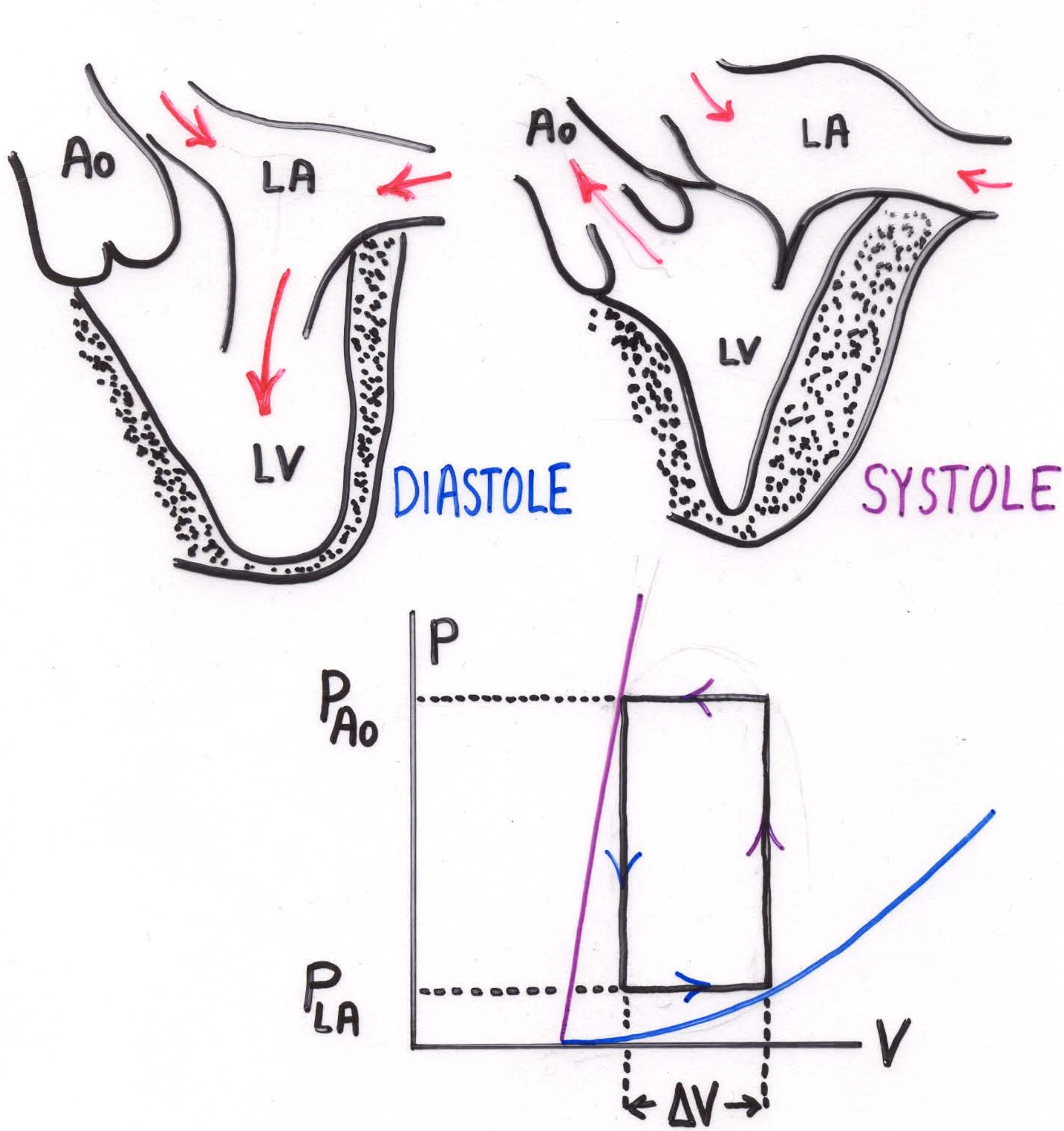
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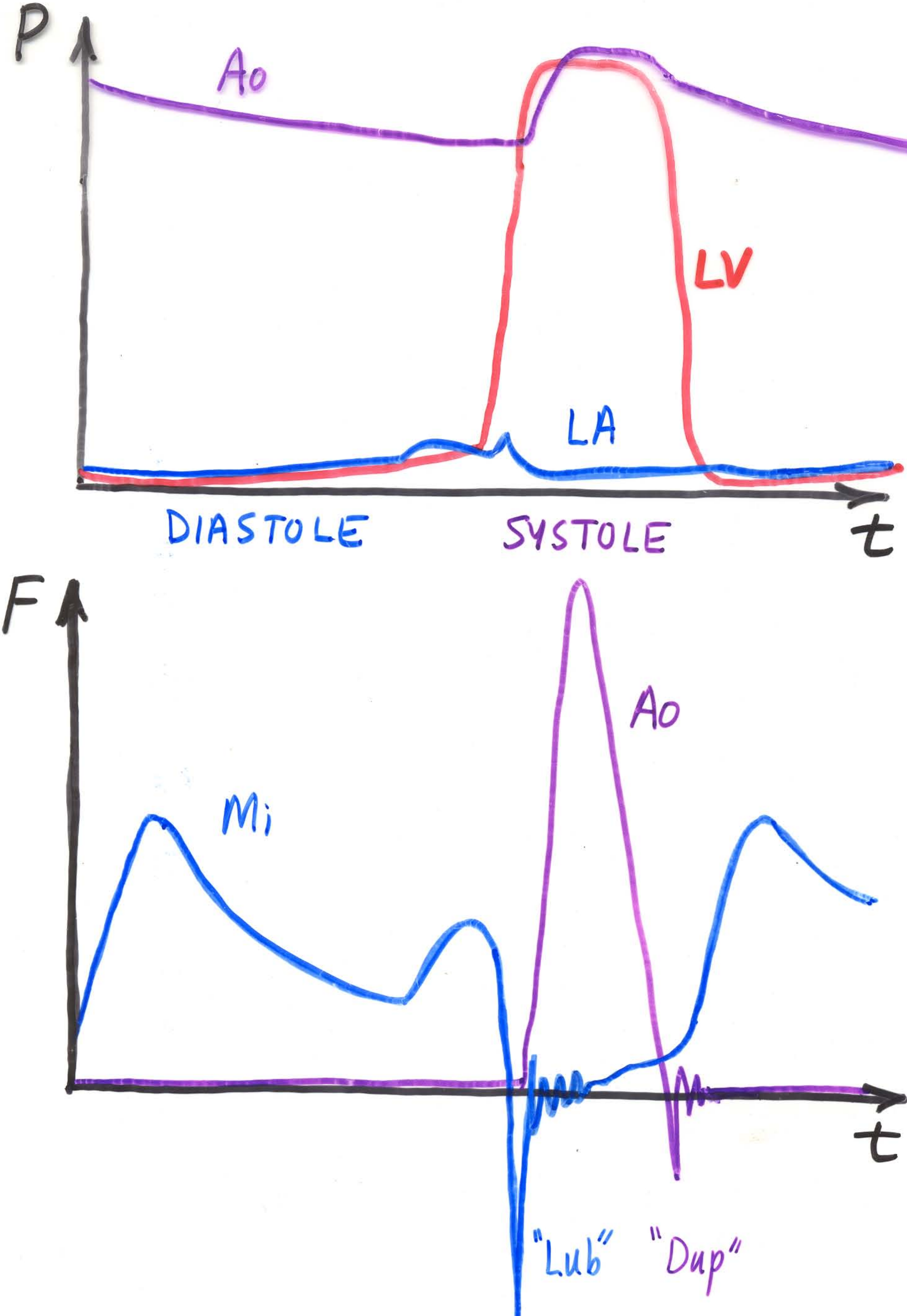
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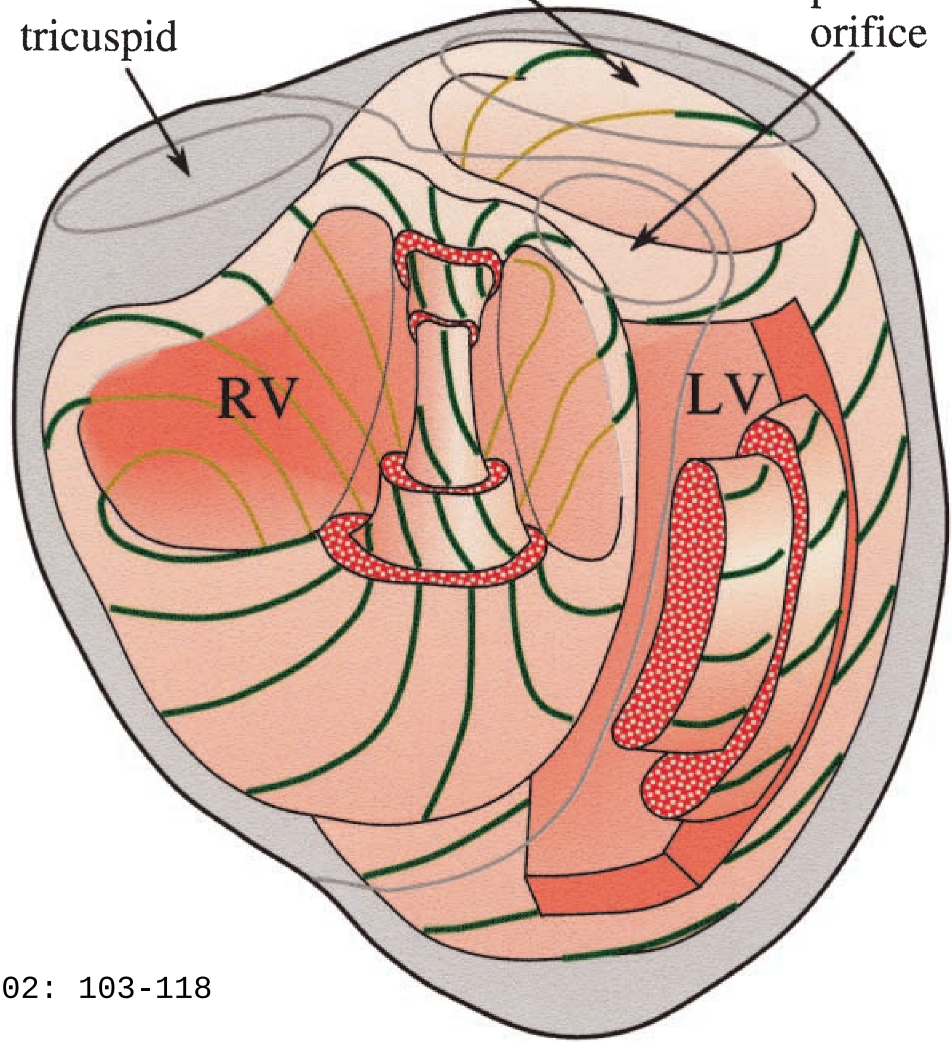
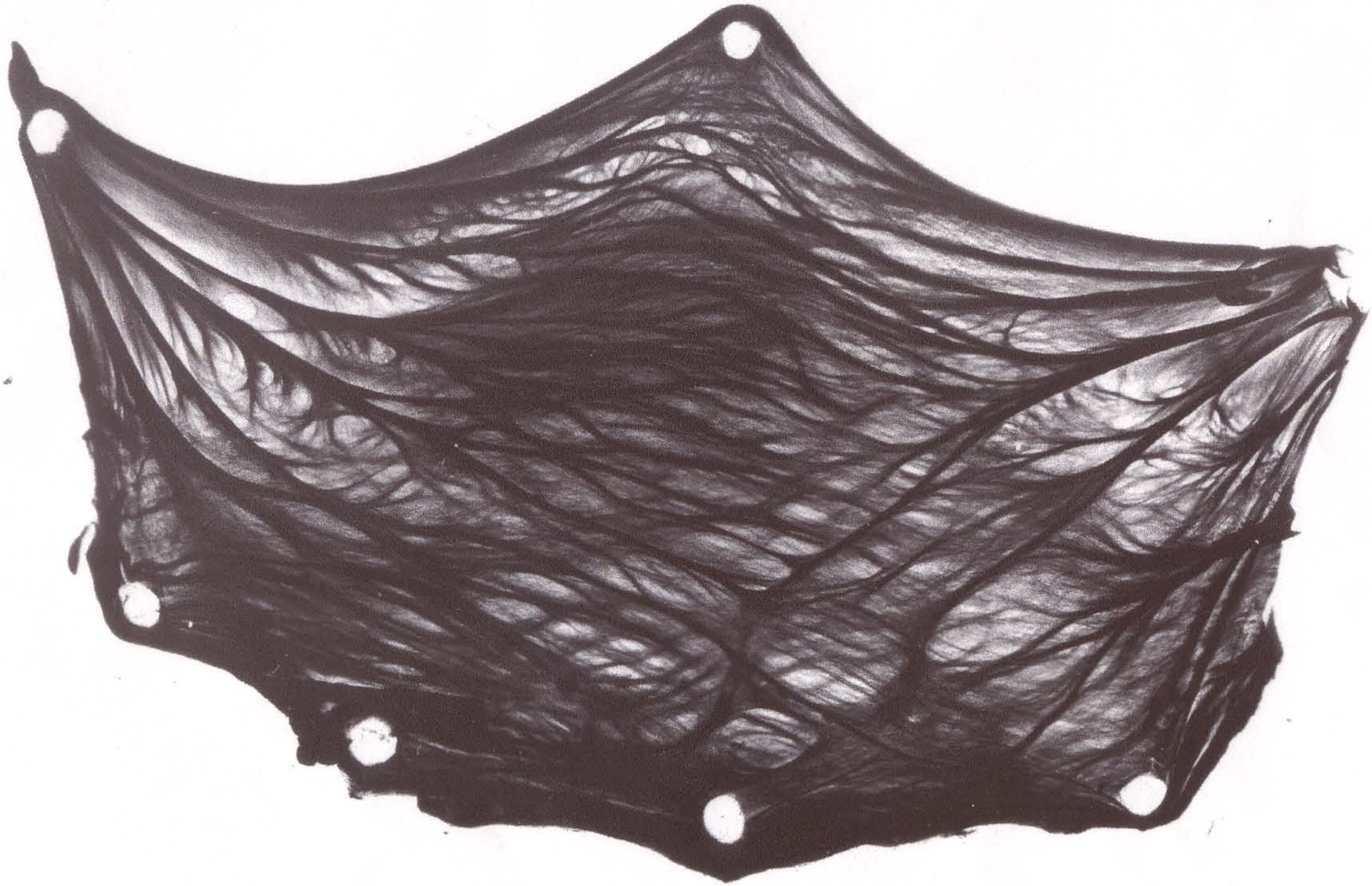
"Lub"
"Dup"

Fig. 9 Schematic description of the fibre architecture of the whole ventricular mass. The lines in green symbolize the geodesic trajectories of the fibres on the nested “pretzels”. Wedges have been cut out of the right and left ventricular walls, in order to show the innermost hulls

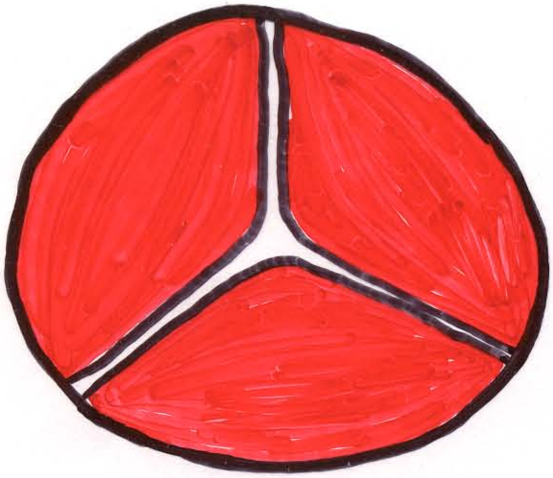


Jouk P-S, et al.
Anat.Embryol.(2000) 202: 103-118

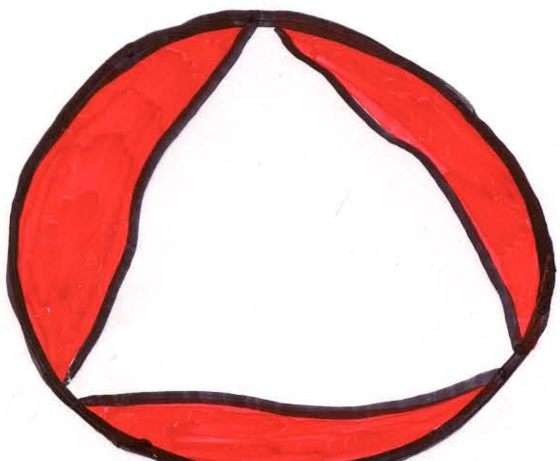


Aortic Leaflet Stained for Collagen
A.A.H.J. Sauren

AORTIC VALVE

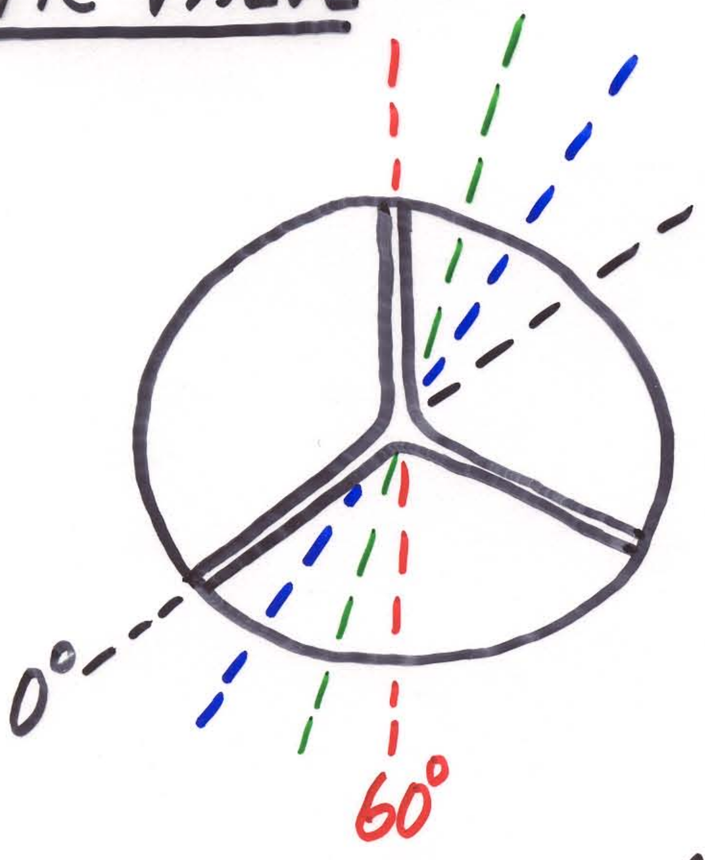


CLOSED

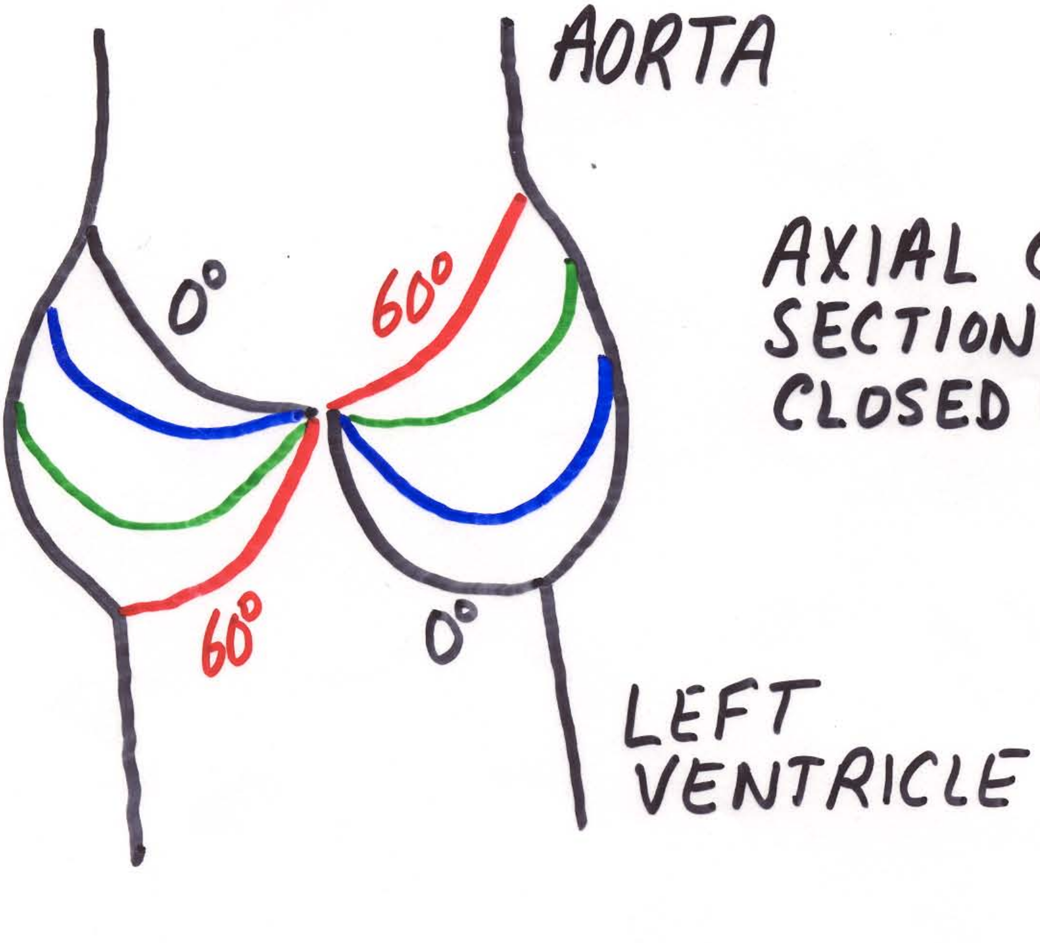


OPEN

AORTIC VALVE

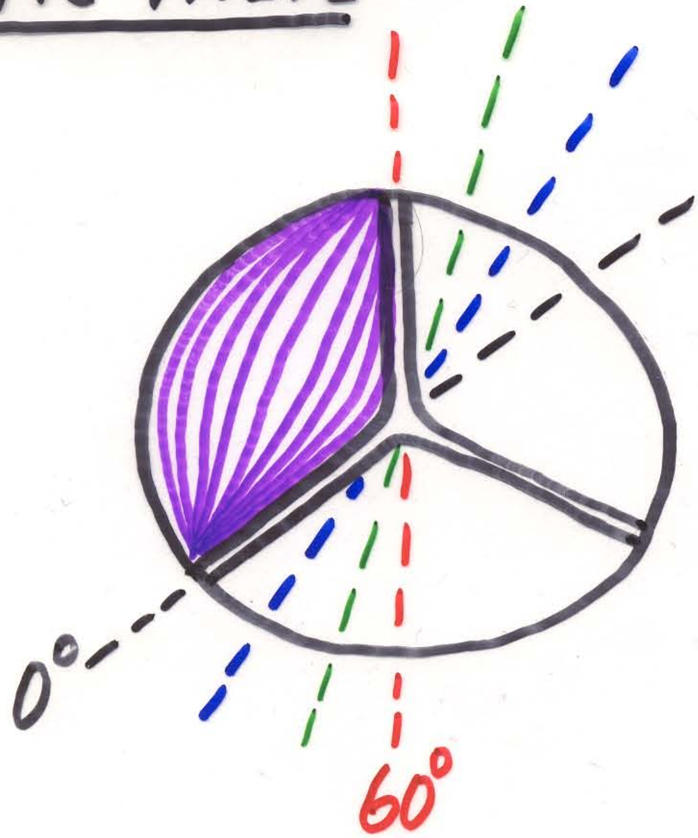


CLOSED VALVE
VIEWED
FROM AORTA

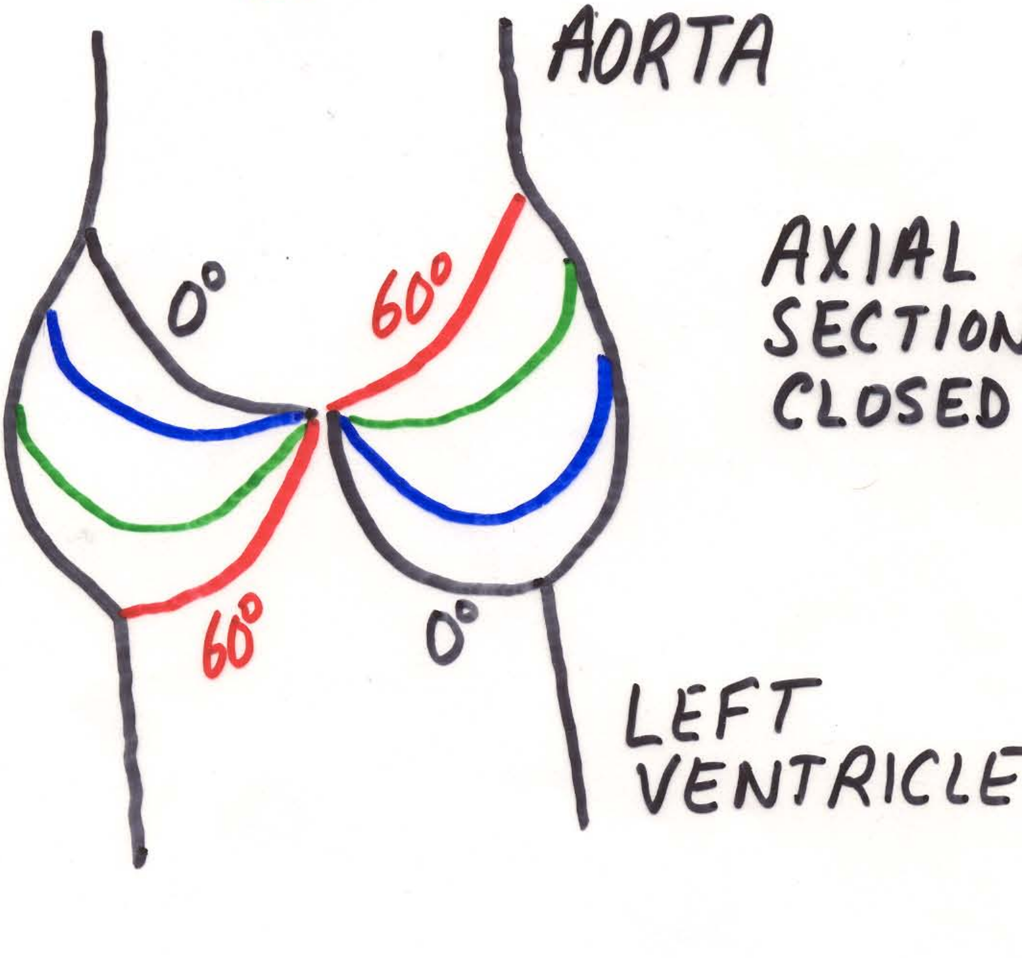


AXIAL CROSS
SECTIONS OF
CLOSED VALVE

AORTIC VALVE

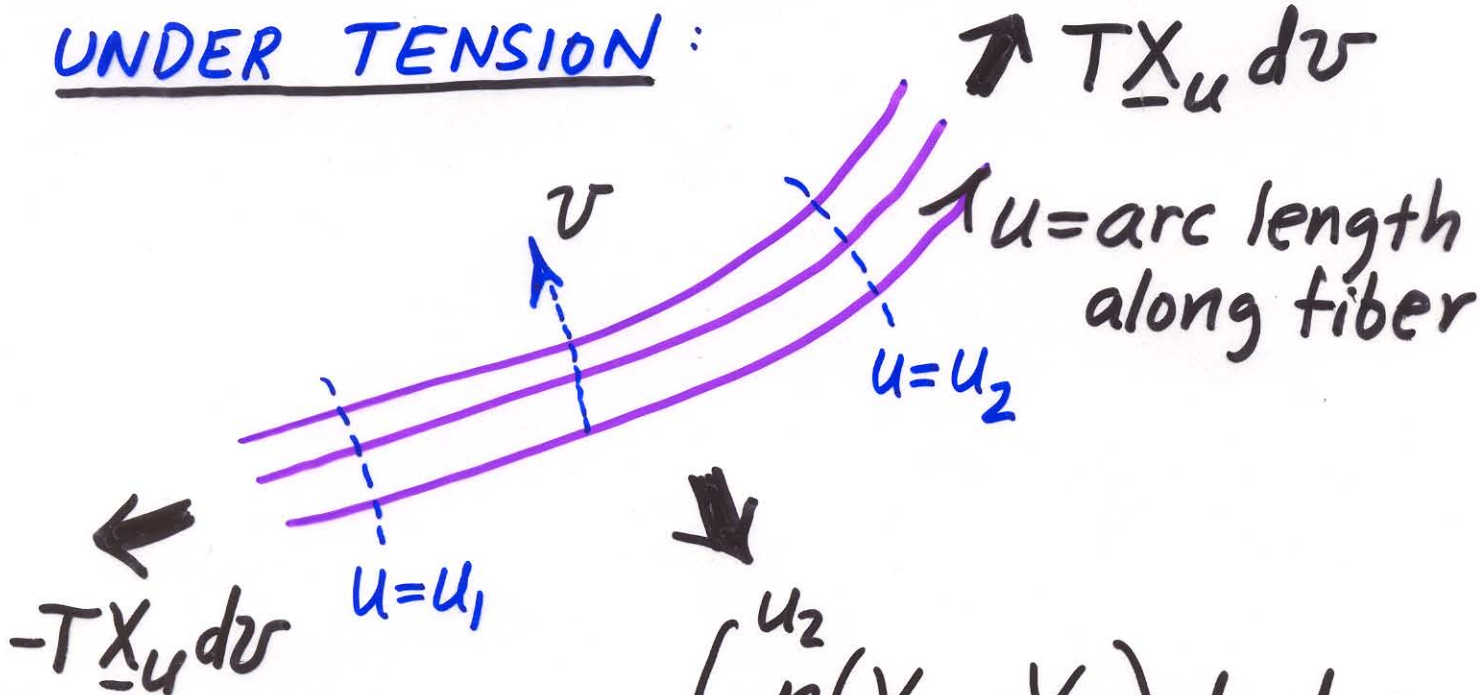


CLOSED VALVE
VIEWED
FROM AORTA



AXIAL CROSS
SECTIONS OF
CLOSED VALVE

UNIFORM PRESSURE LOAD SUPPORTED BY ONE-PARAMETER FAMILY OF FIBERS UNDER TENSION:



$$\int_{u_1}^{u_2} p_0(\underline{X}_u \times \underline{X}_v)\,du\,dv$$

$T(v)\,dv$ = fiber force

p_0 = applied pressure load

$$(T\underline{X}_u)\Big|_{u_1}^{u_2} \cancel{dv} + \int_{u_1}^{u_2} p_0(\underline{X}_u \times \underline{X}_v)\,du\,\cancel{dv} = 0$$

$$\int_{u_1}^{u_2}\left[(T\underline{X}_u)_u + p_0(\underline{X}_u \times \underline{X}_v)\right]du = 0$$

$$\underbrace{\left(T_u \underline{X}_u\right)}_{\text{tangential}} + \underbrace{\left(T\underline{X}_{uu} + p_0(\underline{X}_u \times \underline{X}_v)\right)}_{\text{normal}} = 0$$

It follows that:

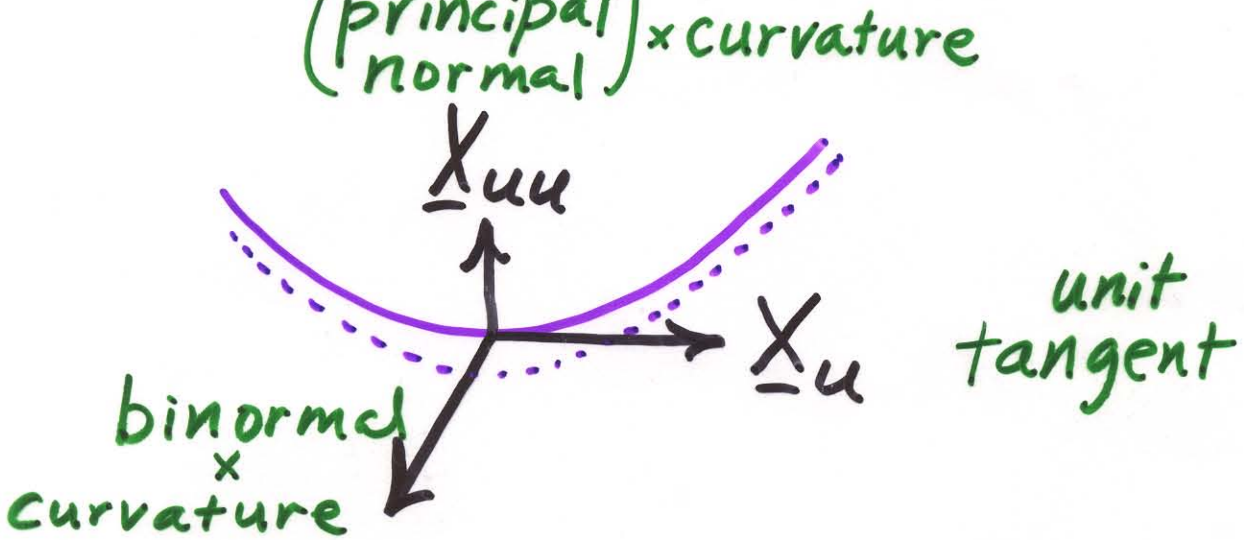
- $T = T(v)$, independent of u.
- Fibers are geodesics.
- Coordinates form an orthogonal net (provided that $u=0$ is chosen orthogonal to fibers).

ELIMINATE MECHANICAL VARIABLES:

Let $$dV = \frac{T(v)\,dv}{\rho_0}$$

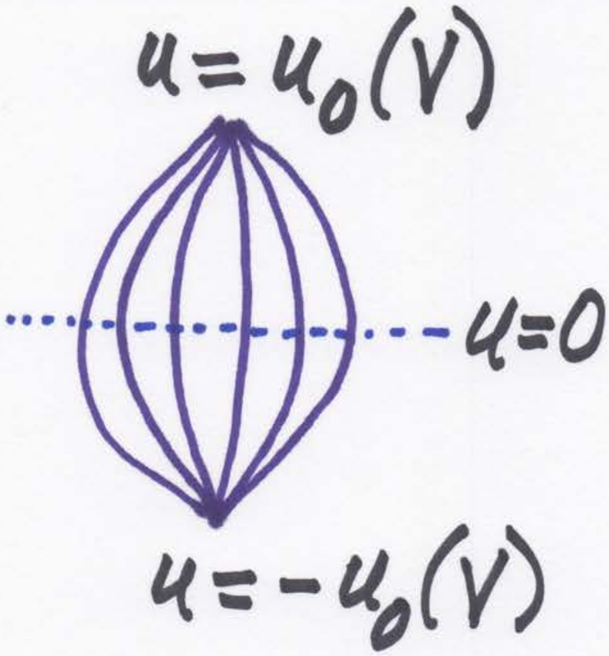
Then $$\underline{X}_{uu} + (\underline{X}_u \times \underline{X}_V) = 0$$

Apply $\underline{X}_u \times$ to solve for $\underline{X}_V$:



$$\underline{X}_V = \underline{X}_u \times \underline{X}_{uu}$$

BOUNDARY CONDITIONS



At the commisures $u = \pm u_0(V)$:

$$0 = \frac{d}{dV} \underline{X}\left(\pm u_0(V), V\right)$$

$$= \pm u_0'(V) \underline{X}_u + \underline{X}_V$$

Since $\underline{X}_u$ and $\underline{X}_v$ are orthogonal and $\underline{X}_u$ is a unit vector, we have

$$u_0'(V) = 0 \Rightarrow$$

u_0 is independent of V, that is, all fibers are the same length!

and

$$X_v(\pm u_0, V) = 0$$

$$\Downarrow$$

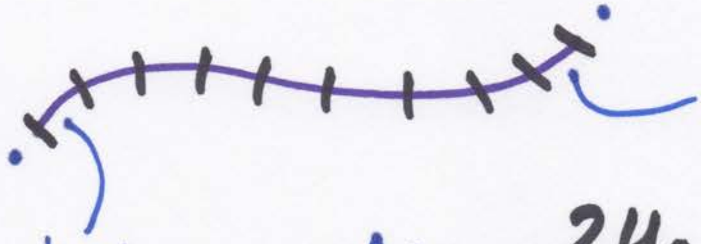
$$(\underline{X}_u \times \underline{X}_{uu})(\pm u_0, V) = 0$$

$$\Downarrow$$

$$\underline{X}_{uu}(\pm u_0, V) = 0$$

i.e., fibers at commisures are straight

NUMERICAL SCHEME (T. Buttke)



$j = N - \frac{1}{2}$

$j = \frac{1}{2}$ $\qquad \Delta u = \dfrac{2u_0}{N}$

Let $\underline{\tau} = \underline{X}_u$ (unit tangent). Then

$$\underline{\tau}_v = \underline{\tau} \times \underline{\tau}_{uu}$$

$$\frac{\underline{\tau}_j^{n+1} - \underline{\tau}_j^{n}}{\Delta v} = \underline{\sigma}_j^n \times \frac{\underline{\sigma}_{j+1}^n - 2\underline{\sigma}_j^n + \underline{\sigma}_{j-1}^n}{(\Delta u)^2}$$

where $\underline{\sigma}_j^n = \dfrac{\underline{\tau}_j^n + \underline{\tau}_j^{n+1}}{2}$

and $j = \frac{1}{2}, \frac{3}{2}, \dots, (N - \frac{1}{2})$

Boundary conditions are enforced by reflection:

$$\underline{\tau}^{n}_{-1/2} = \underline{\tau}^{n}_{1/2}, \quad \underline{\tau}^{n}_{N+\frac{1}{2}} = \underline{\tau}^{n}_{N-\frac{1}{2}}$$

Equations for $\underline{\tau}^{n+1}_{\frac{1}{2}} \cdots \underline{\tau}^{n+1}_{N-\frac{1}{2}}$ are solved by fixed-point iteration, which converges if

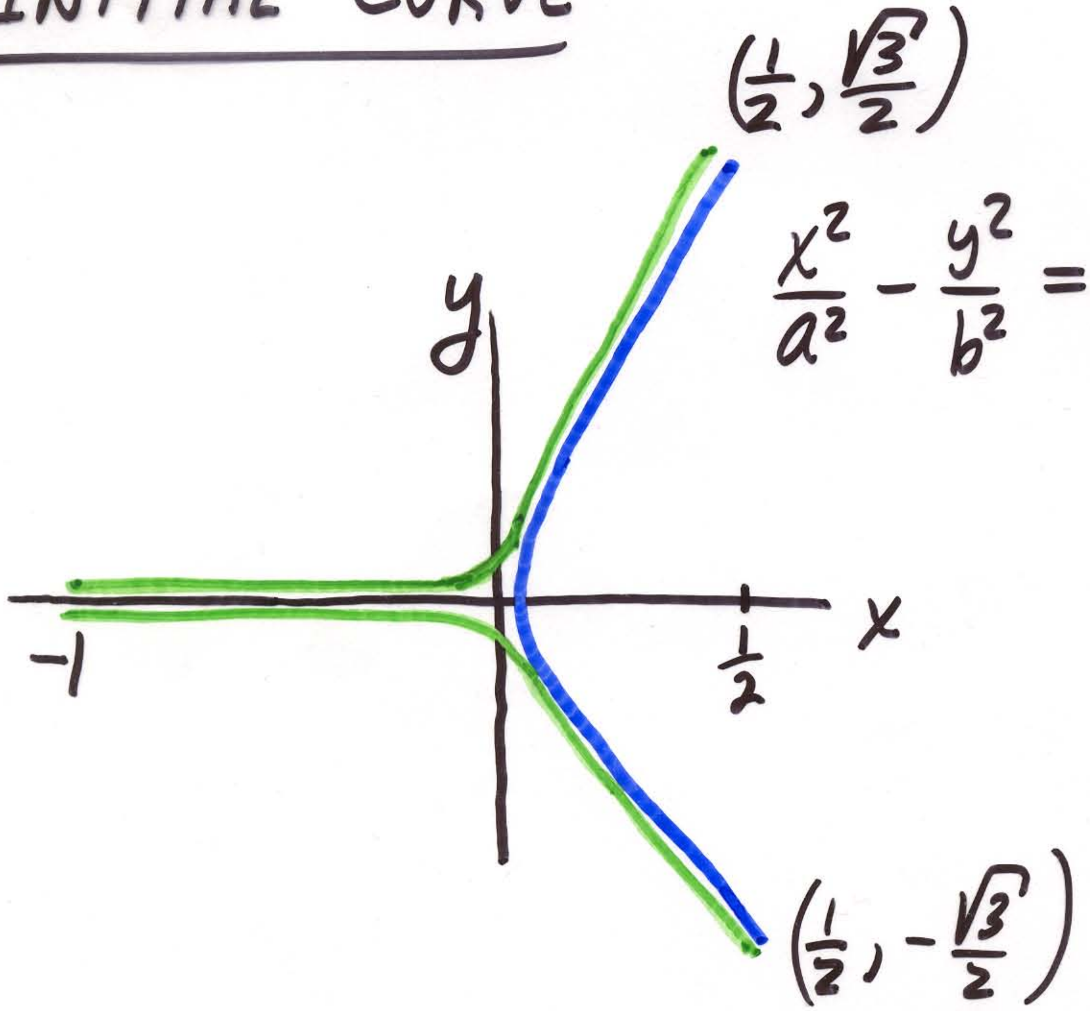
$$\frac{\Delta V}{(\Delta u)^2} \leq \frac{1}{4}$$

After solving for $\underline{\tau}^n$, we construct the n^{th} discretized fiber as

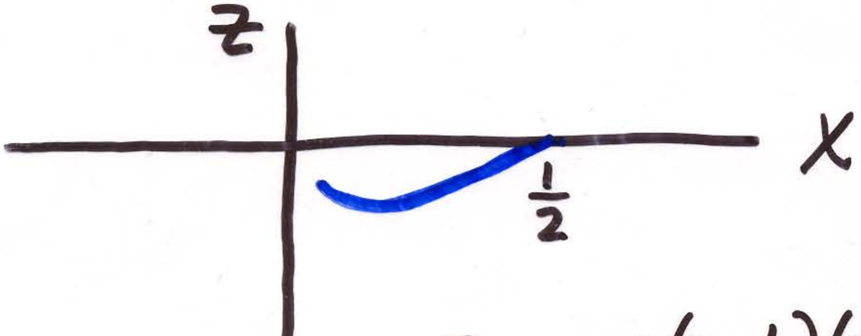
$$\underline{X}_k^n = \underline{X}_0 + (\Delta u)\left(\underline{\tau}_{\frac{1}{2}}^n + \cdots + \underline{\tau}_{k-\frac{1}{2}}^n\right)$$

for $k = 1 \cdots N$. Here, $\underline{X}_0$ is one commisural point and $\underline{X}_N$ is the other one, so it is important that $\underline{X}_N^n$ should be independent of n. This can be proved from our scheme with its reflecting boundary conditions.

INITIAL CURVE

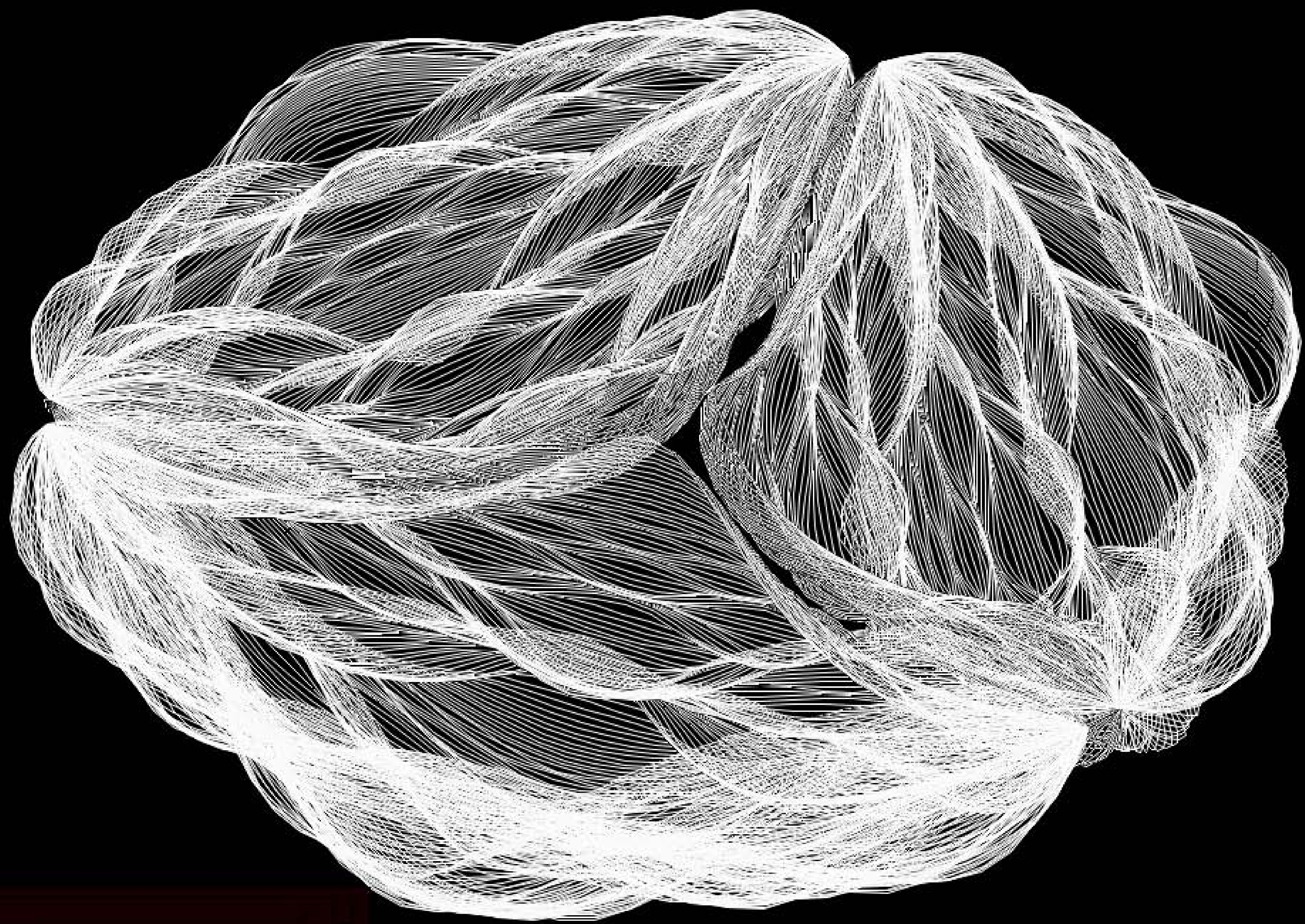


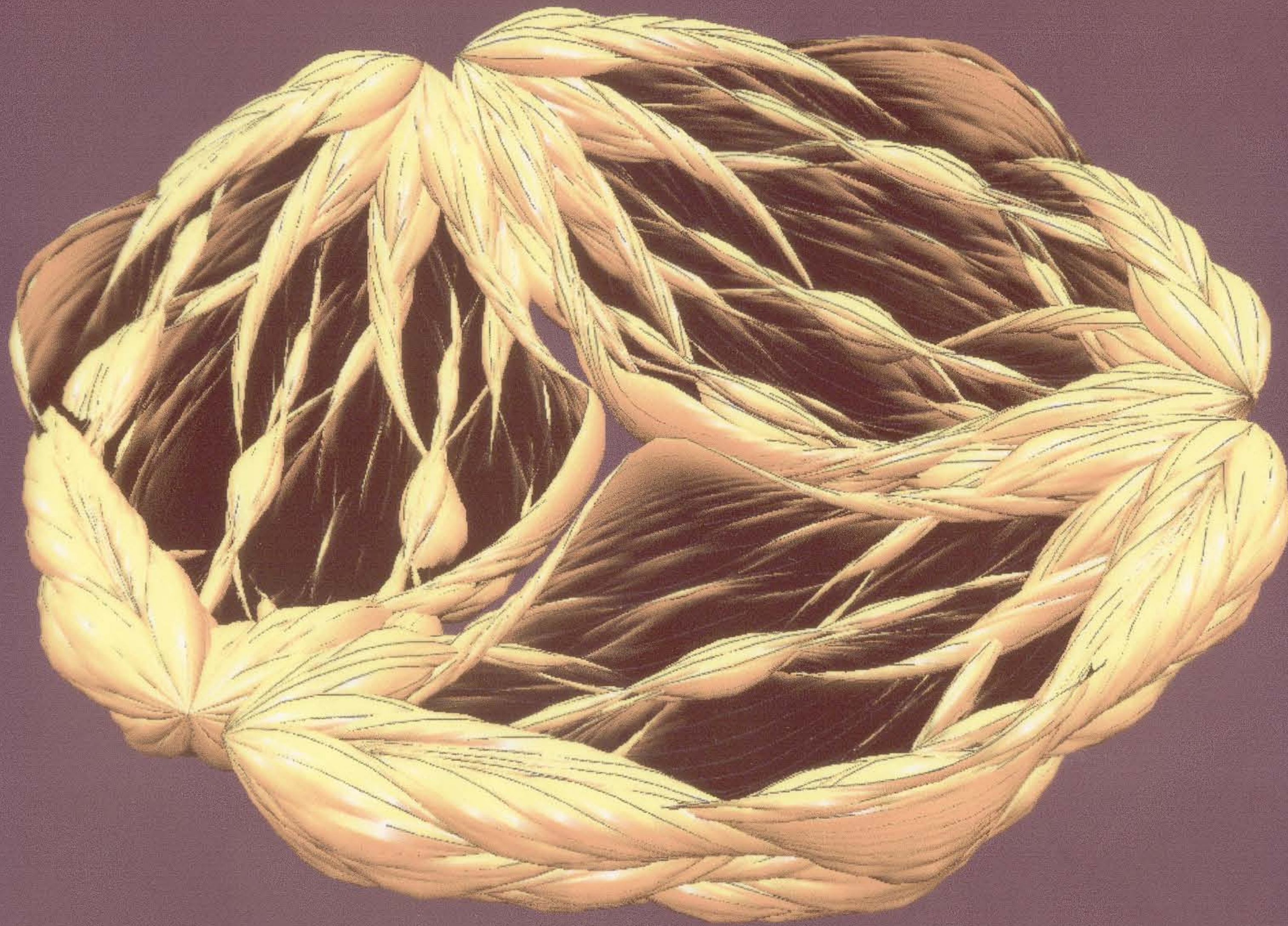
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$



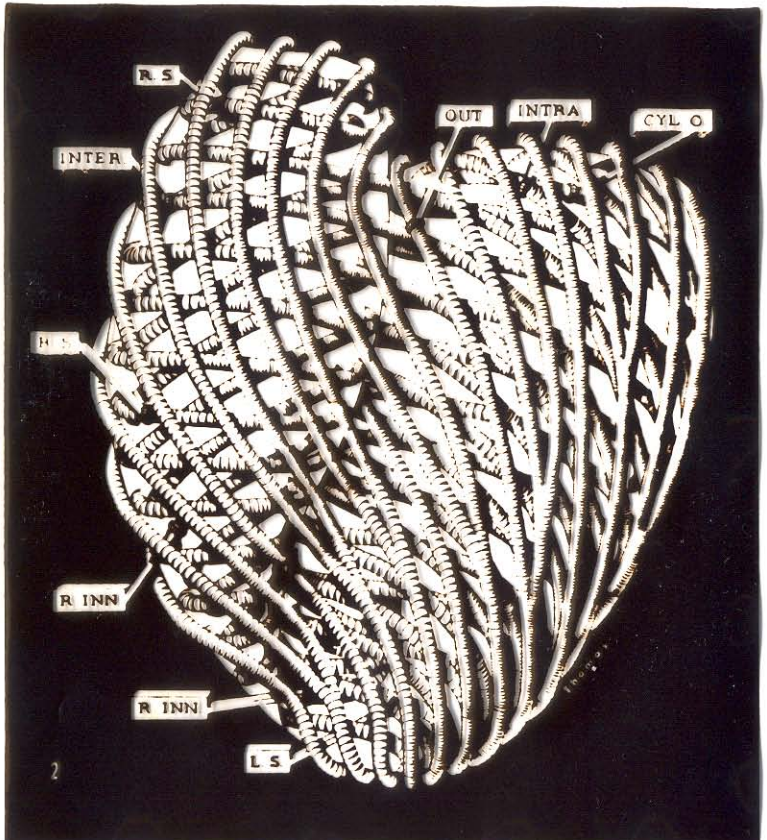
$$z = -cx\left(x-\tfrac{1}{2}\right)(x-1)$$

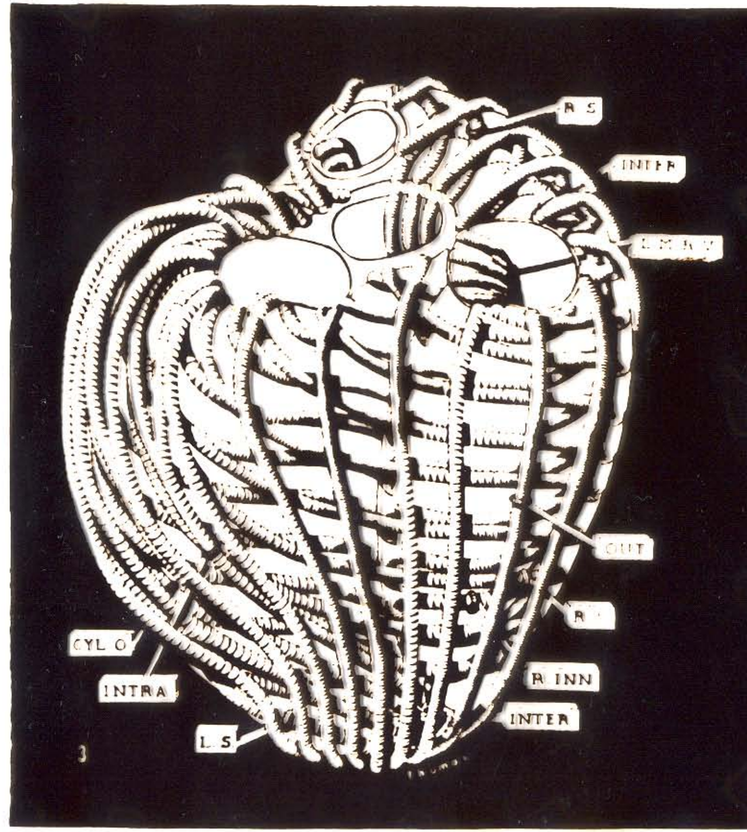
Numerical Solution of the Fiber Architecture Equations of the Aortic Valve
David M. McQueen and Charles S. Peskin





Numerical Solution of the Fiber Architecture Equations
of the Aortic Valve. David M. McQueen & Charles S. Peskin





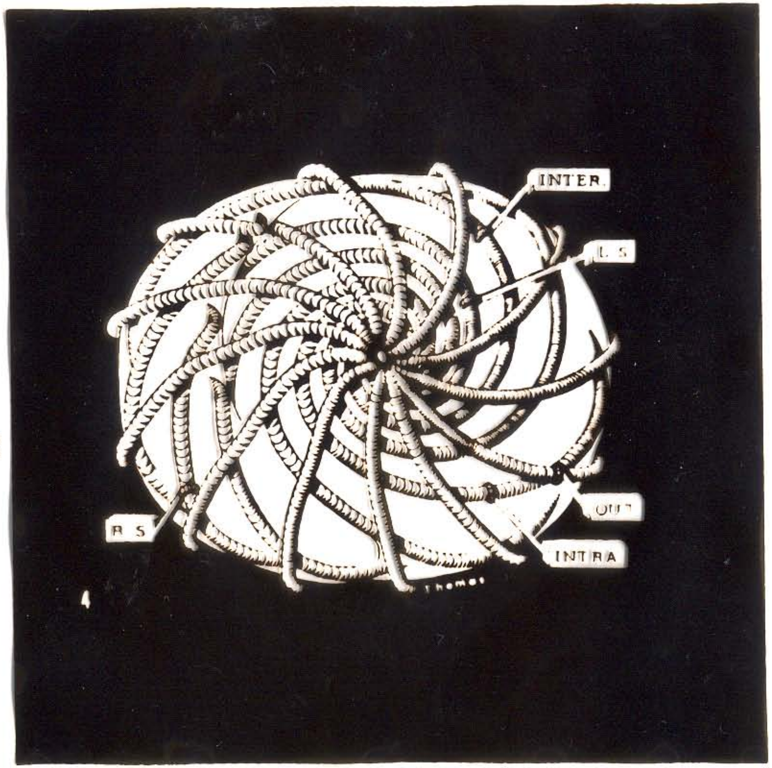
Hog ventricles

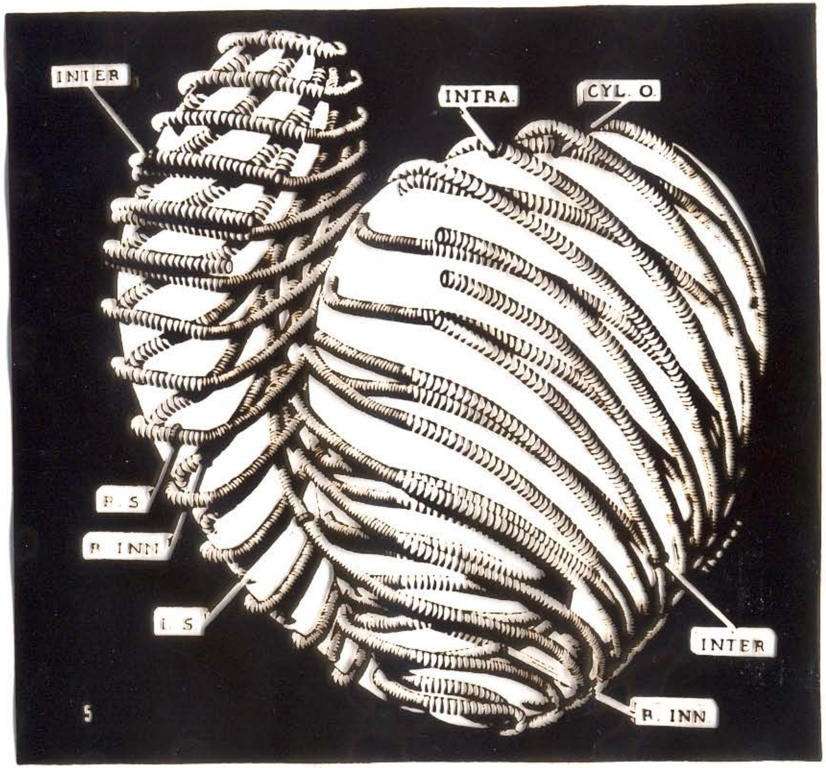
These drawings represent the ventricles before any muscle fascicles were removed.

2 Anterior view.

3 Posterior view.

C. Thomas AM. J. ANATOMY 1957

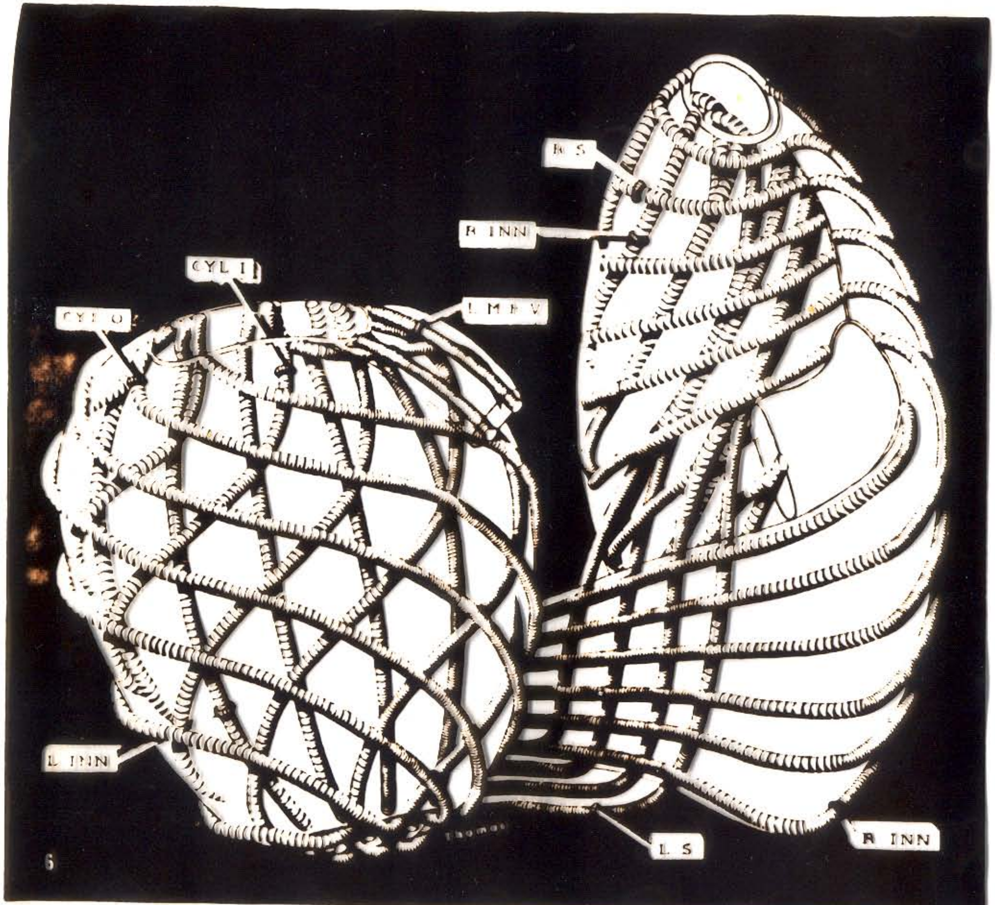


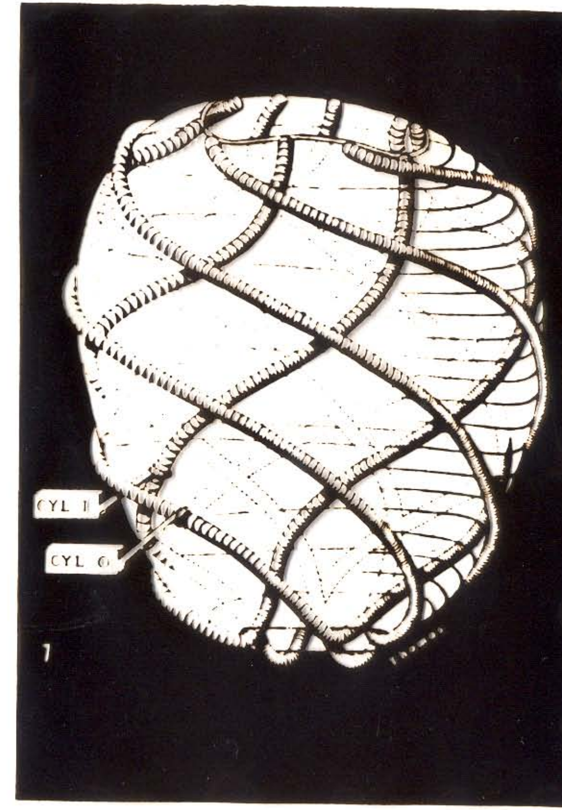


Hog ventricles

4 Apical view. This drawing represents the ventricles before any muscle fascicles were removed. The upper edge of the drawing is the anterior surface of the heart.

5 Anterior view. The outer layer was removed and the interventricular band was transected. Then the ventricles were separated along the anterior sulcus. Note the penetration of the right septal fascicles by the left septal fascicles.





Hog ventricles

6 Posterior view. This drawing represents a more advanced stage of dissection of the heart in figure 5: (1) The interventricular and intraventricular bands were removed. (2) The fibrous base of the right ventricle was separated from the fibrous base of the left ventricle. This included the transection of the longitudinal muscle of the right ventricle. (3) The origins of some of the right inner, right septal, and left septal fascicles were transected from the medial portion of the aortic ring. The line extending from the pulmonary to the right A-V orifice formed part of the medial portion of the aortic ring before the ventricles were separated and represents the origin of these transected fascicles. (4) All the right septal band, except the intraseptal portion, was removed. (5) The ventricles were separated along the posterior sulcus.

7 An isolated intact cylinder. (The mode of isolation is explained in the text.)

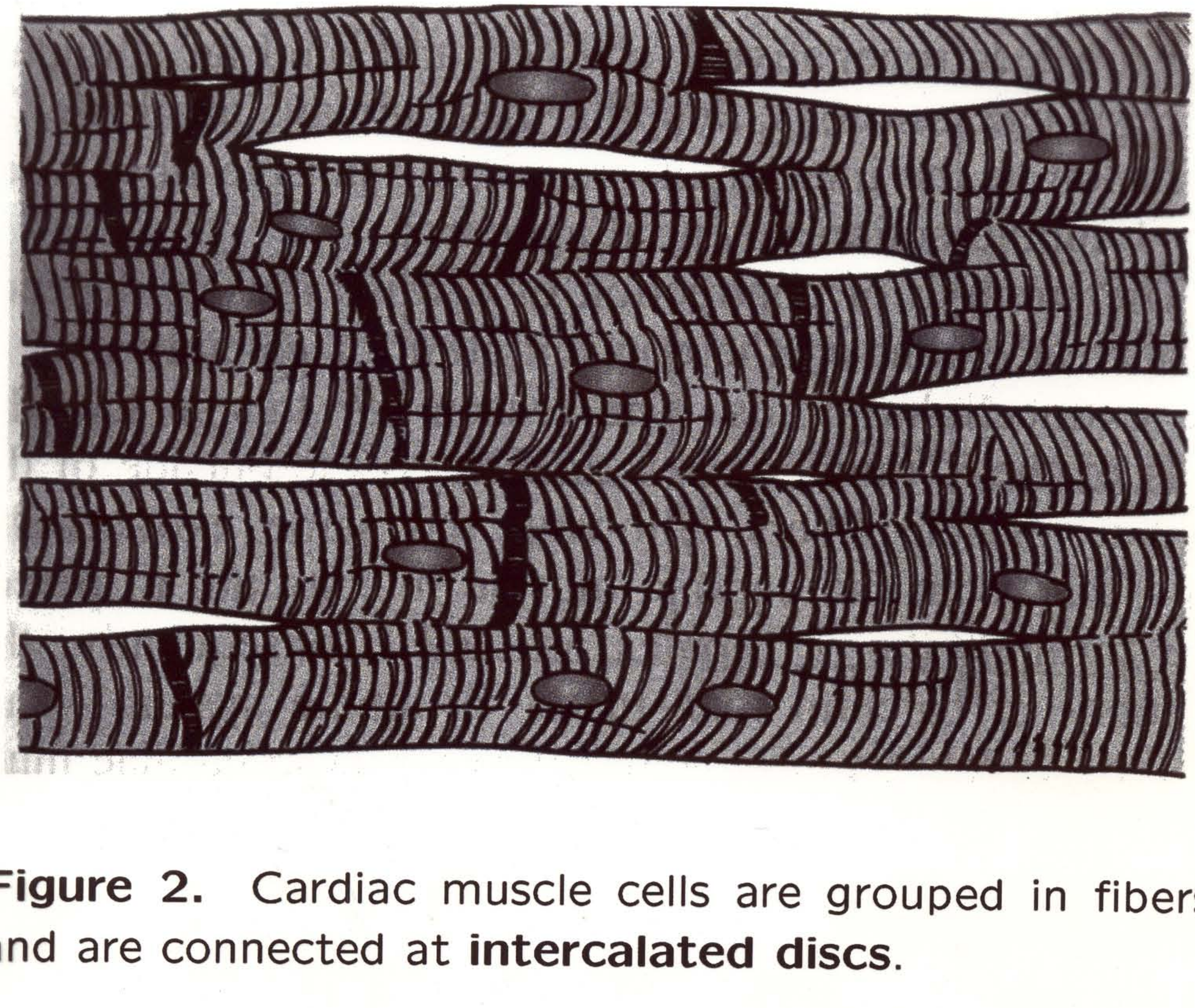


Figure 2. Cardiac muscle cells are grouped in fibers and are connected at **intercalated discs**.

Guyton. Textbook of
Medical Physiology

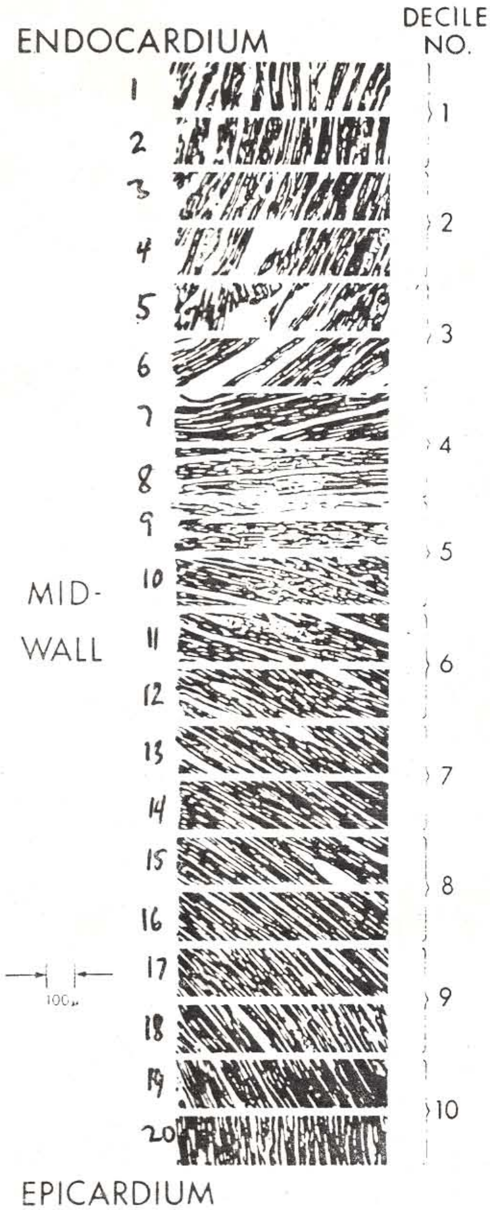


FIGURE 3

Typical sequence of photomicrographs showing fiber angles in successive sections taken from a heart in systole at region Tc. The sections are parallel to the epicardial plane. Fiber angle is +90° at the endocardium, running through 0° at the midwall to −90° at the epicardium. The sequence of numbers refers to deciles of wall thickness.

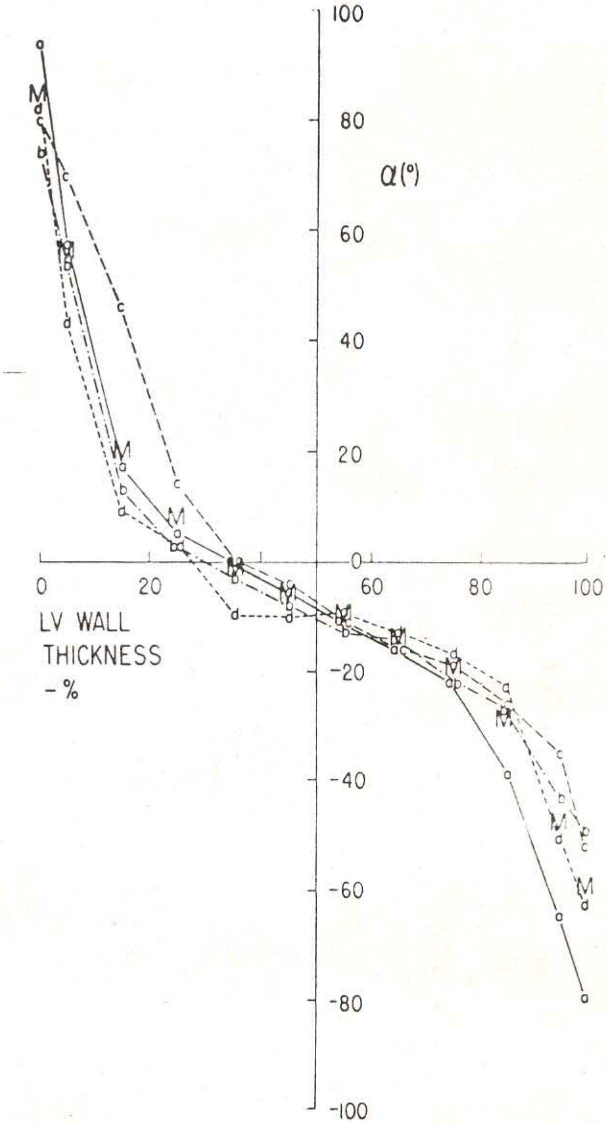


FIGURE 4

Fiber angles for four sampling sites, a through d, in a T-top section from a heart in diastole are plotted as a function of percent wall thickness. Zero percent of wall thickness implies the endocardial surface. M *represents the mean of the data at these four sites.*

within 5 mm of a given sampling point. Shrinkage of the preparation due to dehydrating, embedding, and sectioning by microtome was approximately 18% in the fiber plane parallel to the epicardium.

Results

An example of the change of fiber angle through the wall at a single sampling point in the left ventricle is shown in Figure 3 by

FIBER ARCHITECTURE OF LEFT VENTRICLE

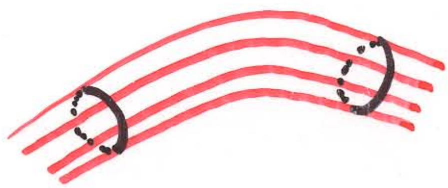
1) Equilibrium: $$\sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j} = 0$$

2) Fiber stress:

$$\sigma_{ij} = -p\,\delta_{ij} + T\,\tau_i \tau_j, \quad |\underline{\tau}| = 1$$

3) Constant-area fiber tubes:

$$\nabla \cdot \underline{\tau} = 0$$



4) Axial symmetry:

5) Thin wall.

EQUATION OF EQUILIBRIUM

$$\nabla p = T(\underline{\tau} \cdot \nabla \underline{\tau}) + \underline{\tau}\, \nabla \cdot (T \underline{\tau})$$

$$= T K \underline{n} + \underline{\tau}\, \nabla \cdot (T \underline{\tau})$$

principal normal

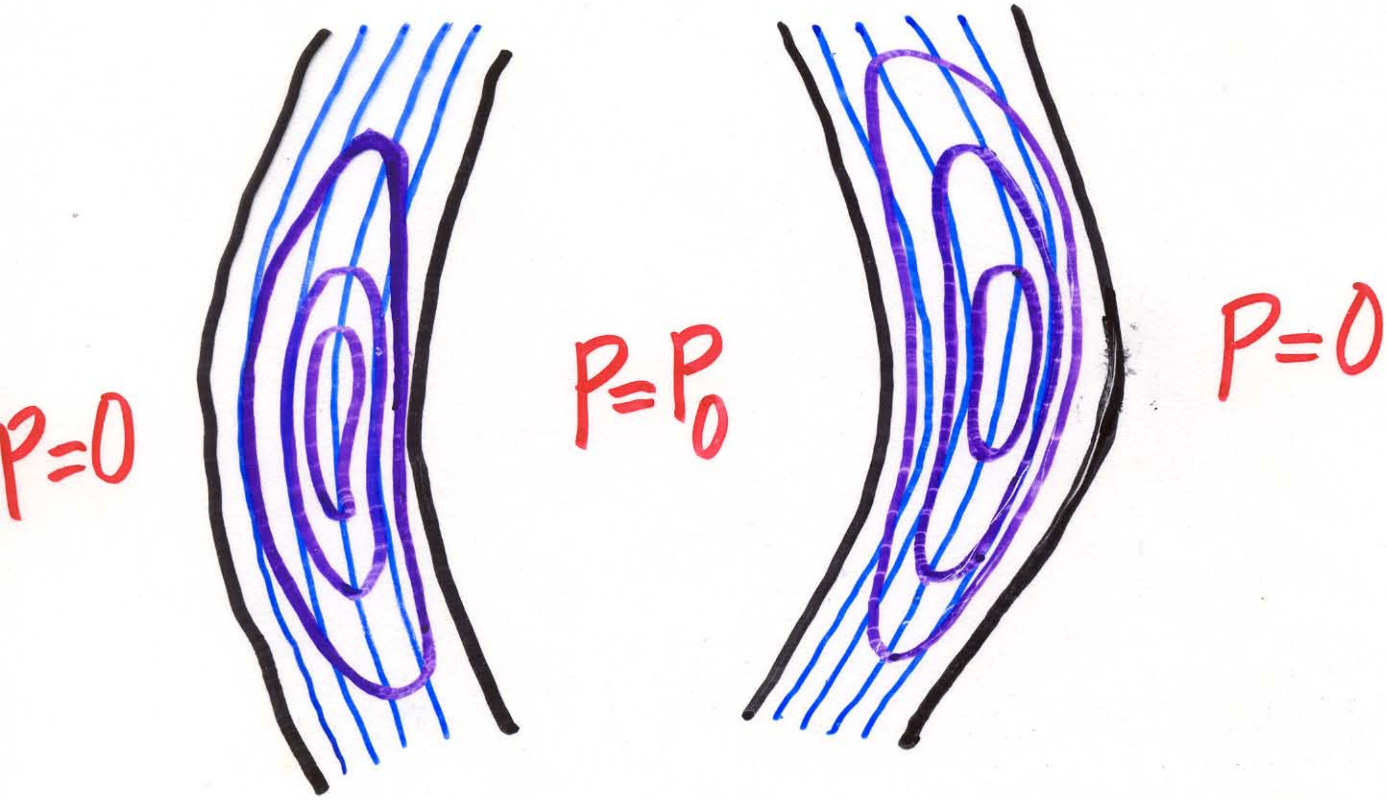
curvature

Special case: $\nabla \cdot (T \underline{\tau}) = 0$

$\underline{\tau} \cdot \nabla p = 0 \Rightarrow$ fibers run on $p = \text{constant}$

$\Downarrow$

$\nabla p = T K \underline{n} \Rightarrow$ fibers are geodesics on $p = \text{constant}$



— Endocardial + Epicardial wall surfaces.

— Surfaces of constant pressure

— Fiber surfaces

THEOREM: If fibers are geodesic, they run (alas) along p=constant!

If:

Eqn. of equilibrium

Axial symmetry

Fibers geodesic on "fiber surfaces"

Non-trivial swirl: $\underline{\tau} \cdot \underline{\hat{\theta}} \neq 0$

Then:

$\nabla \cdot (T\underline{\tau}) = 0$

Fiber surfaces are p=constant

Proof:

$$(\underline{\hat{\theta}} \cdot \nabla p) = TK(\underline{\hat{\theta}} \cdot \underline{n}) + (\underline{\hat{\theta}} \cdot \underline{\tau})\, \nabla \cdot (T\underline{\tau})$$

(zero if fibers are geodesic on any surface of revolution.)

(zero by axial symmetry)

Curvilinear coordinates for asymptotic analysis:

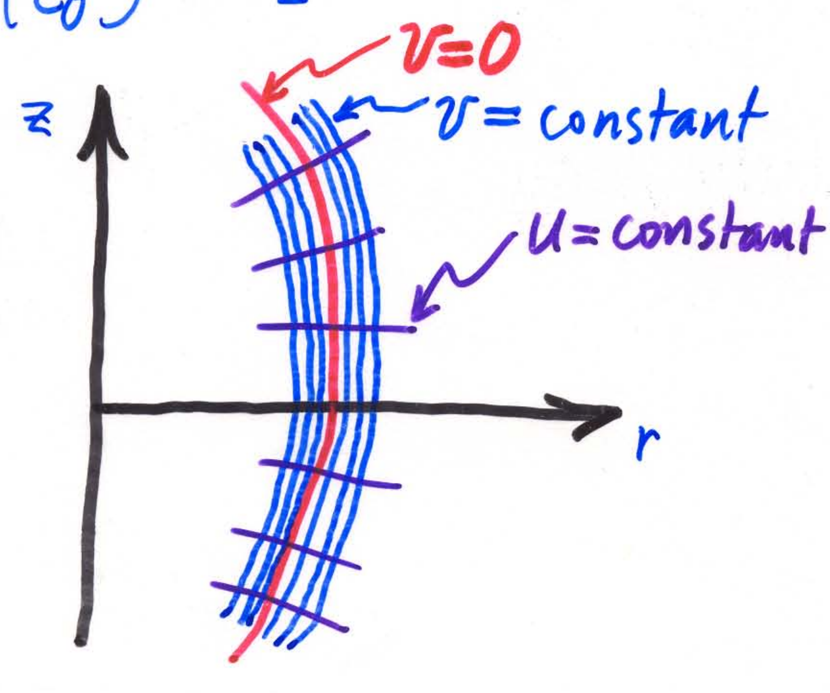
$$\underline{X}(u,v,\theta) = \left(R(u,v)\cos\theta,\ R(u,v)\sin\theta,\ Z(u,v)\right)$$

where

$$R(u,v) = R_0(u) + \varepsilon v Z_0'(u)$$

$$Z(u,v) = Z_0(u) - \varepsilon v R_0'(u)$$

$$(R_0')^2 + (Z_0')^2 = 1$$



Choice of variables for asymptotic analysis:

$$p = P(u,v)$$

$$\varepsilon T = S(u,v)$$

$$\underline{\tau} = \tau_u(u,v)\,\frac{\partial \underline{X}}{\partial u}(u,v,\theta) + \tau_v(u,v)\,\frac{\partial \underline{X}}{\partial v}(u,v,\theta) + \tau_\theta(u,v)\,\frac{\partial \underline{X}}{\partial \theta}(u,v,\theta)$$

where

$$P, S, \tau_u, \tau_v, \tau_\theta = O(1)$$

Note that

$$T = O(\varepsilon^{-1})$$

which is needed to support steep pressure gradient in thin wall

$$\frac{\partial \underline{X}}{\partial v} = O(\varepsilon)$$

which implies that fibers are approximately parallel to middle surface $v=0$

The ε^0 equations:

$$1 = \tau_u^2 + \tau_\theta^2 R_0^2$$

$$0 = \frac{\partial}{\partial u}\left(R_0 \tau_u\right) + \frac{\partial}{\partial v}\left(R_0 \tau_v\right)$$

$$0 = D(S\tau_u) - S\,\tau_\theta^2 R_0 R_0'$$

$$\frac{\partial P}{\partial v} = -S\left[\tau_u^2\left(R_0' Z_0'' - Z_0' R_0''\right) + \tau_\theta^2 R_0 Z_0'\right]$$

$$0 = \left[D(S\tau_\theta)\right] R_0^2 + 2S\,\tau_u \tau_\theta R_0 R_0'$$

where

$$D = \tau_u \frac{\partial}{\partial u} + \tau_v \frac{\partial}{\partial v}$$

Invariants on the fibers (and hence by symmetry on the fiber surfaces):

By combing the 1^{st}, 3^{rd}, & 5^{th} of the ε^0 equations, we find

$$D(S) = 0$$

$$D\left(\tilde{\tau}_\theta R_0^2\right) = 0$$

Note that $\tilde{\tau}_\theta R_0 \sim \underline{\tau} \cdot \underline{\hat{\theta}} =: \cos\alpha$

so we have

$$D\left(R_0 \cos\alpha\right) = 0$$

Here α is the angle between the fiber direction and the circumferential direction, so the result that

$R_0 \cos\alpha$ is constant on a fiber

shows that the fiber is a geodesic on the middle surface. (Clairaut)

But this result holds only in the limit $\varepsilon \to 0$, and we have seen above that the exact equations rule out geodesic paths for the fibers on the fiber surfaces.

Introduction of a stream function:

Since

$$\frac{\partial}{\partial u}\left(R_0 \tau_u\right) + \frac{\partial}{\partial v}\left(R_0 \tau_v\right) = 0$$

$\exists\ \Psi(u,v)$ such that

$$R_0 \tau_u = \frac{\partial \Psi}{\partial v}$$

$$R_0 \tau_v = -\frac{\partial \Psi}{\partial u}$$

Note that $D\Psi = 0$, so Ψ is another invariant of a fiber.

Self-similar solution of the ε^0 equations:

Seek $f(\;)$ such that

$$\sin\alpha = \tau_u = f'\left(\frac{v}{R_0(u)}\right) = f'(V)$$

where

$$f'(0) = 0$$

$v=0$

since $\hat{\tau}_u = 0$ defines the middle surface

but $f'(V) \not\equiv 0$, since that would mean circumferential fibers only. (Also, we may set $f(0) = 0$, since only f' matters.)

We have

$$\frac{\partial \Psi}{\partial v}(u,v) = R_0(u) f'\left(\frac{v}{R_0(u)}\right)$$

$$\Psi(u,v) = R_0^2(u) f\left(\frac{v}{R_0(u)}\right) + g(u)$$

Also

$$\sin^2\alpha \quad + \quad \cos^2\alpha \quad = \quad 1$$

$$\left(f'(V)\right)^2 + \frac{J(\Psi)}{R_0^2(u)} = 1$$

since $R_0 \cos\alpha$ is constant on a fiber and therefore is a function of Ψ.

Therefore,

$$(f'(V))^2 + \frac{J(R_0^2(u)f(V)+g(u))}{R_0^2(u)} = 1$$

From now on, treat u, V as independent variables. Apply $\frac{\partial}{\partial V}$:

$$\cancel{2f'(V)}f''(V)$$

$$+ \cancel{f'(V)}J'(R_0^2(u)f(V)+g(u)) = 0$$

and then $\frac{\partial}{\partial u}$:

$$J''(\quad)(2R_0(u)R_0'(u)f(V)+g'(u)) = 0$$

But

$$2R_0(u)R_0'(u)f(V)+g'(u)=0$$

$$\Rightarrow f(V) = -\frac{g'(u)}{2R_0(u)R_0'(u)}$$

$$\Rightarrow f'(V) \equiv 0$$

Contrary to hypothesis, so the only possibility is

$$J''(\Psi)=0 \Rightarrow J(\Psi)=a\Psi+b$$

where a, b are constant.

Now with $J(\Psi)$ determined,

it is easy to show that

$$f(V) = -\frac{aV^2}{4}$$

$$\Psi(u,v) = -\left(\frac{a}{4}v^2 + \frac{R_0^2(0) - R_0^2(u)}{a}\right)$$

where the constant b has

been chosen to make $\Psi(0,0)=0$.

The fiber angle α is therefore given by

$$\sin\alpha = -\frac{a}{2}\frac{v}{R_0(u)}$$

or

$$\alpha = -\arcsin\left(\frac{a}{2}\frac{v}{R_0(u)}\right)$$

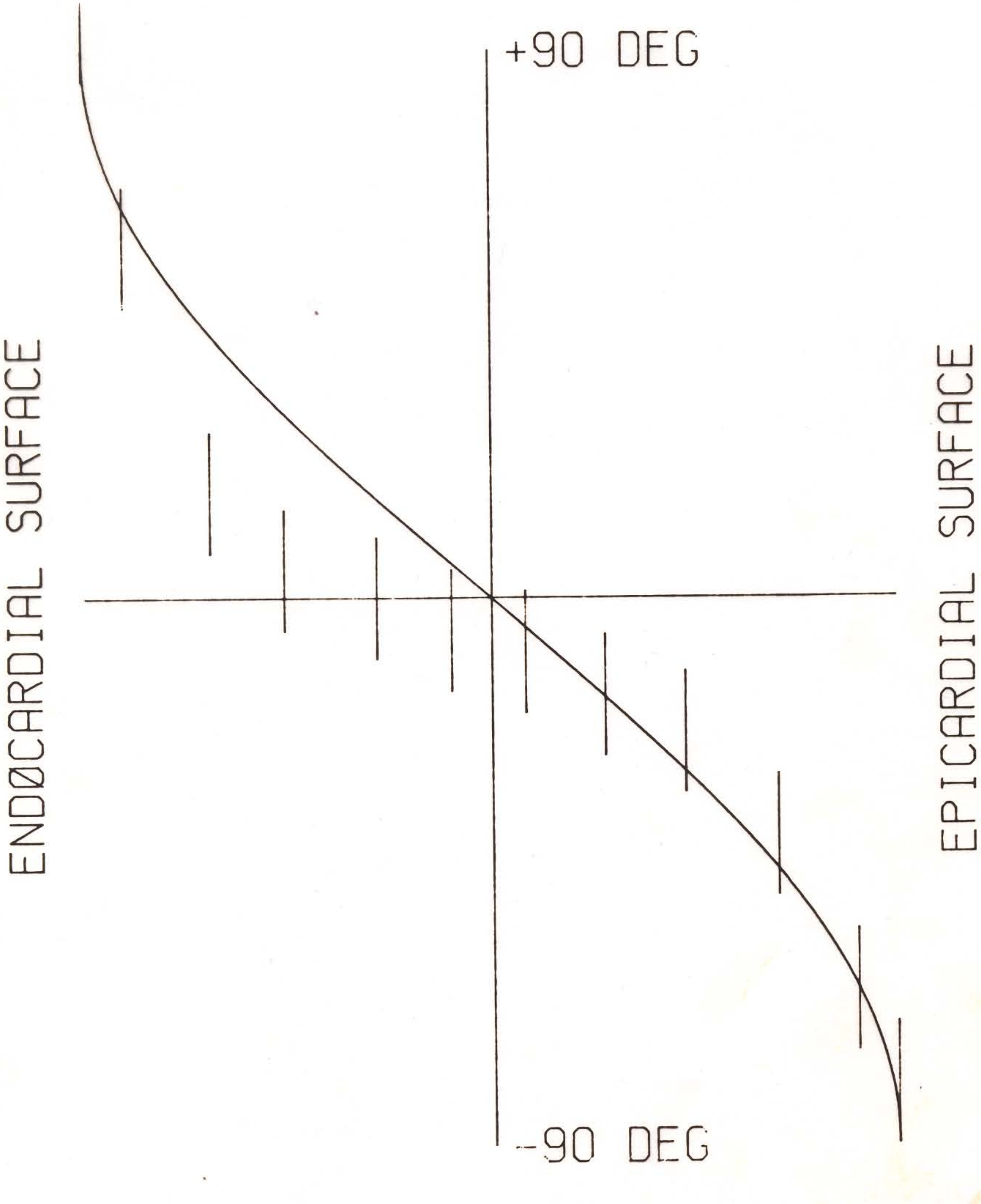
The domain of v is therefore restricted by

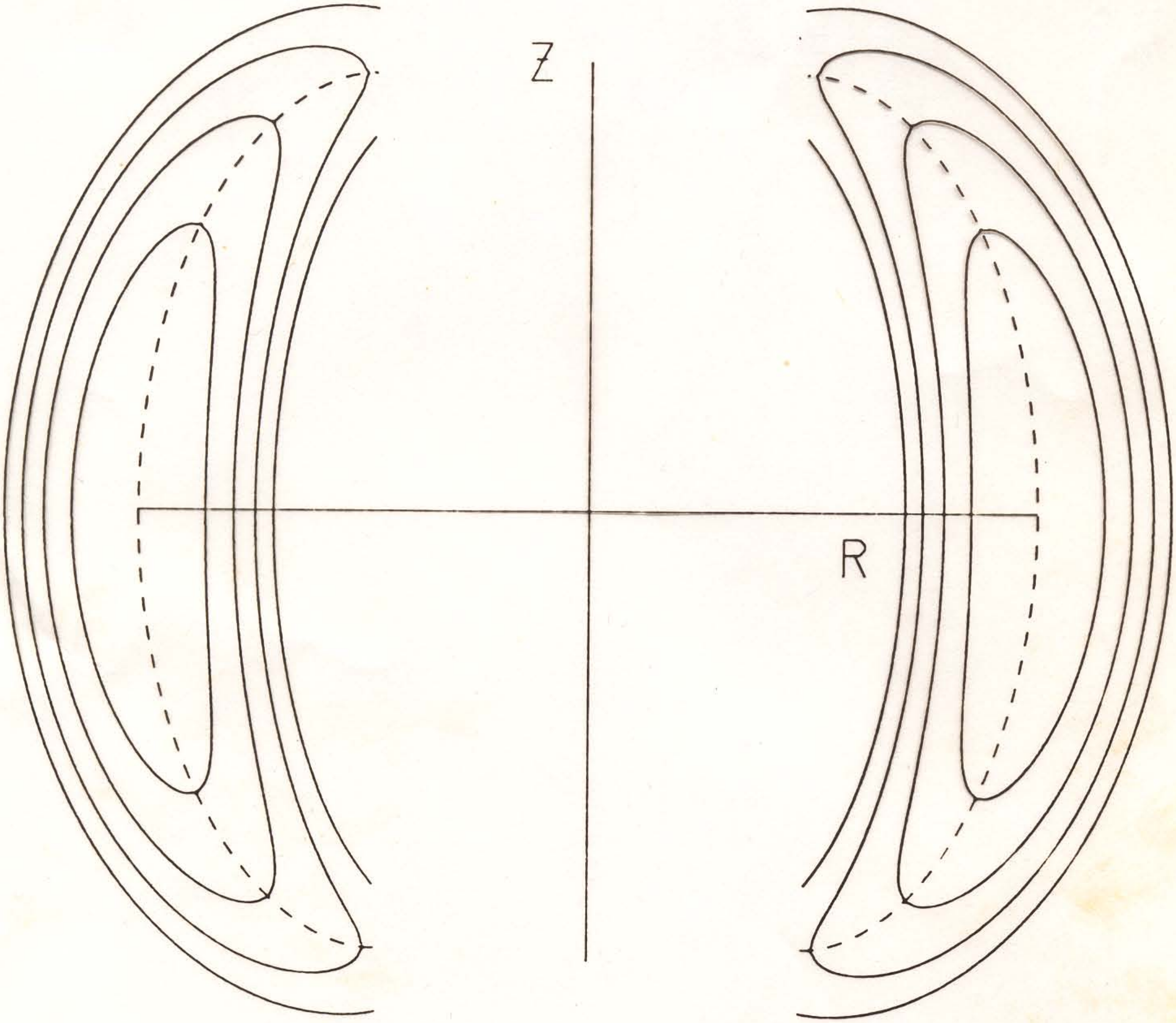
$$|v| \leq \frac{2R_0(u)}{|a|}$$

At the extremes of v, $\alpha = \pm\frac{\pi}{2}$, and these are fiber surfaces, since on them ψ is constant.

Qualitative behavior of the fiber surfaces:

Choose the origin of u so that $R_0(u)$ is maximized by $u=0$. Then $\psi(u,v)$ has a maximum $(a>0)$ or minimum $(a<0)$ at $u=v=0$. In either case, the fiber surfaces in the neighborhood of $u=v=0$ are nested tori.


+90 DEG
-90 DEG
ENDØCARDIAL SURFACE
EPICARDIAL SURFACE


Z
R

Details are in the following papers:

Peskin CS and McQueen DM:
Mechanical equilibrium determines the fractal fiber architecture of the aortic heart valve leaflets.
American Journal of Physiology 266: H319-H328, 1994
http://ajpheart.physiology.org/content/266/1/H319

Stern JV and Peskin CS:
Fractal dimension of an aortic heart valve leaflet.
Fractals - Complex Geometry Patterns and Scaling in Nature and Society 2(3): 461-464, 1994
http://dx.doi.org/10.1142/S0218348X9400065X

Peskin CS:
Fiber architecture of the left ventricular wall: an asymptotic analysis.
Communications on Pure and Applied Mathematics 42: 79-113, 1989
http://dx.doi.org/10.1002/cpa.3160420106