$+(1+\varepsilon e^{\gamma_0 T})\int_0^T \Big(||v||^2 + \varepsilon^2||\nabla v||^2 + \varepsilon^4||\Delta v^l||^2 + \varepsilon^2||\partial_t v^l||^2\Big). \tag{96}$$

To conclude, we use Lemma 6 to estimate higher order derivatives. We get

$$\varepsilon^2||\nabla v^l(T)||^2 + \int_0^T \Big(\varepsilon||\partial_t v^l||^2 + \varepsilon^3||\nabla^2 v||^2\Big) \tag{97}$$

$$\lesssim \varepsilon^2||\nabla v_0||^2 + \int_0^T \Big(\varepsilon^{-1}||v^l||^2 + \varepsilon||\nabla v^l||^2 + \varepsilon||v||^2 + \varepsilon^2||\nabla v||^2 + \varepsilon||F||^2\Big).$$

Finally we can add (97) and (95) times a sufficiently large constant to get

$$||v^l(T)||^2 + \varepsilon^2||\nabla v^l||^2 + \int_0^T \Big(\varepsilon^{-1}||v^l||^2 + \varepsilon||\nabla v^l||^2 + \varepsilon||\partial_t v^l||^2 + \varepsilon^3||\nabla^2 v^l||^2\Big)$$

$$\lesssim e^{\gamma_0 T}||v_0||_{1,\varepsilon}^2 + \varepsilon e^{\gamma_0 T}\int_0^T ||F||^2$$

$$+(1+\varepsilon\, e^{\gamma_0 T})\int_0^T \Big(||v||^2 + \varepsilon^2||\nabla v||^2 + \varepsilon^4||\Delta v^l||^2 + \varepsilon^2||\partial_t v^l||^2\Big)$$

and hence, for ε sufficiently small, and $\varepsilon e^{\gamma_0 T} \leq 1$, we finally get the result of Proposition 8.

3.3 End of the proof of Theorem 4

To get (30), we collect the estimates of Propositions 5, 7, 8. We get for ε, T such that $\varepsilon e^{\gamma_0 T} \leq 1$:

$$||v(T)||^2 + ||\nabla v^{HF}(T)||_{1,\varepsilon}^2 + \int_0^T \Big(\varepsilon||\nabla v||^2 + \varepsilon^{-1}||v^{HF}||^2\Big)$$

$$\lesssim e^{\gamma_0 T}(||v_0||^2 + ||v_0^{HF}||_{1,\varepsilon}^2) + \int_0^T \Big(\varepsilon e^{\gamma_0 T}||F||^2 + ||v||^2 + \varepsilon^2||\nabla v||^2 + |(\kappa^s F, \kappa^s v)|\Big).$$

For ε sufficiently small, this gives

$$||v(T)||^2 + ||\nabla v^{HF}||_{1,\varepsilon}^2 + \int_0^T \Big(\varepsilon||\nabla v||^2 + \varepsilon^{-1}||v^{HF}||^2\Big) \tag{98}$$

$$\lesssim e^{\gamma_0 T}(||v_0||^2 + ||v_0^{HF}||_{1,\varepsilon}^2) + \int_0^T \Big(\varepsilon e^{\gamma_0 T}||F||^2 + ||v||^2 + |(\kappa^s F, \kappa^s v)|\Big)$$

and hence Theorem 4 follows by using the Gronwall inequality.

3.4 Proof of Lemma 9

We represent the exact solution w of (69) as

$$w = \sum_{k\geq 0} w^k$$

where

$$w^0 = G^{app}(F, w_0), \quad R^0 = R^{app}(F, w^0)$$

and for $k \geq 1$ we define recursively w^k and R^k as

$$w^k = -G^{app}(R^{k-1}, 0), \quad R^k = R^{app}(R^{k-1}, 0).$$

Thanks to (70), (71), we easily get by induction that

$$N_T^\varepsilon(w^k) \leq C_{T,\varepsilon}(C^1_{T,\varepsilon})^k ||(F, w_0)||_{T,\varepsilon}, \quad ||R^k||_{T,\varepsilon} \leq (C^1_{T,\varepsilon})^{k+1} ||(F, w_0)||_{T,\varepsilon}$$

and hence, thanks to (72), we get that

$$N_T^\varepsilon(w) \leq C_{T,\varepsilon}\Big(\sum_{k\geq 0}(C^1_{T,\varepsilon})^k\Big) ||(F, w_0)||_{T,\varepsilon} \leq C_{T,\varepsilon} C \, ||(F, w_0)||_{T,\varepsilon}.$$

3.5 Proof of Lemma 10

We start with the proof for $m = 0$. In this section, $\lesssim$ means $\leq C$ where C is independent of Θ.

We can write the solution of (78) under the form

$$G(q,\xi)\mathcal{F}(\tau) = \mathbb{S}_+(\tau, 0, q, \xi)\overline{\overline{\kappa}}^l(\xi) w_0 + \int_0^\tau \mathbb{S}_+(\tau, \tau', q, \xi)\overline{\overline{\kappa}}^l(\xi) F(\tau')\, d\tau'$$

and hence, thanks to (H), we get that

$$|G(q,\xi)\mathcal{F}(\tau)| \lesssim e^{-\alpha\tau}|w_0| + \int_0^\tau e^{-\alpha(\tau-\tau')}|F(\tau')|\, d\tau'.$$

This yields by standard results on convolutions that

$$||G(q,\xi)\mathcal{F}(\tau)||_{\Theta,\infty} \lesssim |w_0| + ||F||_{\Theta,2}$$

and that

$$||G(q,\xi)\mathcal{F}(\tau)||_{\Theta,2} \lesssim |w_0| + ||F||_{\Theta,2} \tag{99}$$

Next, we can reintroduce the pressure and rewrite the equation (78) as

$$\begin{aligned} &\partial_\tau u + V(\tau, Z, q)\cdot \begin{pmatrix} i\xi \\ \partial_Z \end{pmatrix} u + u \cdot \begin{pmatrix} i\xi \\ \partial_Z \end{pmatrix} V(\tau, Z, q) + \begin{pmatrix} i\xi \\ \partial_Z \end{pmatrix} p \\ &\quad + e\times u + |\xi|^2 u - \partial_{ZZ} u = 0 \end{aligned} \tag{100}$$

with the divergence free condition

$$i\xi \cdot u_h + \partial_Z u_3 = 0.$$

Consequently, the standard energy estimate gives

$$|u(\tau)|^2 + \int_0^\tau |\partial_Z u|^2 \lesssim |u_0|^2 + \int_0^\tau |F|^2 + \int_0^\tau |u|^2$$

and since the right hand side is already estimated thanks to (99), we also get

$$||\partial_Z G(q,\xi)\mathcal{F}(\tau)||_{\Theta,2} \lesssim |w_0| + ||F||_{\Theta,2} \tag{101}$$

To estimate higher order derivatives, we use again Lemma 6 (with $\varepsilon = 1$), we get

$$||\partial_Z u||^2_{\Theta,\infty} + ||\partial_\tau u||^2_{\Theta,2} + ||\partial_{ZZ} u||^2_{\Theta,2} \lesssim |\partial_Z u_0|^2 + ||F||^2_{\Theta,2} + ||u||^2_{\Theta,2}$$

and since the right hand side is again already bounded thanks to (99), we get that

$$||\partial_Z G(q,\xi)\mathcal{F}||_{\Theta,\infty} + ||\partial_\tau G(q,\xi)\mathcal{F}||_{\Theta,2} + ||\partial_{ZZ} G(q,\xi)\mathcal{F}||_{\Theta,2} \lesssim |w_0| + ||F||_{\Theta,2}.$$

This ends the proof of (83), (84) for m=0.

The general case follows by induction, we shall just explain how to handle the case $m = 1$. The regularity of the solution of (78) with respect to q follows from standard regularity results for solutions of parabolic equations whose coefficients smoothly depend on a parameter [9]. Taking the differential of (78) with respect to q in the direction h, we find that

$$\Big(\partial_\tau - \mathbb{P}_+(\xi)\mathbb{L}_+(\tau, q, \xi)\Big) D_q u \cdot h = R^1$$

with

$$R^1 = -\Big((D_q V \cdot h) \cdot \begin{pmatrix} i\xi \\ \partial_Z \end{pmatrix} u + u \cdot \begin{pmatrix} i\xi \\ \partial_Z \end{pmatrix} D_q V \cdot h\Big)$$

and $D_q u \cdot h_{/t=0} = 0$. Consequently, we have

$$D_q u \cdot h = G(q,\xi)(R^1, 0).$$

By using (83), (84) for $m = 0$, we get

$$\begin{aligned} & |D_q u \cdot h|_{\Theta,\infty} + |D_q u \cdot h|_{\Theta,2} + |\partial_Z D_q u \cdot h|_{\Theta,2} \\ & + |\partial_Z D_q u \cdot h|_{\Theta,\infty} + |\partial_\tau D_q u \cdot h|_{\Theta,2} + |\partial_{ZZ} D_q u \cdot h|_{\Theta,2} \\ & \lesssim |R^1|_{\Theta,2} \\ & \lesssim |h|\Big(|u|_{\Theta,2} + |\partial_Z u|_{\Theta,2}\Big). \end{aligned}$$

Consequently, we can use again (83), (84) for $m = 0$ to get the result for $m = 1$.

3.6 Estimates of the gradient

The aim of this section is to estimate $||\nabla v(T)||^2$, we first give a crude estimate :

Theorem 11 *Under the assumptions of Theorem 4, we have*

$$\begin{aligned} & \varepsilon^3 ||\nabla v(T)||^2 + \int_0^T \Big(\varepsilon^2 ||\partial_t v||^2 + \varepsilon^4 ||\nabla^2 v||^2 + \varepsilon^2 ||\nabla p||^2\Big) \\ & \lesssim \varepsilon^3 ||\nabla v_0||^2 + e^{\gamma T}\Big(||v_0||^2 + ||v^{HF}||^2_{1,\varepsilon} + \int_0^T |(\kappa^s F, \kappa^s v)| + \varepsilon \int_0^T ||F||^2\Big) \end{aligned} \tag{102}$$

Note that this estimate is relatively crude since we have only a control of $\varepsilon^3||\nabla v(T)||^2$ whereas, because of the size of the boundary layers, we would expect a control of $\varepsilon^2||\nabla v(T)||^2$ as we had for the large frequency part of the solution. Nevertheless, this estimate will be useful in section 4. The reason is that in the proof we do not use in an optimal way the structure of the singular term $\varepsilon^{-1}e \times v$.

Proof of Theorem 11

To get (102), it suffices to use Lemma 6, then multiply the estimate (35) by ε and finally use (30). $\square$

To get better estimates of some components of ∇v, we shall rewrite the equation (28) under an equivalent form which is classically used in fluid mechanics. We define $\eta = \partial_1 v_2 - \partial_2 v_1$ and $w = v_3$. Note that η is the third component of the curl of v. Taking the curl of (28) and using that v is divergence-free, we easily get that the equation for η is given by

$$\begin{aligned} &\partial_t \eta - \frac{\partial_z w}{\varepsilon} - \varepsilon \Delta \eta \\ &= \partial_1 \Big(u^{app} \cdot \nabla v_2 + v \cdot \nabla u_2^{app} + F_2 \Big) - \partial_2 \Big(u^{app} \cdot \nabla v_1 + v \cdot \nabla u_1^{app} + F_1 \Big). \end{aligned} \tag{103}$$

For the equation on w, we first derive the equation for the pressure. We take the divergence of (28) to get

$$\Delta p = \frac{\eta}{\varepsilon} + \nabla \cdot \Big(F - u^{app} \cdot \nabla v - v \cdot \nabla u^{app} \Big) \tag{104}$$

and next, we take the Laplacian in the third component of (28) and we use (104) to get

$$\partial_t \Delta w + \frac{\partial_z \eta}{\varepsilon} - \varepsilon \Delta^2 w = \Big(\Delta \circ \pi_3 - \partial_z \nabla \cdot \Big) \Big(- u^{app} \cdot \nabla v - v \cdot \nabla u^{app} + F \Big) \tag{105}$$

where π_3 stands for the projection on the third component i.e. $\pi_3(v) = v_3$. Next, thanks to (2) and the fact that $\nabla \cdot w = 0$, we notice that the boundary condition for (103), (105) is given by

$$\eta_{/\partial\Omega} = w_{/\partial\Omega} = \partial_z w_{/\partial\Omega} = 0. \tag{106}$$

For the system (103), (105), we can prove :

Theorem 12 *Under the same assumptions as in Theorem 4, we have*

$$\varepsilon^2(||\eta(T)||^2 + ||\nabla w(T)||^2) \lesssim e^{\gamma T} \Big(||v_0||^2_{1,\varepsilon} + \int_0^T |(\kappa^s F, \kappa^s v| + \varepsilon \int_0^T ||F||^2 \Big) \tag{107}$$

and also

$$\varepsilon^2(||\nabla_h v_h(T)||^2 + ||\nabla w(T)||^2) \lesssim e^{\gamma T} \Big(||v_0||^2_{1,\varepsilon} + \int_0^T |(\kappa^s F, \kappa^s v)| + \varepsilon \int_0^T ||F||^2 \Big) \tag{108}$$

for some $\gamma > 0$.

Proof

To estimate the solution of (103), (105), we multiply (103) by η and (105) by $-w$ and we add the two equations. We use (106) to get thanks to integration by parts that

$$\begin{aligned}&\frac{d}{dt}\frac{1}{2}\Big(||\eta||^2+||\nabla w||^2\Big)+\varepsilon(||\nabla\eta||^2+||\Delta w||^2)-\varepsilon^{-1}\Big((\partial_z w,\eta)+(\partial_z\eta,w)\Big)\\&\lesssim\Big(||\nabla v||+\varepsilon^{-1}||v\,\partial_Z u^b||+||F||\Big)\Big(||\nabla\eta||+||\Delta w||\Big).\end{aligned}$$

The crucial fact in the above identity is that

$$(\partial_z w,\eta)+(\partial_z\eta,w)=0$$

so that the singular term vanishes. Consequently, we can use the Young inequality, the estimate (38) and multiply by ε^2 to get

$$\begin{aligned}&\varepsilon^2(||\eta(T)||^2+||\nabla w(T)||^2)+\varepsilon^3\int_0^T(||\nabla\eta||^2+||\Delta w||^2)\\&\lesssim\varepsilon^2(||\eta_0||^2+||\nabla w_0||^2)+\int_0^T\Big(\varepsilon||\nabla v||^2+||v||^2+\varepsilon||F||^2\Big).\end{aligned}$$

Since the right hand side of the above estimate was already estimated in (30), we get (107).

To get (108), we use that $\nabla\cdot v=0$, to get

$$\Delta_h v_2=\partial_1\eta-\partial_2\partial_z w,\quad \Delta_h v_1=-\partial_2\eta-\partial_1\partial_z w,\quad \Delta_h=\partial_1^2+\partial_2^2.$$

This immediately yields that

$$||\nabla_h v_i||\lesssim||\eta||+||\partial_z w||,\quad i=1,\,2$$

and hence (108) follows from (107).

4 Higher order conormal derivatives

The estimates of Theorem 4 are sufficient to get a nonlinear stability result when it is possible to construct a very accurate approximate solution. Indeed, we can very easily estimate weighted derivatives under the form $\varepsilon^{|\alpha|}\partial_y^\alpha$. It suffices to apply the operator $\varepsilon^{|\alpha|}\partial_y^\alpha$ to (28), to rewrite the obtained equation as

$$\mathcal{T}^\varepsilon(\varepsilon^{|\alpha|}\partial_y^\alpha v,\varepsilon^{|\alpha|}\partial_y^\alpha p)=\varepsilon^{|\alpha|}\partial_y^\alpha F+\mathcal{C}$$

where $\mathcal{C}$ is a well-controlled commutator and then to apply Theorem 4. The drawback of this approach is that this yields by Sobolev embedding a bad control of the L^∞ norm which is needed to prove the nonlinear stability by a fixed point argument. Since here, we have been only able to construct an approximate solution with $||R^\varepsilon||\lesssim\varepsilon$ we cannot use this rough approach to conclude.

In order to conclude, we would want to prove as it was done in [25], [27] that we can estimate tangential derivatives i.e $\partial_y v$ where v is still the solution of (28) without loss. This means that we want to estimate $||\partial_y^\alpha v||$ and not $\varepsilon^{|\alpha|}||\partial_y^\alpha v||$. Here a new difficulty appears which was not present in the well prepared case. Indeed, when we apply ∂_y to (28), we have in particular to handle the commutator

$$[\partial_y, (u^{int} + u^b) \cdot \nabla]v = \partial_y u^{int} \cdot \nabla v + \partial_y u^b \cdot \nabla v.$$

The second term has the same property as in the well-prepared case since $u_3^b = 0$, we get that

$$||\partial_y u^b \cdot \nabla v||^2 \lesssim ||\nabla_h v||^2$$

and hence this term can be handled by a Gronwall type argument since it involves only first order tangential derivatives. The main difficulty comes from the term $\partial_y u^{int} \cdot \nabla v$. In the well prepared case, we have $u_3^{int} = 0$ and hence the same argument as above is valid. Nevertheless, here we do not have $u_3^{int} = 0$, we only have $u_{3/\partial\Omega}^{int} = 0$. Consequently, we have the estimate

$$||\partial_y u_3^{int} \partial_z v|| \lesssim ||\varphi \, \partial_z v||$$

where $\varphi(z)$ is a smooth bounded function which behaves as z and $1 - z$ in the vicinity of $z = 0$ and $z = 1$ respectively. The usual method in the case of initial boundary value problems for viscous conservation laws (see [23], [18], for example) is to work in conormal spaces and to consider simultaneously the derivatives in the directions tangent to the boundary and the additional vector field $\varphi\partial_z$. Note that, it is legitimate to apply this last vector field $\varphi\partial_z$ to the equation since $(\varphi\partial_z v)_{/\partial\Omega} = 0$. Nevertheless, in the case of our singularly perturbed incompressible Navier-Stokes equation, it does not seem easy to use readily this method. Indeed $\varphi\partial_z v$ does not verify the incompressibility condition and moreover $\nabla \cdot (\varphi\partial_z v)$ is not small. To overcome this difficulty, we shall use that u_3^{int} is highly oscillating in time. We recall that u_3^{int} is defined by

$$u_3^{int} = \sum_{\mathbf{k}} b(t, \mathbf{k}) e^{i\lambda(\mathbf{k})\tau} e^{ik\cdot y} 2i \sin(k_3 z) = \sum_{\mathbf{k},\, k_3 \neq 0} b(t, \mathbf{k}) e^{i\lambda(\mathbf{k})\tau} e^{ik\cdot y} 2i \sin(k_3 z).$$

Thanks to this definition, we introduce

$$W_j^\varepsilon(t, x) = \sum_{\mathbf{k},\, k_3 \neq 0} k_j \frac{b(t, \mathbf{k})}{\lambda(\mathbf{k})} e^{i\lambda(\mathbf{k})t/\varepsilon} e^{ik\cdot y} 2i \sin(k_3 z).$$

Note that we have

$$\partial_t W_j^\varepsilon = \varepsilon^{-1} \partial_j u_3^{int} + \mathcal{O}(1). \tag{109}$$

and that

$$W_{j/\partial\Omega}^\varepsilon = 0. \tag{110}$$

Next, we introduce the vector fields

$$Z_j = \partial_j + \varepsilon W_j^\varepsilon(t, x)\partial_z,\ j = 1,\, 2, \quad Z_3 = \varepsilon\, \Gamma \varphi(z) \partial_z$$

where $\Gamma \geq 1$ will be chosen sufficiently large. Since $|W_3^\varepsilon| \lesssim \varphi$ thanks to (110), we have

$$|\partial_j v| \lesssim |Z_j v| + |Z_3 v| \tag{111}$$

and hence we can recover good estimates on $\partial_y v$ from estimates on Zv. The main property of these vector fields as we shall see below is that they have good commutation property with respect to $\mathcal{T}$.

Consequently, let us define the weighted norms

$$
\begin{aligned}
&||v||_m^2 = \sum_{|\alpha|\le m} ||Z_3^{\alpha_3} Z_2^{\alpha_2} Z_1^{\alpha_1} v||^2,\\
&Y_m(v) = ||v||_m^2 + \varepsilon^2 ||\nabla_h v_h||_m^2 + \varepsilon^2 ||\nabla v_3||_m^2 + \varepsilon^2 ||\nabla v^{HF}||^2 + \varepsilon^3 ||\nabla v||^2,\\
&Y_{T,m}(v) = \sup_{[0,T]} Y_m(v(T)),\\
&D_{T,m}(v,p) = \int_0^T \Big(\varepsilon^{-1} ||v^{HF}||_m^2 + \varepsilon ||\nabla v||_m^2 + \varepsilon^2 ||\partial_t v||_m^2 + \varepsilon^2 ||\nabla p||_m^2 + \varepsilon^4 ||\nabla^2 v||_m^2\Big).
\end{aligned}
$$

Note that the norm $||\cdot||_m$ that we have just defined is equivalent to the norm $||\cdot||_m$ defined in (18) because of (111), this is why we have abusively used the same notation. In order to deal with the source term in an optimal way, we also use the notation

$$
(u,v)_m = \sum_{|\alpha|\le m} \Big|\Big(Z_3^{\alpha_3} Z_2^{\alpha_2} Z_1^{\alpha_1} u, Z_3^{\alpha_3} Z_2^{\alpha_2} Z_1^{\alpha_1} v\Big)\Big|.
$$

Our main result is :

Theorem 13 *Under the same assumptions as in Theorem 4, we have for every m*

$$
Y_{T,m}(v) + D_{T,m}(v,p) \lesssim e^{\gamma T}\Big(Y_m(v_0) + \int_0^T (\kappa^s F, \kappa^s v)_m + \varepsilon \int_0^T ||F||_m^2\Big). \tag{112}
$$

Proof of Theorem 13

We shall prove (112) by induction on m. In the proof, the harmless numbers contained in $\lesssim$ are also independent of $\Gamma \ge 1$.

Note that for $m = 0$, the estimate (112) follows by collecting (102), (30), (108). To present the main idea without too much technicalities, we first give the proof of (112) for $m = 1$. At first, let us study what happens to (28) when we apply the vector field Z_i for $i = 1$, 2. The case where we apply Z_3 is easier because of the ε weight in the vector field will not be detailled. Moreover, most of the terms which appear in the computation are similar to the ones which appear when we apply Z_i to the equation since $\varepsilon W_i^\varepsilon \partial_z$ behaves in the same way as Z_3.

For $i = 1, 2$, we get

$$
\mathcal{T}^\varepsilon(Z_i v, Z_i p) = Z_i F - \mathcal{C}^Z \tag{113}
$$

where

$$
\mathcal{C}^Z = \mathcal{C}_1^Z + \mathcal{C}_2^Z,
$$

$$
\begin{aligned}
\mathcal{C}_1^Z &= Z_i u^{app} \cdot \nabla v - \varepsilon \partial_t W_i^\varepsilon \partial_z v\\
\mathcal{C}_2^Z &= \varepsilon u^{app} \cdot \nabla W_i^\varepsilon v + v \cdot Z_i \nabla u^{app} + \varepsilon \partial_z p \nabla W_i^\varepsilon + \varepsilon \Delta W_i^\varepsilon v + 2\varepsilon \nabla W_i^\varepsilon \cdot \nabla v.
\end{aligned}
$$

Note that thanks to the crucial property (109), we have, using the notation $v_h = (v_1, v_2)$ for vectors of $v \in \mathbb{R}^3$, that

$$
\begin{aligned}
\mathcal{C}_1^Z &= (\partial_i u_3^{int} - \varepsilon \partial_t W_i^\varepsilon)\partial_z v + \partial_i(u^b + u^r)\cdot \nabla v + \partial_i u_h^{int} \cdot \nabla_h v - \varepsilon W_i^\varepsilon \partial_z u^{app} \cdot \nabla v\\
&= \mathcal{O}(\varepsilon)\partial_z v + \partial_i(u^b + u^r)\cdot \nabla v + \partial_i u_h^{int} \cdot \nabla_h v - \varepsilon W_i^\varepsilon \partial_z u^{app} \cdot \nabla v.
\end{aligned} \tag{114}
$$

By using that

$$\varphi(z)\partial_z u^b = \mathcal{O}(1)$$

since $\varphi(z)$ vanishes on the boundary and by using also the inequality (38), we get that

$$\int_0^T ||\mathcal{C}^Z||^2 \lesssim \int_0^T \Big(\varepsilon^2||\nabla v||^2 + \varepsilon^2||\nabla p||^2 + ||\nabla_h v||^2\Big) \lesssim D_{T,0}(v,p) + \int_0^T ||v||_1^2 \tag{115}$$

where in the second inequality we have used the property (111).

Finally, let us notice that

$$\nabla \cdot \Big(Z_i v\Big) = \varepsilon \nabla W_i^\varepsilon \cdot \partial_z v := d_i. \tag{116}$$

A difficulty comes from the nonvanishing divergence of $Z_i v$. To estimate the solution of (113), we follow the same scheme as in the proof of Theorem 4. We use the same localization in frequencies. We begin with the small frequencies which is actually the more difficult. We apply $\kappa^s(\varepsilon D_y)$ to (113) to get

$$\mathcal{T}^\varepsilon(\kappa^s Z_i v, \kappa^s Z_i p) = \kappa^s Z_i F - \kappa^s \mathcal{C}^Z + \mathcal{C}^s \tag{117}$$

where the commutator $\mathcal{C}^s$ satisfies the estimate

$$||\mathcal{C}^s||^2 \lesssim \varepsilon^2||\nabla v||_1^2 + ||v||_1^2. \tag{118}$$

We use the estimates (115), (118) and the standard energy estimate for (117) to obtain

$$\begin{aligned} &||\kappa^s Z_i(T)||^2 + \varepsilon \int_0^T ||\kappa^s \nabla Z_i v||^2 + (\kappa^s \nabla Z_i p, \kappa^s Z_i v) \\ &\lesssim S^\varepsilon + D_{T,0}(v,p) + \int_0^T \Big(||v||_1^2 + \varepsilon^2||\nabla v||_1^2 + (F,v)_1\Big) \end{aligned}$$

where the singular term S^ε is defined as

$$S^\varepsilon = \Big|(Z_i v \cdot \partial_z u^b, Z_i v)\Big|.$$

As in the proof of Proposition 7, we can estimate the singular term

$$S^\varepsilon \lesssim \varepsilon r ||\nabla Z_i v||^2$$

and hence we can absorb it in the left hand side. Next, we have to be careful with the term involving the pressure since $Z_i v$ is not divergence free. We write thanks to integration by parts and (116)

$$\begin{aligned} (\kappa^s \nabla Z_i p, \kappa^s Z_i v) &= \varepsilon(\kappa^s Z_i p, \kappa^s \nabla W_i^\varepsilon \cdot \partial_z v) \\ &= -\varepsilon(\kappa^s \partial_z Z_i p, \kappa^s \nabla W_i^\varepsilon \cdot v) - \varepsilon(\kappa^s \partial_z Z_i p, \kappa^s \nabla \partial_z W_i^\varepsilon \cdot v) \end{aligned}$$

and hence we get

$$\Big|(\kappa^s \nabla Z_i p, \kappa^s Z_i v)\Big| \lesssim \varepsilon ||\nabla p||_1 ||v||_1.$$

so that we finally find

$$\begin{aligned} &||\kappa^s Z_i(T)||^2 + \varepsilon \int_0^T ||\kappa^s \nabla Z_i v||^2 \\ &\lesssim D_{T,0}(v,p) + \int_0^T \Big(||v||_1^2 + \varepsilon^2||\nabla v||_1^2 + \varepsilon||\nabla p||_1\, ||v||_1 + (F,v)_1\Big). \end{aligned} \tag{119}$$

In a similar way, by combining the previous argument and the arguments in the proof of proposition 5, we get in the high frequency region

$$||\kappa^L Z_i(T)||^2 + \int_0^T \Big(\varepsilon^{-1}||\kappa^L Z_i v||^2 + \varepsilon||\kappa^L \nabla Z_i v||^2\Big) \tag{120}$$
$$\lesssim D_{T,0}(v,p) + \int_0^T \Big(||v||_1^2 + \varepsilon^2||\nabla v||_1^2 + +\varepsilon||\nabla p||_1\,||v||_1 + (F,v)_1\Big).$$

It remains the medium frequency estimates. In this range of frequency, we can lift the nonzero divergence and use the result of Proposition 8. Indeed, let us first establish some usefull estimates on d_i. Thanks to (116), we have

$$\int_0^T ||d_i||^2 \lesssim \varepsilon^2 \int_0^T ||\nabla v||^2 \lesssim \varepsilon D_{T,0}(v,p) \tag{121}$$

and

$$\int_0^T ||\nabla d_i||^2 \lesssim \varepsilon^2 \int_0^T ||\nabla^2 v||^2 \lesssim \varepsilon^{-2} D_{T,0}(v,p). \tag{122}$$

Moreover, we notice that

$$||d_i||^2 \lesssim \frac{1}{\Gamma^2}||Z_3 v||^2 + \varepsilon^2||\partial_z v_3||^2 \lesssim \frac{1}{\Gamma^2}||Z_3 v||^2 + \varepsilon^2||\nabla_h v_h||^2 \lesssim \Big(\frac{1}{\Gamma^2} + \varepsilon^2\Big)Y_1(v) \tag{123}$$

and hence, by taking the time derivative, we also have

$$\int_0^T ||\partial_t d_i||^2 \lesssim \int_0^T ||\partial_t v||_1^2. \tag{124}$$

Now let us choose as before D_i which satisfies the boundary condition (2) and such that

$$\nabla \cdot D_i = d_i. \tag{125}$$

By using (121), (122), (123), (124), we get

$$\int_0^T \Big(||D_i||^2 + ||\nabla D_i||^2\Big) \lesssim \varepsilon D_{T,0}(v,p), \tag{126}$$

$$\int_0^T ||\nabla^2 D_i||^2 \lesssim \varepsilon^{-2} D_{T,0}(v,p), \tag{127}$$

$$||\nabla D_i||^2 \lesssim \Big(\frac{1}{\Gamma^2} + \varepsilon^2\Big)||v||_1^2, \tag{128}$$

$$\int_0^T \Big(||\partial_t D_i||^2 + ||\nabla \partial_t D_i||^2\Big) \lesssim \int_0^T ||\partial_t v||_1^2. \tag{129}$$

To estimate $\kappa^l Z_i v$, we shall consider the equation satisfied by

$$u = Z_i v - D_i. \tag{130}$$

A very usefull remark already used to get (60) is that

$$\varepsilon^{-2}||\kappa^l D_i||^2 \lesssim ||\kappa^l \nabla D_i||^2. \tag{131}$$

Combined with (126)-(129), this gives very good estimate on D_i.

Thanks to (113), we get

$$\mathcal{T}^\varepsilon(\kappa^l u, Z_i p) = \kappa^l Z_i F - \kappa^l \mathcal{C}^Z + \mathcal{C}^l + \mathcal{R}^l, \quad \nabla\cdot(\kappa^l u) = 0 \tag{132}$$

where $\mathcal{C}^l$ is the commutator $[\kappa^l, \mathcal{T}^\varepsilon]$ and hence still satisfies the estimate (118) and $\mathcal{R}^l$ is defined by

$$\mathcal{R}^l = \mathcal{T}^\varepsilon(\kappa^l D_i, 0). \tag{133}$$

Consequently, by combining (126), (127), (129) and (131), we get the estimate

$$\int_0^T ||\mathcal{R}^l||^2 \lesssim \int_0^T \Big(\varepsilon^2||\partial_t v||_1^2\Big) + \varepsilon^2 D_{T,0}(v,p). \tag{134}$$

Next, since u solves (132), we can use the result of Proposition 8 to get

$$\begin{aligned} &||\kappa^l u(T)||^2 + \varepsilon^2||\nabla(\kappa^l u)(T)||^2 + \int_0^T \Big(\varepsilon^{-1}||\kappa^l u||^2 + \varepsilon||\kappa^l \nabla u||^2\Big) \\ &\lesssim e^{\gamma_0 T}(||\kappa^l u_0||^2 + \varepsilon^2||\nabla(\kappa^l u_0)||^2) + \varepsilon e^{\gamma_0 T}\int_0^T \Big(||F||_1^2 + ||\kappa^l \mathcal{C}^Z||^2 + ||\mathcal{C}^l||^2 + ||\mathcal{R}^l||^2\Big). \end{aligned}$$

Now, we can use (126), (128) and (131) and the fact that $u = Z_i v - D_i$ plus the estimates (115), (118) and (134). From now on, we restrict ε and T such that $\varepsilon e^{\gamma_0 T} \leq 1$. This yields

$$\begin{aligned} &||\kappa^l Z_i v(T)||^2 + \varepsilon^2||\nabla(\kappa^l Z_i v)(T)||^2 + \int_0^T \Big(\varepsilon^{-1}||\kappa^l Z_i v||^2 + \varepsilon||\nabla(\kappa^l Z_i v)||^2\Big) \\ &\lesssim e^{\gamma_0 T}(||v_0||_1^2 + \varepsilon^2||\kappa^l\nabla v_0||_1^2) + D_{T,0}(v,p) + \int_0^T \Big(\varepsilon e^{\gamma_0 T}||F||_1^2 + ||v||_1^2 + \varepsilon^2||\nabla v||_1^2 + \varepsilon^3||\partial_t v||_1^2\Big). \end{aligned} \tag{135}$$

Note that by combining (119), (120), (135), we have actually proven that

$$\begin{aligned} &||Zv||^2 + \int_0^T \Big(\varepsilon||\nabla Zv||^2 + \varepsilon^{-1}||Zv^{HF}||^2\Big) \\ &\lesssim e^{\gamma_0 T} Y_1(v_0) + D_{T,0}(v,p) + \int_0^T \Big(||v||_1^2 + \varepsilon e^{\gamma_0 T}||F||_1^2\Big) \end{aligned} \tag{136}$$

Next, since u solves

$$\mathcal{T}^\varepsilon(u, Z_i p) = Z_i F - \mathcal{C}^Z + \mathcal{R}, \quad \nabla\cdot u = 0 \tag{137}$$

where

$$\mathcal{R} = \mathcal{T}^\varepsilon(D_i, 0),$$

we get thanks to the result of Lemma 11 that

$$\begin{aligned} &\varepsilon^3||\nabla u(T)||^2 + \int_0^T \Big(\varepsilon^2||\partial_t u||^2 + \varepsilon^4||\nabla^2 Zv||^2 + \varepsilon^2||\nabla Zp||^2\Big) \\ &\lesssim \varepsilon^3||\nabla u_0||^2 + \int_0^T \Big(||u||^2 + \varepsilon^2||\nabla u||^2 + \varepsilon^2||\mathcal{C}^Z||^2 + \varepsilon^2||\mathcal{R}||^2\Big) \end{aligned}$$

and hence by using (130) and (126), (127), (128), (129), we find

$$\varepsilon^3||\nabla Zv(T)||^2 + \int_0^T \Big(\varepsilon^2||\partial_t Zv||^2 + \varepsilon^4||\nabla^2 Zv||^2 + \varepsilon^2||\nabla Zp||^2\Big) \tag{138}$$
$$\lesssim \varepsilon^3(||v_0||_1^2 + ||\nabla v_0||_1^2) + D_{T,0}(v,p) + \int_0^T \Big(||v||_1^2 + \varepsilon^2||\nabla Zv||^2 + D_{T,0}(v,p) + \varepsilon^2||F||^2\Big)$$

To conclude, we can add (112) for $m = 0$ and a large constant (independent of ε) times (136) plus (138) to get

$$||v(T)||_1^2 + \varepsilon^3||\nabla v(T)||_1^2 + D_{T,1}(v,p)$$
$$\lesssim Y_1(v_0) + \int_0^T \Big((F,v)_1 + \varepsilon||\nabla p||_1||v||_1 + ||v||_1^2 + \varepsilon||F||_1^2\Big).$$

Next, we use the Young inequality to write for every $\delta > 0$

$$\varepsilon||\nabla p||_1\,||v||_1 \leq \frac{\delta}{2}\varepsilon^2||\nabla p||_1^2 + C(\delta)||v||_1^2$$

and we choose δ sufficiently small to absorb $\varepsilon^2||\nabla p||_1^2$ in $D_{T,1}(v,p)$ so that we get

$$||v(T)||_1^2 + \varepsilon^3||\nabla v(T)||_1^2 + D_{T,1}(v,p)$$
$$\lesssim e^{\gamma_0 T}Y_1(v_0) + \int_0^T \Big((F,v)_1 \ + ||v||_1^2 + \varepsilon e^{\gamma_0 T}||F||_1^2\Big).$$

and we conclude by using the Gronwall inequality as in the end of the proof of Theorem 4.

It remains to estimate $||\nabla_h Z_i v_h||$ and $||\nabla v_3||$. We use again (137). The result of Theorem 12 gives

$$\varepsilon^2(||\nabla_h u_h(T)||^2 + ||\nabla w(T)||^2) \lesssim ||u_0||_{1,\varepsilon}^2 + \int_0^T \Big(\varepsilon||F||_1^2 + \varepsilon||\mathcal{C}^Z||^2 + \varepsilon||\mathcal{R}||^2 + ||u||^2 + \varepsilon||\nabla u||^2\Big)$$

and hence we can use that $u = Z_i v - D_I$ and (126)-(129) and (115) to get

$$\varepsilon^2(||\nabla_h Z_i v(T)||^2 + ||\nabla Z_i v_3(T)||^2) \lesssim Y_1(v_0) + D_{T,1}(v,p) + ||v(T)||_1^2 + \int_0^T \Big(\varepsilon||F||^2 + ||v||_1^2\Big)$$

and we can conclude since all the terms in the right-hand side have already been estimated.

We have given the proof for $m = 1$, the general case follows by induction, this is left to the reader.

5 Nonlinear stability

In this section we prove the main theorem 2. We introduce a new notation, namely for $f \in H^m_{anis}(\Omega)$, we denote for $z \in (0,1)$

$$|f|_m^2(z) = \sum_{\alpha\in\mathbb{N}^3,\,|\alpha|\leq m} |Z^\alpha f|_{L^2(\mathbb{T}_a^2)}$$

where the integration only takes place in the y variable. An important remark is that for every $f \in H^m_{anis}(\Omega)$, we have by Sobolev imbedding

$$|f(.,z)|_{L^\infty(\mathbb{T}^2_a)} \leq |f(.,z)|_m \quad m \geq 2.$$

Consequently by Leibnitz formula, we find

$$|fg|_m \lesssim |f|_m \, |g|_m, \quad m \geq 2.$$

5.1 Proof of Theorem 2

For the proof of theorem 2, we recall that $u^{app} = u^{int} + u^b + u^r$, then we can see that u^{app} is an approximate solution to (1) with an error term F^ε which has size $\sqrt{\varepsilon}$ in $L^\infty(0,T;L^2(\Omega))$. Moreover, we can describe more precisely the structure of F^ε. Namely, we have

$$F^\varepsilon = \partial_t u^{app} + \frac{e \times u^{app}}{\varepsilon} - \varepsilon \Delta u^{app} + u^{app} \cdot \nabla u^{app} + \nabla p \tag{139}$$

and we can see that F^ε can be decomposed as

$$F^\varepsilon = F^{\varepsilon,1} + F^{\varepsilon,2}$$

with two types of terms. The first term $F^{\varepsilon,1}$ contains boundary layer terms such as $u^{app} \cdot \nabla u^b$ and hence has an L^2 norm of size $\sqrt{\varepsilon}$ and is concentrated near $z = 0$ or $z = 1$. The other type is $F^{\varepsilon,2}$ which has an L^2 norm of size ε and which comes for instance from the time derivative of $\mathcal{Y}$.

For each $\varepsilon > 0$, the existence theory for the Navier-Stokes system yields the existence and uniqueness of a solution u^ε to (1) in $L^\infty(0,T^\varepsilon;H^1(\Omega))$ with the initial data $u^{\varepsilon,0}$ on some time interval $T^\varepsilon > 0$. Proving that the time $T^\varepsilon > 0$ can be taken uniform in ε and the convergence of $u^\varepsilon - \mathcal{L}(\frac{t}{\varepsilon})w$ to zero will be done together. We set $v = u^\varepsilon - u^{app}$ where we recall that $u^{app} = u^{int} + u^b + u^r$. Note that v depends on ε, but we drop this dependence in ε in the notation. We find that v solves

$$\begin{cases} \partial_t v + u^{app} \cdot \nabla v + v \cdot \nabla u^{app} + \frac{e \times v}{\varepsilon} - \varepsilon \Delta v + \nabla p = -F^\varepsilon - v \cdot \nabla v \\ \\ \mathrm{div}(v) = 0, \qquad v = 0 \text{ on } \partial\Omega \qquad v(t=0) = u^{\varepsilon,0} - u^{app}(t=0). \end{cases} \tag{140}$$

We start with the case $3/4 < \alpha \leq 1$. Let us define

$$T^\varepsilon = \sup\{t_0 \mid \forall t \in [0,t_0],\, Y_m(v(t)) \leq C_0^2 \varepsilon^{2\alpha}\}$$

for some big constant C_0. We recall that $Y_m(v(0)) \leq c^2 \varepsilon^{2\alpha}$ for some constant c, hence by continuity $T^\varepsilon > 0$ with the choice $C_0 > c$. Notice that (140) can be written as $\mathcal{T}^\varepsilon(v,p) = -(F^\varepsilon + v \cdot \nabla v)$. Hence, we can use Theorem 13 to deduce that for $0 < T < T^\varepsilon$, we have

$$Y_{T,m}(v) + D_{T,m}(v,p) \lesssim e^{\gamma T}\Big(Y_m(v_0) + \int_0^T (\kappa^s(F^\varepsilon + v \cdot \nabla v), \kappa^s v)_m + \tag{141}$$

$$\varepsilon \int_0^T ||(F^\varepsilon + v \cdot \nabla v)||^2_m\Big). \tag{142}$$

We have to estimate the different terms appearing on the right hand side of (141). We recall that F^ε can be written as $F^\varepsilon = F^{\varepsilon,1} + F^{\varepsilon,2}$ (for simplicity of notation, we assume that $F^{\varepsilon,1}$ is a boundary layer at $z = 0$, the term at $z = 1$ can be treated in a similar way by replacing z by $1 - z$). We have

$$\begin{aligned}
&\int_0^T |\int_\Omega (\kappa^s(F^{\varepsilon,1} + F^{\varepsilon,2}), \kappa^s v)_m| \\
&\quad \lesssim \int_0^T ||F^{\varepsilon,2}||_m^2 + ||v||_m^2 + \int_0^T \int_{\mathbb{T}_a^2} \Big(\int_0^1 |z^{1/2} F^{\varepsilon,1}|_m dz \Big(\int_0^1 |\partial_z v|_m^2 dz \Big)^{1/2} \Big) dy ds \\
&\quad \lesssim \int_0^T ||F^{\varepsilon,2}||_m^2 + ||v||_m^2 + C\varepsilon^{3/2} \int_0^T ||\nabla v||_m ds \\
&\quad \lesssim \int_0^T ||v||_m^2 + C(1+T)\varepsilon^2 + \frac{\varepsilon}{8} \int_0^T ||\nabla v||_m^2 ds
\end{aligned}$$

Moreover, we also have

$$\begin{aligned}
&\int_0^T |\int_\Omega (\kappa^s(v \cdot \nabla v), \kappa^s v)_m| \\
&\quad \le C \int_0^T ||v||_m^{3/2} ||\nabla v||_m^{3/2} ds \\
&\quad \le C T^{1/4} C_0^{3/2} \varepsilon^{3/2\alpha - 3/4} \Big(\varepsilon \int_0^T ||\nabla v||_m^2 \Big)^{3/4} \\
&\quad \le C T C_0^6 \varepsilon^{6\alpha - 3} + \frac{1}{8} \varepsilon \int_0^T ||\nabla v||_m^2 ds
\end{aligned}$$

The first term in the above estimate yields the restriction $6\alpha - 3 > 2\alpha$ that is $\alpha > 3/4$.

Besides,

$$\begin{aligned}
&\varepsilon \int_0^T ||v \cdot \nabla v||_m^2 \lesssim \varepsilon \int_0^T \int_0^1 |v_h|_m^2 \, |\nabla_h v|_m^2 + |v_3|_m^2 \, |\partial_z v|_m^2 \, dz dt \\
&\le \varepsilon \int_0^T ||\partial_z v_h||_m \, ||v_h||_m \, ||\nabla_h v||_m^2 + ||\partial_z v_3||_m \, ||v_3||_m \, ||\partial_z v||_m^2 \, dt \\
&\le \varepsilon \sup_{[0,T]} \Big(||v_h||_m \, (||\nabla_h v||_m + ||\partial_z v_3||_m \Big) \int_0^T ||\nabla v||_m^2 \\
&\le \varepsilon^{-1} \sup_{[0,T]} Y_m(v(t))^2 \, D_{T,m}(v,p) \\
&\le C_0^2 \, \varepsilon^{2\alpha - 1} \, D_{T,m}(v,p)
\end{aligned}$$

and this term can be absorbed in the left hand side if ε is small enough since $\alpha > 1/2$. Finally, we have

$$\varepsilon \int_0^T ||F^\varepsilon||_m^2 \lesssim CT\varepsilon^2.$$

Hence, by Gronwall lemma, we deduce that for all $T > 0$, there exists an ε_0 such that for $\varepsilon < \varepsilon_0$, we can take $T^\varepsilon > T$ and we have

$$\sup_{0 \le t \le T} ||v(t)||^2 + \int_0^T \varepsilon ||\nabla v||^2 \le C_T \varepsilon^{2\alpha} \tag{143}$$

if $\alpha > 3/4$.

Remark 14 *The proof of Theorem 2 shows that for* $\alpha = 3/4$*, we have a similar result but only on a finite time interval* $(0, T_0)$ *where* T_0 *depends on the initial data* w^0 *and the constant* c *appearing in (19). In some sense, it appears from the proof of the theorem that the value* $\alpha = 3/4$ *is critical. However, it may seem more natural to have a critical result at the value* $\alpha = 1/2$*. But we were not able to prove the theorem 2 when* $1/2 < \alpha < 3/4$*. The main problem comes from estimating the nonlinear term* $\int_0^T | \int_\Omega (\kappa^s(v.\nabla v), \kappa^s v)_m|$.

Appendix

A Simple results about operator valued semi-classical pseudo-differential calculus

We consider smooth symbols $A(y,\xi)$. Here, for each y, ξ, $A(y,\xi)$ is an operator from H_ξ to $L^2(\mathbb{R}_+)$ where H_ξ is a closed subspace of $L^2(\mathbb{R}_+)$. We only need to consider operators associated with symbols of degree zero which basically verify

$$|\partial_y^\alpha \partial_\xi^\beta A(y,\xi)|_{\mathcal{L}(H_\xi, L^2(\mathbb{R}_+)} \leq C_{\alpha,\beta}. \tag{144}$$

We associate to A a semi-classical pseudo-differential operator acting on functions on $\mathbb{T}^2_a \times \mathbb{R}_+$ defined by

$$A(y,\varepsilon\partial_y)w = Op_A w(y,z) = \sum_{k\in\mathbb{Z}^2} e^{ik\cdot y} A(y,\varepsilon k)\hat{w}(k,\cdot)(z) \tag{145}$$

where $\hat{w}(k,z)$ are the Fourier coefficients of $w(\cdot,z)$ that is to say :

$$w(y,z) = \sum_k e^{ik\cdot y}\hat{w}(k,z), \quad \hat{w}(k,z) = \int_{\mathbb{T}^2_a} e^{-ik\cdot y} w(y,z)dy,$$

here we assume that dy is normalized such that $\int_{\mathbb{T}^2_a} dy = 1$. We shall only give the proof of the properties that we have used, for more detail, we refer for example to the book [7].

A.1 Continuity in L^2

Let us define

$$|A|_{M,0} = \sup_{y,\xi} \sup_{|\alpha|\leq M} |\partial_y^\alpha A(y,\xi)|_{\mathcal{L}(H_\xi, L^2(\mathbb{R}_+))}.$$

We also introduce the space $H \subset L^2(\mathbb{T}^2_a \times \mathbb{R}_+)$

$$H = \{w \in L^2(\mathbb{T}^2_a \times \mathbb{R}_+), \quad \hat{w}(k,\cdot) \in H_{\varepsilon k}\}.$$

We have the following result

Lemma 15 *There exists $C > 0$ such that for every ε, $\varepsilon \in (0,1)$,*

$$\forall w \in H, \quad ||Op_A w||_{L^2(\mathbb{T}^2_a\times\mathbb{R}_+)} \leq C|A|_{M,0}||w||_{L^2(\mathbb{T}^2_a\times\mathbb{R}_+)}$$

for $M > 2$.

Proof

We follow the method to prove the boundedness of pseudo-differential operator in [30]. We expand $A(y, \varepsilon k)\hat{w}(k, \cdot)$ in Fourier series :

$$A(y, \varepsilon k)\hat{w}(k, \cdot) = \sum_l e^{il\cdot y}\hat{A}(l, \varepsilon k)\hat{w}(k, \cdot), \quad \hat{A}(l, \varepsilon k) = \int_{\mathbb{T}_a^2} e^{-il\cdot y}A(y, \varepsilon k)\, dy.$$

Since $A(y, \varepsilon k)$ is smooth in y, we have

$$(1 + |l|^2)^N \hat{A}(l, \varepsilon k) = \int_{\mathbb{T}_a^2} (I - \Delta_y)^N (e^{-il\cdot y})A(y, \varepsilon k)\, dy = \int_{\mathbb{T}_a^2} e^{-il\cdot y}(I - \Delta_y)^N A(y, \varepsilon k)\, dy$$

and hence we get

$$|\hat{A}(l, \varepsilon k)|_{\mathcal{L}(L^2(\mathbb{R}_+)} \lesssim \frac{1}{(1 + |l|^2)^N} |A|_{2N,0} \tag{146}$$

Since

$$Op_A w = \sum_l e^{il\cdot y}\Big(\sum_k e^{ik\cdot y}\hat{A}(l, \varepsilon k)\hat{w_k}\Big),$$

we get by using the Bessel identity that for $N > 1$

$$||Op_A w|| \lesssim \sum_l \Big|\Big|\sum_k e^{ik\cdot y}\hat{A}(l, \varepsilon k)\hat{w_k}\Big|\Big| = \sum_l \Big(\sum_k |\hat{A}(l, \varepsilon k)w_k|^2\Big)^{\frac{1}{2}}$$

and hence thanks to (146), we obtain

$$||Op_A w|| \lesssim |A|_{2N,0} \sum_l \frac{1}{(1 + |l|^2)^N} \sum_k |w_k|^2$$

which finally gives by a new use of Bessel identity

$$||Op_A w|| \lesssim |A|_{N,0}\, ||w||.$$

This ends the proof.

A.2 Product

Here we only need to study the product of a differential operator and of a pseudo-differential operator of order 0 which is obvious and the product of a bounded Fourier multiplier and a pseudo-differential operator of order zero. Note that here we want to prove that the residual is small in ε.

Lemma 16 *Let $B(\xi) \in \mathcal{L}(L^2)$ and $A(y, \xi)$ two symbols then*

$$Op_B Op_A w = Op_{BA} w + \varepsilon\, Op_R w, \quad \forall w \in H$$

where there exists C such that for every $\varepsilon \in (0, 1)$,

$$\forall w \in H, \quad ||Op_R w||_{(L^2(\mathbb{T}_a^2 \times \mathbb{R}_+))} \leq C|B|_{0,1}|A|_{M,0}||w||_H$$

for $M > 5$ where

$$|B|_{0,1} = \sup_\xi |\nabla_\xi B(\xi)|_{\mathcal{L}(L^2(\mathbb{R}_+)}.$$

Proof

We write

$$\begin{aligned} Op_B Op_A w(y,\cdot) &= \sum_{k,l} \int_{\mathbb{T}^2_a} e^{ik(y-y')} e^{ily'} B(\varepsilon k) A(y', \varepsilon l) \hat{w}(l,\cdot)\, dy' \\ &= \sum_l e^{ily} \sum_k e^{i(k-l)y} B(\varepsilon k) \Big(\int_{\mathbb{T}^2_a} e^{-iky'} A(y', \varepsilon l) dy' \Big) \hat{w}(l,\cdot) \\ &= \sum_l e^{ily} \sum_k e^{i(k-l)y} B(\varepsilon k) \hat{A}(k, \varepsilon l) \hat{w}(l,\cdot) \\ &= \sum_l e^{ily} \Big(\sum_k e^{iky} B(\varepsilon(k+l)) \hat{A}(k,\varepsilon l) \Big) \hat{w}(l,\cdot). \end{aligned}$$

Consequently, the symbol of $Op_B Op_A$ is C defined by

$$C(y,\varepsilon l) = \sum_k e^{iky} B(\varepsilon(k+l)) \hat{A}(k,\varepsilon l)$$

Note that the object is well-defined since B is uniformly bounded and $\hat{A}(k,\varepsilon l)$ is fastly decreasing in k. By Taylor expansion, we can write

$$B(\varepsilon(k+l)) = B(\varepsilon l) + \varepsilon B^1(\varepsilon l, \varepsilon k)\cdot k, \quad B^1(\xi,\zeta) = \int_0^1 DB(\xi + t\zeta)\, dt$$

and hence we get

$$Op_B Op_A w(y,\cdot) = Op_{BA} w(y,\cdot) + \varepsilon Op_R w(y,\cdot)$$

where R is defined by

$$R(y,\xi) = \sum_k e^{ik\cdot y} B^1(\xi,\varepsilon k)\cdot k \hat{A}(k,\xi).$$

Since B^1 is uniformly bounded and $\hat{A}$ is fastly decreasing, we easily get that R satisfies (144). More precisely, we have

$$|R|_{M,0} \lesssim |B|_{0,1}\, |A|_{M+4,0}.$$

We end the proof by using Lemma 15.

A.3 Version with time dependence

Here, we consider the case where for each y, ξ, $A(y,\xi)$ is an operator from $L^2(0,T;H_\xi)$ or $L^\infty(0,T;H_\xi)$ to $L^2((0,T)\times\mathbb{R}_+)$ or $L^\infty((0,T);L^2(\mathbb{R}_+))$. Let us set

$$||w||_{T,2} = ||w||_{L^2([0,T],L^2(\mathbb{T}^2_a\times\mathbb{R})_+)}, \quad ||w||_{T,\infty} = ||w||_{L^\infty([0,T],L^2(\mathbb{T}^2_a\times\mathbb{R})_+)}$$

and

$$\begin{aligned} ||A||_{M,0,T,2} &= \sup_{y,\xi} \sup_{|\alpha|\le M} |\partial_y^\alpha A(y,\xi)|_{\mathcal{L}(L^2([0,T],H_\xi),L^2([0,T],L^2(\mathbb{R}_+)))} \\ ||A||_{M,0,T,\infty} &= \sup_{y,\xi} \sup_{|\alpha|\le M} |\partial_y^\alpha A(y,\xi)|_{\mathcal{L}(L^\infty([0,T],H_\xi),L^2([0,T],L^2(\mathbb{R}_+)))}, \\ ||B||_{0,1,T,2} &= \sup_\xi ||\nabla_\xi B||_{\mathcal{L}(L^2([0,T],L^2(\mathbb{R}_+)))}, \\ ||B||_{0,1,T,\infty} &= \sup_\xi ||\nabla_\xi B||_{\mathcal{L}(L^\infty([0,T],L^\infty(\mathbb{R}_+)))}. \end{aligned}$$

Then we have the following properties :

Lemma 17 *There exists $C > 0$ such that*

$$\forall w \in L^2([0,T],H), \quad ||Op_A w||_{T,2} \leq C||A||_{M,0,T,2}||w||_{T,2},$$
$$||Op_A w||_{T,\infty} \leq C||A||_{M,0,T,\infty}||w||_{T,2},$$

and

$$Op_B Op_A w = Op_{BA} w + \varepsilon Op_R$$

with

$$\forall w \in L^2([0,T],H), \quad ||Op_R w||_{T,2} \leq C||B||_{0,1,T,2}||A||_{M,0,T,2}||w||_{T,2},$$
$$||Op_R w||_{T,\infty} \leq C||B||_{0,1,T,\infty}||A||_{M,0,T,\infty}||w||_{T,2}.$$

The proof can be obtained by the same method as in the previous version and we shall not detail it.

B The Leray Projection

B.1 The case of a half space

In this section, we study the symbol $\mathbb{P}_+(k)$ for $k \neq 0$. We look for a decomposition

$$v = u + \begin{pmatrix} ikp \\ \partial_z p \end{pmatrix}, \; z > 0, \quad ik \cdot u_h + \partial_z u_3 = 0, \quad u_3(0) = 0 \tag{147}$$

and we set $\mathbb{P}_+(k)v = u$. It is convenient to introduce also $\mathbb{Q}_+(k) = Id - \mathbb{P}_+(k)$. We have the following properties :

Lemma 18 *i) The operator $\mathbb{P}_+(k)$ can be written for every $v \in L^2(\mathbb{R}_+)$ as*

$$\mathbb{P}_+(k)v(z) = I_h v(z) - \int_{\mathbb{R}_+} K_+(k,z,z')v(z')\,dz', \quad \forall z \geq 0$$

where

$$I_h v(z) = \begin{pmatrix} v_h(z) \\ 0 \end{pmatrix}$$

and there exists $C > 0$ such that the matrix $K_+(k,z,z')$ satisfies the estimate

$$\forall k \neq 0, \quad |K_+(k,z,z')| \leq C|k|\Big(e^{-|k|\,|z-z'|} + e^{-|k|(z+z')}\Big) \tag{148}$$

ii) There exists $C > 0$ such that the operators $\mathbb{P}_+(k)$ satisfy for $k \neq 0$ the uniform estimates

$$\forall v \in L^2(\mathbb{R}_+), \quad |\mathbb{P}_+(k)v|_{L^2(\mathbb{R}_+)} \leq C|v|_{L^2(\mathbb{R}_+)} \tag{149}$$

iii) Let $\kappa(\xi)$ be a smooth bounded function which vanishes in the vicinity of zero and $\chi(z)$ another smooth bounded function then there exists $C > 0$ such that we have the uniform estimate

$$\forall \varepsilon \in (0,1), \forall v \in L^2(\mathbb{R}_+), \quad \Big|\Big[\chi(z), \mathbb{P}_+(k)\kappa(\varepsilon k)\Big]v\Big|_{L^2(\mathbb{R}_+)} \lesssim \varepsilon |v|_{L^2(\mathbb{R}_+)}. \tag{150}$$

The precise expression of K_+ will be given in the proof.

Proof

Note that ii) is a direct consequence of i) and the Schur Lemma since

$$\sup_{z'\geq 0}\int_{z\geq 0}|K_+(k,z,z')|\leq C, \quad \sup_{z\geq 0}\int_{z'\geq 0}|K_+(k,z,z')|\leq C.$$

Let us prove iii), we have

$$\begin{aligned}\Big[\chi(z),\mathbb{P}_+(k)\kappa(\varepsilon k)\Big]v &= -\Big[\chi(z),\mathbb{Q}_+(k)\kappa(\varepsilon k)\Big]v\\ &= \int_{z'}K_+(k,z,z')\kappa(\varepsilon k)\Big(\chi(z)-\chi(z')\Big)v(z')\,dz'\end{aligned}$$

and hence since χ is smooth, we get thanks to (148)

$$\begin{aligned}&\Big|\int_{z'}K_+(k,z,z')\kappa(\varepsilon k)\Big(\chi(z)-\chi(z')\Big)v(z')\,dz'\Big|\\ &\lesssim \int\Big(|k|e^{-|k|\,|z-z'|}|z-z'|+|k|e^{-|k|(z+z')}(z+z')\Big)\kappa(\varepsilon k)|v(z')|\,dz'\\ &\lesssim \int\Big(e^{-\frac{|k|}{2}\,|z-z'|}+e^{-\frac{|k|}{2}(z+z')}\Big)\kappa(\varepsilon k)|v(z')|\,dz'\\ &\lesssim \int\Big(e^{-\frac{c}{\varepsilon}|z-z'|}+e^{-\frac{c}{\varepsilon}(z+z')}\Big)|v(z')|\,dz'\end{aligned}$$

for some $c>0$, where in the two last lines, we have used the inequality $Xe^{-X}\lesssim e^{-\frac{X}{2}}$ for $X\geq 0$ and the fact that on the support of κ, we have $\varepsilon|k|\geq c>0$. The result follows by a new use of the Schur Lemma since

$$\sup_{z'\geq 0}\int_{z\geq 0}e^{-\frac{c}{\varepsilon}|z-z'|}+e^{-\frac{c}{\varepsilon}(z+z')}\,dz\leq C\varepsilon, \quad \sup_{z\geq 0}\int_{z'\geq 0}e^{-\frac{c}{\varepsilon}|z-z'|}+e^{-\frac{c}{\varepsilon}(z+z')}\,dz'\leq C\varepsilon.$$

It remains to prove i). The result follows by an explicit computation. Thanks to (147), we find that p solves the ODE

$$\Big(\partial_{zz}-|k|^2\Big)p=ikv_h+\partial_z v_3, \quad \partial_z p(0)=v_3(0).$$

The unique bounded solution is given by

$$\begin{aligned}p(z)=&-\int_z^{+\infty}\Big(ik\cdot v_h(z')+\partial_z v_3(z')\Big)e^{-|k|z'}\frac{\cosh(|k|z)}{|k|}\,dz'\\ &-\int_0^z\Big(ik\cdot v_h(z')+\partial_z v_3(z')\Big)e^{-|k|z}\frac{\cosh(|k|z')}{|k|}\,dz'-\frac{v_3(0)}{|k|}e^{-|k|z}\end{aligned}$$

and hence, after an integration by part, we find

$$\begin{aligned}p=&-\int_z^{+\infty}ik\cdot v_h(z')e^{-|k|z'}\frac{\cosh(|k|z)}{|k|}+v_3(z')e^{-|k|z'}\cosh(|k|z)\,dz'\\ &-\int_0^z ik\cdot v_h(z')e^{-|k|z}\frac{\cosh(|k|z')}{|k|}-v_3(z')e^{-|k|z}\sinh(|k|z')\,dz'.\end{aligned}$$

Note that this also yields

$$\partial_z p = v_3(z) - \int_z^{+\infty} ik\cdot v_h(z')e^{-|k|z'}\sinh(|k|z) + v_3(z')|k|e^{-|k|z'}\sinh(|k|z)\,dz'$$
$$- \int_0^z -ik\cdot v_h(z')e^{-|k|z}\cosh(|k|z') + v_3(z')|k|e^{-|k|z}\sinh(|k|z').$$

Finally, since

$$\mathbb{P}_+ v = v - \begin{pmatrix} ikp \\ \partial_z p \end{pmatrix} = I_h v - \int_0^{+\infty} K_+(k,z,z')v(z')\,dz',$$

it suffices to read the expression of K_+, we find

$$K_+(k,z,z') = \begin{pmatrix} \frac{k_1^2}{|k|}e^{-|k|z'}\cosh(|k|z) & \frac{k_1k_2}{|k|}e^{-|k|z'}\cosh(|k|z) & -ik_1e^{-|k|z'}\cosh(|k|z) \\ \frac{k_1k_2}{|k|}e^{-|k|z'}\cosh(|k|z) & \frac{k_2^2}{|k|}e^{-|k|z'}\cosh(|k|z) & -ik_2e^{-|k|z'}\cosh(|k|z) \\ -ik_1e^{-|k|z'}\sinh(|k|z) & -ik_2e^{-|k|z'}\sinh(|k|z') & -|k|e^{-|k|z'}\sinh(|k|z) \end{pmatrix}, \ 0 \le z < z' \tag{151}$$

and

$$K_+(k,z,z') = \begin{pmatrix} \frac{k_1^2}{|k|}e^{-|k|z}\cosh(|k|z') & \frac{k_1k_2}{|k|}e^{-|k|z}\cosh(|k|z') & ik_1e^{-|k|z}\sinh(|k|z') \\ \frac{k_1k_2}{|k|}e^{-|k|z}\cosh(|k|z') & \frac{k_2^2}{|k|}e^{-|k|z}\cosh(|k|z') & ik_2e^{-|k|z}\sinh(|k|z') \\ ik_1e^{-|k|z}\cosh(|k|z') & ik_2e^{-|k|z}\cosh(|k|z') & -|k|e^{-|k|z}\sinh(|k|z') \end{pmatrix}, \ z > z' \ge 0. \tag{152}$$

The estimate (148) follows immediately from the above expressions.

B.2 The case of a strip.

We now study the operator $\mathbb{P}(k)$, for $k \neq 0$. We now look for the decomposition

$$v = u + \begin{pmatrix} ikp \\ \partial_z p \end{pmatrix}, \ z \in (0,1), \quad ik\cdot u_h + \partial_z u_3 = 0, \quad u_3(0) = 0,\ u_3(1) = 0.$$

Lemma 19 *i) The operator $\mathbb{P}(k)$ can be written for every $v \in L^2(\mathbb{R}_+)$ as*

$$\mathbb{P}(k)v(z) = I_h v(z) - \int_{(0,1)} K(k,z,z')v(z')\,dz', \quad \forall z \ge 0$$

where

$$I_h v(z) = \begin{pmatrix} v_h(z) \\ 0 \end{pmatrix}$$

and there exists $C > 0$ such that the matrix $K(k,z,z')$ satisfies the estimate

$$\forall k \neq 0, \quad |K_+(k,z,z')| \le C|k|\Big(e^{-|k|\,|z-z'|} + e^{-|k|(1-z+1-z')}\Big) \tag{153}$$

ii) There exists $C > 0$ such that the operators $\mathbb{P}(k)$ satisfy for $k \neq 0$ the uniform estimates

$$\forall v \in L^2(\mathbb{R}_+), \quad |\mathbb{P}(k)v|_{L^2(0,1)} \le C|v|_{L^2(0,1)} \tag{154}$$

iii) Let $\kappa(\xi)$ a smooth function bounded function which vanishes in the vicinity of zero and $\chi(z)$ another smooth bounded function then there exists $C > 0$ we have the uniform estimate

$$\forall \varepsilon \in (0,1), \forall v \in L^2(0,1), \quad \Big|\Big[\chi(z), \mathbb{P}_+(k)\kappa(\varepsilon k)\Big]v\Big|_{L^2(0,1)} \lesssim \varepsilon|v|_{L^2(0,1)}. \tag{155}$$

Proof

We now solve the ODE

$$\partial_{zz}p - |k|^2 p = ik\cdot v_h + \partial_z v_3,\ z\in(0,1), \quad \partial_z p(0) = v_3(0),\ \partial_z p(1) = v_3(1).$$

The explicit resolution gives

$$\begin{aligned} p = &\int_0^z \Big(ik\cdot v_h + \partial_z v_3\Big)\big)\frac{\sinh(|k|(z-z'))}{|k|}\,dz' \\ &-\int_0^1 \Big(ik\cdot v_h + \partial_z v_3\Big)\frac{\cosh(|k|(1-z')}{|k|\sinh|k|}\cosh(|k|z)\,dz' \\ &-v_3(0)\frac{\cosh(|k|(z-1)}{|k|\sinh|k|} + v_3(1)\frac{\cosh(|k|z)}{|k|\sinh|k|} \end{aligned}$$

and hence after an integration by parts, we find

$$\begin{aligned} p = &-\int_z^1 \Big(ik\cdot v_h \frac{\cosh(|k|(1-z'))\cosh(|k|z)}{|k|\sinh|k|} + v_3\frac{\sinh(|k|(1-z'))}{\sinh|k|}\cosh(|k|z)\Big)dz' \\ &+\int_0^z \Big(ik\cdot v_h\Big(\frac{\sinh(|k|(z-z'))}{|k|} - \frac{\cosh(|k|(1-z'))}{|k|\sinh|k|}\cosh(|k|z)\Big) \\ &\quad + v_3\Big(\cosh(|k|(z-z')) - \frac{\sinh(|k|(1-z'))}{\sinh|k|}\cosh(|k|z)\Big)\Big)\,dz'. \end{aligned}$$

Note that we have used the fact that

$$v_3(0)\Big(\frac{\cosh|k|}{|k|\sinh|k|}\cosh(|k|z) - \frac{\sinh(|k|z)}{|k|} - \frac{\cosh(|k|(z-1)}{|k|\sinh|k|}\Big) = 0.$$

Finally, we can rewrite the pressure under the more convenient form

$$\begin{aligned} p = &-\int_z^1 i\frac{k}{|k|}\cdot v_h\Big(\frac{e^{-|k|z'}\cosh(|k|z)}{1-e^{-2|k|}} + \frac{e^{-|k|(2-z')}\cosh(|k|z)}{1-e^{-2|k|}}\Big)\,dz' \\ &-\int_z^1 v_3\Big(\frac{e^{-|k|z'}\cosh(|k|z)}{1-e^{-2|k|}} - \frac{e^{-|k|(2-z')}\cosh(|k|z)}{1-e^{-2|k|}}\Big)\,dz' \\ &-\int_0^z i\frac{k}{|k|}\cdot v_h\Big(\frac{e^{-|k|z}\cosh(|k|z')}{1-e^{-2|k|}} + \frac{e^{-|k|(2-z)}\cosh(|k|z')}{1-e^{-2|k|}}\Big)\,dz' \\ &+\int_0^z v_3\Big(\frac{e^{-|k|z}\sinh(|k|z')}{(1-e^{-2|k|})} + \frac{e^{-|k|(2-z)}\sinh(|k|z')}{1-e^{-2|k|}}\Big)\,dz'. \end{aligned}$$

By taking the derivative with respect to z, we also find

$$\begin{aligned} \partial_z p = &\, v_3(z) - \int_z^1 i\frac{k}{|k|}\cdot v_h|k|\Big(\frac{e^{-|k|z'}\sinh(|k|z)}{1-e^{-2|k|}} + \frac{e^{-|k|(2-z')}\sinh(|k|z)}{1-e^{-2|k|}}\Big)\,dz' \\ &-\int_z^1 v_3|k|\Big(\frac{e^{-|k|z'}\sinh(|k|z)}{1-e^{-2|k|}} - \frac{e^{-|k|(2-z')}\sinh(|k|z)}{1-e^{-2|k|}}\Big)\,dz' \\ &-\int_0^z i\frac{k}{|k|}\cdot v_h|k|\Big(-\frac{e^{-|k|z}\cosh(|k|z')}{1-e^{-2|k|}} + \frac{e^{-|k|(2-z)}\cosh(|k|z')}{1-e^{-2|k|}}\Big)\,dz' \\ &+\int_0^z v_3|k|\Big(-\frac{e^{-|k|z}\sinh(|k|z')}{(1-e^{-2|k|})} + \frac{e^{-|k|(2-z)}\sinh(|k|z')}{1-e^{-2|k|}}\Big)\,dz'. \end{aligned}$$

Consequently, since

$$\mathbb{P}v = v - \begin{pmatrix} ikp \\ \partial_z p \end{pmatrix} = I_h v - \int_0^{+\infty} K(k,z,z')v(z')\,dz',$$

we find

$$K(k,z,z') =$$

$$\frac{1}{1-e^{-2|k|}} \begin{pmatrix} \frac{k_1^2}{|k|}a_+(k,z,z') & \frac{k_1k_2}{|k|}a_+(k,z,z') & -ik_1a_-(k,z,z') \\ \frac{k_1k_2}{|k|}a_+(k,z,z') & \frac{k_2^2}{|k|}a_+(k,z,z') & -ik_2a_-(k,z,z') \\ -ik_1b_+(k,z,z') & -ik_2b_+(k,z,z') & -|k|b_-(k,z,z') \end{pmatrix}, \; 0 \le z < z', \tag{156}$$

$$= \frac{1}{1-e^{-2|k|}} \begin{pmatrix} \frac{k_1^2}{|k|}a_+(k,z',z') & \frac{k_1k_2}{|k|}a_+(k,z',z) & -ik_1b_+(k,z',z) \\ \frac{k_1k_2}{|k|}a_+(k,z',z) & \frac{k_2^2}{|k|}a_+(k,z',z) & -ik_2b_+(k,z',z) \\ ik_1a_-(k,z',z) & ik_2a_-(k,z',z) & -|k|b_-(k,z',z) \end{pmatrix}, \; 0 \le z' < z \tag{157}$$

where

$$\begin{aligned} a_+(k,z,z') &= e^{-|k|z'}\cosh(|k|z) + e^{-|k|(2-z')}\cosh(|k|z), \\ a_-(k,z,z') &= e^{-|k|z'}\cosh(|k|z) - e^{-|k|(2-z')}\cosh(|k|z), \\ b_+(k,z,z') &= e^{-|k|z'}\sinh(|k|z) + e^{-|k|(2-z')}\sinh(|k|z), \\ b_-(k,z,z') &= e^{-|k|z'}\sinh(|k|z) - e^{-|k|(2-z')}\sinh(|k|z). \end{aligned}$$

Thanks to (156), (157), the estimate (153) follows easily and next, we obtain ii) and iii) as previously.

B.3 Estimate of $\mathbb{P} - \mathbb{P}_+$

Lemma 20 *Let $\kappa(\xi)$ a smooth function bounded function which vanishes in the vicinity of zero and $\chi_1(z)$, $\chi_2(z)$ two smooth bounded function which are compactly supported in $[0,1)$, then there exists $C>0$ and $c>0$ such that we have the uniform estimate*

$$\forall \varepsilon \in (0,1),\, \forall v \in L^2(\mathbb{R}_+), \quad |\chi_1\Big(\mathbb{P} - \mathbb{P}_+\Big)\kappa(\varepsilon k)\chi_2 v|_{L^2(\mathbb{R}_+)} \le Ce^{-\frac{c}{\varepsilon}}|v|_{L^2(\mathbb{R}_+)} \tag{158}$$

Proof

We can use the explicit expressions given by (156), (157) and (151), (152). This yields

$$\begin{aligned} &|\chi_1(z)\Big(\mathbb{P} - \mathbb{P}_+\Big)\kappa(\varepsilon k)\chi_2(z)v| \\ &\le \int_0^{+\infty} \chi_1(z)\chi_2(z')\kappa(\varepsilon k)|v(z')|\Big(|k|e^{-|k||z-z'|}(\frac{1}{1-e^{-2|k|}} - 1) + |k|e^{-2|k|+|k|z+|k|z'}\Big)\,dz' \end{aligned}$$

and hence if the support of $\kappa(\xi)$ is in $|\xi| \ge r > 0$ and the support of χ and χ' are in $[0,\delta]$, $\delta < 1$, we find

$$\begin{aligned} &|\chi_1(z)\Big(\mathbb{P} - \mathbb{P}_+\Big)\kappa(\varepsilon k)\chi_2(z)v| \\ &\le \int_0^{+\infty} \chi_1(z)\chi_2(z')\kappa(\varepsilon k)|v(z')|\Big(|k|e^{-|k||z-z'|}e^{\frac{-2r}{\varepsilon}} + |k|e^{-2|k|(1-\delta)}\Big)\,dz' \\ &\lesssim \int_0^{+\infty} \chi_1(z)\chi_2(z')\kappa(\varepsilon k)|v(z')|e^{-\frac{r}{\varepsilon}}\,dz'. \end{aligned}$$

The estimate (158) follows by using again the Schur Lemma.

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