On the Numerical Stability of Target Point Forces in the Immersed Boundary Method

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Joint work (in progress) with Mengjian Hua

For related work, see

Stockie JM & Wetton BR:
Analysis of Stiffness in the Immersed Boundary Method and Implications for Time-stepping Schemes.
Journal of Computational Physics 154: 41-64, 1999

We consider the time-dependent Stokes equations with an immersed boundary that is held near a specified configuration by spring-like forces. The equations are

(1) $$\rho \frac{\partial \underline{u}}{\partial t} + \nabla p = \mu \Delta \underline{u} + \underline{f}$$

(2) $$\nabla \cdot \underline{u} = 0$$

(3) $$\underline{f}(\underline{x},t) = \int_S \underline{F}(\underline{s},t)\, \delta(\underline{x} - \underline{X}(\underline{s},t))\, d\underline{s}$$

(4) $$\frac{\partial \underline{X}}{\partial t}(s,t) = \int_\Omega \underline{u}(\underline{x},t)\, \delta(\underline{x} - \underline{X}(\underline{s},t))\, d\underline{x}$$

(5) $$\underline{F}(s,t) = K\left(\underline{X}^0(\underline{s}) - \underline{X}(\underline{s},t)\right)$$

The fluid domain is

(6) $$\Omega = (0, L_0)^3$$

with periodic boundary conditions, and the domain of the material coordinates $\underline{s} = (s_1, s_2)$ is

(7) $$S = (0, L_0)^2$$

also with periodic boundary conditions

The target position of the immersed boundary is given by

(8) $$\underline{X}^0(s_1, s_2) = (s_1, s_2, 0)$$

Let

(9) $$Z_N = \{0, 1, \cdots, N-1\}$$

and let all integer arithmetic be modulo N
let

(10) $$h = L_0 / N$$

and introduce the grids

(11) $$\Omega_h = \{ \underline{j} h : \underline{j} \in Z_N^3 \}$$

(12) $$S_h = \{ \underline{k} h : \underline{k} \in Z_N^2 \}$$

Note that we use the same meshwidth h in both grids. Also, for $\underline{k} h \in S_h$

(13) $$\underline{X}^0(\underline{k}h) = (k_1 h, k_2 h, 0) \in \Omega_h$$

The spatial discretization of equations (1-5) is as follows:

(14) $$\rho \frac{d\underline{u}}{dt} + \nabla_h p = \mu \Delta_h \underline{u} + \underline{f}$$

(15) $$\nabla_h \cdot \underline{u} = 0$$

(16) $$\underline{f}(\underline{x},t) = \sum_{\underline{s}} \underline{F}(\underline{s},t)\, \delta_h(\underline{x}-\underline{X}^0(\underline{s}))\, h^2$$

(17) $$\frac{d\underline{X}}{dt}(\underline{s},t) = \sum_{\underline{x}} \underline{u}(\underline{x},t)\, \delta_h(\underline{x}-\underline{X}^0(\underline{s}))\, h^3$$

(18) $$\underline{F}(\underline{s},t) = K\left(\underline{X}^0(\underline{s}) - \underline{X}(\underline{s},t)\right)$$

Note that the functions δ_h are evaluated at $\underline{x} - \underline{X}^0(\underline{s})$ instead of $\underline{x} - \underline{X}(\underline{s},t)$.

For K large, we expect $\underline{X}(\underline{s},t)$ to be approximately equal to $\underline{X}^0(\underline{s})$, and by using $\underline{X}^0(\underline{s})$ we make the whole problem linear.

In equations (14-18), the variables $\underline{u}, p, \underline{f}$ are functions of $(\underline{x}, t)$ for $\underline{x} \in \Omega_h$, and the variables $\underline{X}, \underline{F}$ are functions of $(\underline{s}, t)$ for $s \in S_h$.

Also the sums over $\underline{s}$ and $\underline{x}$ are understood to be sums over $\underline{s} \in S_h$ and $\underline{x} \in \Omega_h$, respectively.

The difference operators ∇_h and Δ_h are defined as follows:

$$(19) \quad (\nabla_h p)_\alpha(\underline{x}) = \frac{p(\underline{x}+h\,\underline{e}_\alpha) - p(\underline{x}-h\,\underline{e}_\alpha)}{2h}$$

$\alpha = 1, 2, 3$

(20) $$(\nabla_h \cdot \underline{u})(\underline{x}) = \sum_{\alpha=1}^{3} \frac{u_\alpha(\underline{x}+h\underline{e}_\alpha) - u_\alpha(\underline{x}-h\underline{e}_\alpha)}{2h}$$

(21) $$(\Delta_h \underline{u})(\underline{x}) = \sum_{\alpha=1}^{3} \frac{\underline{u}(\underline{x}+h\underline{e}_\alpha) - 2\underline{u}(\underline{x}) + \underline{u}(\underline{x}-h\underline{e}_\alpha)}{h^2}$$

In these definitions $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$ is the standard basis of $\mathbb{R}^3$, and $\underline{x} \in \Omega_h$.

In equations (16-17), the functions δ_h are defined by

(22) $$\delta_h(\underline{x}) = \frac{1}{h^3} \phi\left(\frac{x_1}{h}\right) \phi\left(\frac{x_2}{h}\right) \phi\left(\frac{x_3}{h}\right)$$

and we will only need to evaluate ϕ for $j \in Z_N$, where it takes the values

(23) $$\phi(j) = \begin{cases} 1/2 , & j = 0 \\ 1/4 , & j = 1 \text{ or } N-1 \\ 0 , & \text{otherwise} \end{cases}$$

(Recall here that $j = N-1$ is equivalent to $j = -1$, since arithmetic on j is modulo N.)

Equations (16-18) can now be written more explicitly as follows:

(24) $$\underline{f}(\underline{j}h, t) = \sum_{\underline{k} \in Z_N^2} \underline{F}(\underline{k}h, t)\, \phi(j_1 - k_1)\, \phi(j_2 - k_2) \frac{\phi(j_3)}{h}$$

$$\frac{d\underline{X}}{dt}(\underline{k}h,t) = \sum_{\underline{j}\in Z_N^3} \underline{u}(\underline{j}h,t)\,\phi(j_1-k_1)\,\phi(j_2-k_2)\,\phi(j_3) \tag{25}$$

$$\underline{F}(\underline{k}h,t) = K\Big((k_1h, k_2h, 0) - \underline{X}(\underline{k}h,t)\Big) \tag{26}$$

We can eliminate $\underline{X}$ from our system by differentiating with respect to t in (26) and then making use of (25). The result is

$$\frac{d\underline{F}}{dt}(\underline{k}h,t) = -K\sum_{\underline{j}\in Z_N^3} \underline{u}(\underline{j}h,t)\,\phi(j_1-k_1)\,\phi(j_2-k_2)\,\phi(j_3) \tag{27}$$

Our system is now reduced to the four equations (14), (15), (24), (27).

For discretization in time, we first write down the Runge-Kutta scheme described in the first lecture of this course, but this scheme will turn out to be equivalent to a simpler scheme because we have here dropped the nonlinear terms of the Navier-Stokes equations and also because of the centering of the functions δ_h at fixed locations.

In writing this scheme it will be helpful to use the notation

$$(28) \quad \underline{U}(\underline{k}h) = \sum_{\underline{j} \in \mathbb{Z}_N^3} \underline{u}(\underline{j}h)\, \phi(j_1 - k_1)\, \phi(j_2 - k_2)\, \phi(j_3)$$

with superscripts as needed to indicate the time step

The Runge-Kutta scheme is then as follows

(29) $$\underline{F}^{n+1/2} = \underline{F}^{n} - \frac{\Delta t}{2} K \underline{U}^{n}$$

(30) $$\underline{f}^{n+1/2}(\underline{j}h) = \sum_{\underline{k} \in Z_N^2} \underline{F}^{n+1/2}(\underline{k}h)\,\phi(j_1-k_1)\,\phi(j_2-k_2)\,\frac{\phi(j_3)}{h}$$

(31) $$\begin{cases} \rho \dfrac{\underline{u}^{n+1/2}-\underline{u}^{n}}{\Delta t/2} + \nabla_h \tilde{p}^{\,n+1/2} = \mu \Delta_h \underline{u}^{n+1/2} + \underline{f}^{n+1/2} \\ \nabla_h \cdot \underline{u}^{n+1/2} = 0 \end{cases}$$

(32) $$\begin{cases} \rho \dfrac{\underline{u}^{n+1}-\underline{u}^{n}}{\Delta t} + \nabla_h p^{n+1/2} = \mu \Delta_h \dfrac{\underline{u}^{n+1}+\underline{u}^{n}}{2} + \underline{f}^{n+1/2} \\ \nabla_h \cdot \underline{u}^{n+1} = 0 \end{cases}$$

(33) $$\underline{F}^{n+1} = \underline{F}^{n} - (\Delta t) K \underline{U}^{n+1/2}$$

To see how this scheme simplifies, we first note that equations (31) & (32), which have the same source term, are independent of each other, since $\underline{u}^{n+1/2}$ does not appear in (32) as it would if we were considering the Navier-Stokes equations.

Moreover, if we have the solution of (32), then we also have the solution of (31), simply by setting

$$(34)\quad \begin{cases} \underline{u}^{n+1/2} = \dfrac{\underline{u}^{n} + \underline{u}^{n+1}}{2} \\ \\ \tilde{p}^{n+1/2} = p^{n+1/2} \end{cases}$$

Thus, we can dispense with (31), and also we can rewrite (33) as

$$(35)\quad \underline{F}^{n+1} = \underline{F}^{n} - \frac{\Delta t}{2} K\left(\underline{U}^{n} + \underline{U}^{n+1}\right)$$

$$= \underline{F}^{n+1/2} - \frac{\Delta t}{2} K \underline{U}^{n+1}$$

Lowering n by 1 in equation (35), we get

(36) $$\underline{F}^{n} = \underline{F}^{n-1/2} - \frac{\Delta t}{2} K \underline{U}^{n}$$

and substitution of this into (29) gives

(37) $$\underline{F}^{n+1/2} = \underline{F}^{n-1/2} - (\Delta t) K \underline{U}^{n}$$

Thus, we have eliminated the variables $\underline{u}^{n+1/2}$, $\tilde{p}^{n+1/2}$, $\underline{F}^{n}$, and what we are left with is a <u>leapfrog</u> scheme that we can write out as follows:

(38) $$\underline{F}^{n+1/2}(\underline{k}h) = \underline{F}^{n-1/2}(\underline{k}h) - (\Delta t) K \sum_{\underline{j} \in Z_N^3} \underline{u}^n(\underline{j}h)\,\phi(j_1-k_1)\,\phi(j_2-k_2)\,\phi(j_3)$$

(39) $$\underline{f}^{n+1/2}(\underline{j}h) = \sum_{\underline{k} \in Z_N^2} \underline{F}^{n+1/2}(\underline{k}h)\,\phi(j_1-k_1)\,\phi(j_2-k_2)\,\frac{\phi(j_3)}{h}$$

(40) $$\begin{cases} \rho \dfrac{\underline{u}^{n+1} - \underline{u}^n}{\Delta t} + \nabla_h p^{n+1/2} = \mu \Delta_h \dfrac{\underline{u}^{n+1} + \underline{u}^n}{2} + \underline{f}^{n+1/2} \\ \nabla_h \cdot \underline{u}^{n+1} = 0 \end{cases}$$

To analyze the behavior of this scheme, we introduce the discrete Fourier transform, defined on d space dimensions by

(41) $$\hat{a}(\underline{m}) = \sum_{\ell \in Z_N^d} e^{-i\frac{2\pi}{N}(\underline{\ell}\cdot\underline{m})} a(\ell)$$

(42) $$a(\ell) = \frac{1}{N^d} \sum_{\underline{m} \in Z_N^d} e^{i\frac{2\pi}{N}(\underline{\ell}\cdot\underline{m})} \hat{a}(\underline{m})$$

It is arbitrary where to put the normalizing factor N^{-d}, and it is slightly more convenient here to put it in the inverse transform.

When we apply the discrete Fourier transform to a function of $\underline{x}$ or $\underline{s}$, we are actually applying it to the restriction of that function to the lattice $(h Z_N)^3$ or $(h Z_N)^2$ regarded as a function of $\underline{j}$ or $\underline{k}$.

Thus, for example,

$$(43) \quad \hat{\underline{u}}(\underline{m}) = \sum_{\underline{j} \in Z_N^3} e^{-i \frac{2\pi}{N} (\underline{j} \cdot \underline{m})} \underline{u}(\underline{j} h)$$

$$(44) \quad \underline{u}(\underline{j} h) = \frac{1}{N^3} \sum_{\underline{m} \in Z_N^3} e^{i \frac{2\pi}{N} (\underline{j} \cdot \underline{m})} \hat{\underline{u}}(\underline{m})$$

The operators ∇_h and Δ_h become multiplication operators in Fourier space

(45) $$(\hat{\nabla}_h)_\alpha(\underline{m}) = \frac{i}{h}\sin\left(\frac{2\pi}{N}m_\alpha\right), \quad \alpha=1,2,3$$

(46) $$\hat{\Delta}_h(\underline{m}) = -\frac{4}{h^2}\sum_{\alpha=1}^{3}\sin^2\left(\frac{\pi m_\alpha}{N}\right)$$

We also need to evaluate $\hat{\phi}$. From (23), and from (41) with $d=1$,

(47) $$\hat{\phi}(m) = \frac{1}{2} + \frac{1}{4}\left(e^{-i\frac{2\pi}{N}m} + e^{-i\frac{2\pi}{N}(N-1)m}\right)$$

$$= \frac{1}{2} + \frac{1}{4}\left(e^{-i\frac{2\pi}{N}m} + e^{i\frac{2\pi}{N}m}\right)$$

$$= \frac{1}{2}\left(1 + \cos\frac{2\pi}{N}m\right)$$

$$= \cos^2\left(\frac{\pi}{N}m\right)$$

To apply the Fourier transform to equation (38), we introduce again the notation $\underline{U}^n$, and we also make use of the evenness of ϕ (since our integer arithmetic is mod N) to write

(48) $$\underline{U}^n(\underline{k}h) = \sum_{\underline{j} \in Z_N^3} \underline{u}^n(\underline{j}h)\,\phi(k_1 - j_1)\,\phi(k_2 - j_2)\,\phi(-j_3)$$

$$= \frac{1}{N^3} \sum_{\underline{m} \in Z_N^3} \sum_{\underline{j} \in Z_N^3} \underline{u}^n(\underline{j}h)\, e^{-i\frac{2\pi}{N}(\underline{j}\cdot\underline{m})}$$

$$\hat{\phi}(m_1)\,\hat{\phi}(m_2)\,\hat{\phi}(m_3)\; e^{i\frac{2\pi}{N}(k_1 m_1 + k_2 m_2)}$$

$$= \frac{1}{N^2} \sum_{(m_1, m_2) \in Z_N^2} e^{i\frac{2\pi}{N}(k_1 m_1 + k_2 m_2)}$$

$$\hat{\phi}(m_1)\,\hat{\phi}(m_2)\,\frac{1}{N} \sum_{m_3 \in Z_N} \hat{\phi}(m_3)\,\hat{\underline{u}}^n(\underline{m})$$

Equation (48) shows that

$$(49) \quad \underline{\hat{U}}^n(m_1,m_2) = \frac{\hat{\phi}(m_1)\hat{\phi}(m_2)}{N} \sum_{m_3 \in \mathbb{Z}_N} \hat{\phi}(m_3)\, \underline{\hat{u}}^n(\underline{m})$$

and therefore, when we apply the discrete Fourier transform to both sides of equation (38), we get

$$(50) \quad \underline{\hat{F}}^{n+1/2}(m_1,m_2) = \underline{\hat{F}}^{n-1/2}(m_1,m_2)$$

$$- (\Delta t) K \frac{\hat{\phi}(m_1)\hat{\phi}(m_2)}{N} \sum_{m_3 \in \mathbb{Z}_N} \hat{\phi}(m_3)\, \underline{\hat{u}}^n(\underline{m})$$

To apply the discrete Fourier transform on equation (39), we simply multiply both sides by

(51) $$e^{-i\frac{2\pi}{N}\underline{j}\cdot\underline{m}}$$

and sum over $\underline{j} \in Z_N^3$. On the right-hand side, we use

(52) $$e^{-i\frac{2\pi}{N}j_1 m_1} = e^{-i\frac{2\pi}{N}(j_1-k_1)}\, e^{-i\frac{2\pi}{N}k_1}$$

(53) $$e^{-i\frac{2\pi}{N}j_2 m_2} = e^{-i\frac{2\pi}{N}(j_2-k_2)}\, e^{-i\frac{2\pi}{N}k_2}$$

and shift indices in the sums over j_1 and j_2. In this way, we get

(54) $$\hat{\underline{f}}^{n+1/2}(\underline{m}) = \frac{1}{h}\,\hat{\phi}(m_1)\,\hat{\phi}(m_2)\,\hat{\phi}(m_3)\,\hat{\underline{F}}^{n+1/2}(m_1, m_2)$$

After application of the discrete Fourier transform, the system (40) becomes

$$(55)\quad \begin{cases} \rho \dfrac{\hat{\underline{u}}^{n+1} - \hat{\underline{u}}^{n}}{\Delta t} + \hat{\underline{\nabla}}_h \hat{p}^{n+1/2} = \mu \hat{\Delta}_h \dfrac{\hat{\underline{u}}^{n+1} + \hat{\underline{u}}^{n}}{2} + \hat{\underline{f}}^{n+1/2} \\ \hat{\underline{\nabla}}_h \cdot \hat{\underline{u}}^{n+1} = 0 \end{cases}$$

This is a separate system of 4 equations in the 4 unknowns $\hat{\underline{u}}^{n+1}(\underline{m})$, $\hat{p}^{n+1/2}(\underline{m})$ for each $\underline{m}$. We can eliminate $\hat{p}^{n+1/2}$ and solve for $\hat{\underline{u}}^{n+1}$ by introducing the 3×3 matrix

$$(56)\quad \hat{P}(m) = I - \frac{\left(\hat{\underline{\nabla}}_h(m)\right)\left(\hat{\underline{\nabla}}_h(m)\right)^*}{\left(\hat{\underline{\nabla}}_h(m)\right)^*\left(\hat{\underline{\nabla}}_h(m)\right)}$$

In the definition of $\hat{P}(\underline{m})$, $*$ denotes the Hermitian conjugate of a matrix and I is the 3×3 identity matrix. Also we are now considering 3-vectors as columns, so from (45)

$$\hat{\nabla}_h(\underline{m}) = \frac{i}{h}\begin{pmatrix} \sin\left(\frac{2\pi}{N} m_1\right) \\ \sin\left(\frac{2\pi}{N} m_2\right) \\ \sin\left(\frac{2\pi}{N} m_3\right) \end{pmatrix} \tag{57}$$

It follows, since $-i^2 = +1$, that the matrix elements of $\hat{P}$ are real and are given by

$$\hat{P}_{\alpha\beta}(\underline{m}) = \delta_{\alpha\beta} - \frac{\sin\left(\frac{2\pi}{N} m_\alpha\right)\sin\left(\frac{2\pi}{N} m_\beta\right)}{\sum_{\gamma=1}^{3} \sin^2\left(\frac{2\pi}{N} m_\gamma\right)} \tag{58}$$

It is easy to show that $\hat{P}(\underline{m})\hat{\nabla}_h(\underline{m}) = 0$, and also that $\hat{P}(\underline{m})\underline{\hat{u}} = \underline{\hat{u}}$ for all $\underline{\hat{u}}$ such that $\hat{\nabla}_h(\underline{m}) \cdot \underline{\hat{u}} = 0$. Moreover, $\hat{P}(\underline{m})$ commutes with $\hat{\Delta}_h(\underline{m})$ since $\hat{\Delta}_h(\underline{m})$ is a scalar operator (i.e., a scalar multiple of the identity when applied to a vector). It follows that (55), together with $\hat{\nabla}_h(\underline{m}) \cdot \underline{u}^n(\underline{m}) = 0$ (from the previous time step), implies

$$(59) \quad \rho \frac{\underline{\hat{u}}^{n+1} - \underline{\hat{u}}^n}{\Delta t} = \mu \hat{\Delta}_h \frac{\underline{u}^{n+1} + \underline{u}^n}{2} + \hat{P} \hat{f}^{n+1/2}$$

and this can be solved for $\underline{\hat{u}}^{n+1}$:

$$\hat{\underline{u}}^{n+1}(\underline{m}) = \tag{60}$$

$$\frac{\left(1 + \frac{\Delta t}{2}\frac{\mu}{\rho}\hat{\Delta}_h(\underline{m})\right)\hat{\underline{u}}^n(\underline{m}) + \frac{\Delta t}{\rho}\hat{P}(\underline{m})\hat{\underline{f}}^{n+1/2}(\underline{m})}{1 - \frac{\Delta t}{2}\frac{\mu}{\rho}\hat{\Delta}_h(\underline{m})}$$

In summary, our leapfrog scheme is described in Fourier space by equations (50), (54), and (60). It is a separate scheme for each pair (m_1, m_2) but different m_3 are coupled together by the sum over m_3 in equation (50).

To study the stability of this scheme, we look for a solution in which all variables are multiplied by a possibly complex number z at each time step. We are considering each pair (m_1, m_2) separately, and z may depend on (m_1, m_2). The type of solution that we seek is defined by

$$
(61) \quad \left\{ \begin{aligned}
\underline{\hat{u}}^n(\underline{m}) &= z^n(m_1, m_2)\, \underline{\hat{u}}^0(\underline{m}) \\
\underline{\hat{F}}^{n+1/2}(m_1, m_2) &= z^n(m_1, m_2)\, \underline{\hat{F}}^{1/2}(m_1, m_2) \\
\underline{\hat{f}}^{n+1/2}(\underline{m}) &= z^n(m_1, m_2)\, \underline{\hat{f}}^{1/2}(\underline{m})
\end{aligned} \right.
$$

Although most of our superscripts are merely labels, the superscript on z is actually a power. The dependence of z on (m_1, m_2) will be left understood in the following.

Now we substitute (61) into the equations of our Fourier-transformed leapfrog scheme and cancel common factors of z^n. The results are

(62) $$\left(1-\frac{1}{z}\right)\hat{\underline{F}}^{1/2}(m_1, m_2) =$$

$$-(\Delta t) K \frac{\hat{\phi}(m_1)\hat{\phi}(m_2)}{N} \sum_{m_3 \in Z_N} \hat{\phi}(m_3)\, \hat{\underline{u}}^0(\underline{m})$$

(63) $$\hat{\underline{f}}^{1/2}(\underline{m}) = \frac{1}{h}\hat{\phi}(m_1)\hat{\phi}(m_2)\hat{\phi}(m_3)\hat{\underline{F}}^{1/2}(m_1, m_2)$$

(64) $$z\,\hat{\underline{u}}^0(\underline{m}) =$$

$$\frac{\left(1+\frac{\Delta t}{2}\frac{\mu}{\rho}\hat{\Delta}_h(\underline{m})\right)\hat{\underline{u}}^0(\underline{m}) + \frac{\Delta t}{\rho}\hat{P}(\underline{m})\,\hat{\underline{f}}^{1/2}(\underline{m})}{1-\frac{\Delta t}{2}\frac{\mu}{\rho}\hat{\Delta}_h(\underline{m})}$$

Equation (64) can be solved for $\hat{\underline{u}}^0(\underline{m})$:

$$\hat{\underline{u}}^0(\underline{m}) = \frac{\frac{\Delta t}{\rho}\hat{P}(\underline{m})\,\hat{\underline{f}}^{1/2}(\underline{m})}{(z-1)-(z+1)\frac{\Delta t}{2}\frac{\mu}{\rho}\hat{\Delta}_h(\underline{m})} \tag{65}$$

Now combining (62), (63), and (65), and recalling that $Nh = L_0$, we get an equation for $\hat{\underline{F}}^{1/2}(m_1,m_2)$ in terms of itself:

$$\left(\frac{z-1}{z}\right)\hat{\underline{F}}^{1/2}(m_1,m_2) = -\frac{(\Delta t)^2 K}{\rho L_0}\left(\hat{\phi}(m_1)\hat{\phi}(m_2)\right)^2 \left(\sum_{m_3\in Z_N}\frac{\left(\hat{\phi}(m_3)\right)^2\hat{P}(\underline{m})}{(z-1)-(z+1)\frac{\Delta t}{2}\frac{\mu}{\rho}\hat{\Delta}_h(\underline{m})}\right)\hat{\underline{F}}^{1/2}(m_1,m_2) \tag{66}$$

We can rewrite (66) as

(67) $$\left(\frac{(z-1)^2}{z} I + A\right) \underline{\hat{F}}^{1/2} = 0$$

where

(68) $$A = \frac{(\Delta t)^2 K}{\rho L_0} \left(\hat{\phi}(m_1)\,\hat{\phi}(m_2)\right)^2 \sum_{m_3 \in Z_N} \frac{\left(\hat{\phi}(m_3)\right)^2 \hat{P}(\underline{m})}{1 - \left(\frac{z+1}{z-1}\right) \frac{\Delta t}{2} \frac{\mu}{\rho} \hat{\Delta}_h(\underline{m})}$$

Thus, equation (67) has non-trivial solutions if and only if

(69) $$\det\left(\frac{(z-1)^2}{z} I + A\right) = 0$$

Equation (69) is equivalent to the statement that

(70) $$\frac{(z-1)^2}{z} + \lambda = 0$$

where λ is an eigenvalue of A.

Consider now the special case $\mu = 0$. Then A does not depend on z, so its eigenvalues do not depend on z. Also A is a real symmetric matrix, so its eigenvalues are real, and since A is a linear combination of projection matrices with positive weights, the eigenvalues of A are clearly non-negative. Thus we consider equation (70) with $\lambda \geq 0$. This is the quadratic equation

(71) $$z^2 - (2-\lambda)z + 1 = 0$$

The product of the roots is 1, and the discriminant is

(72) $$(2-\lambda)^2 - 4 = -4\lambda + \lambda^2$$

For $\lambda = 0$, the only solution is $z = 1$, and the system is stable

For $0 < \lambda < 4$, the solutions are not real; they are complex conjugates and since their product is 1, they are on the unit circle, and the system is stable

For $\lambda = 4$, the only solution is $z = -1$ and the system is stable.

For $\lambda > 4$, the solutions are real and distinct. Since their product is 1, one of them is outside the unit circle, and the system is unstable.

From the above result we can easily derive a sufficient condition for stability in the inviscid case. Since $\hat{P}(\underline{m})$ is a projection matrix for each $\underline{m}$ and since the factors $\hat{\phi}$ are all bounded from above by 1, it is obvious that

(73) $$\|A\| \leq \frac{(\Delta t)^2 K}{\rho L_0} N = \frac{(\Delta t)^2 K}{\rho h}$$

where $\| \ \|$ denotes the operator norm.

Thus, every eigenvalue of λ likewise satisfies

(74) $$\lambda \leq \frac{(\Delta t)^2 K}{\rho h}$$

so if

(75) $$\frac{(\Delta t)^2 K}{\rho h} \leq 4$$

then all of the eigenvalues of A are ≤ 4 and the system is stable.

If $\mu > 0$, then we expect that all solutions z lie strictly inside the unit circle for Δt positive and sufficiently small. The mechanism of instability then has to be some solution z crossing the unit circle for some (m_1, m_2).

We claim that this can only happen at $z = -1$. To see this, note that for $z = e^{i\theta}$, we have

(76) $$\frac{(z-1)^2}{z} = \frac{e^{2i\theta} - 2e^{i\theta} + 1}{e^{i\theta}}$$

$$= \left(e^{i\theta} - 2 + e^{-i\theta}\right) = 2(\cos\theta - 1)$$

which is real, but

$$(77)\qquad \frac{z+1}{z-1} = \frac{e^{i\theta}+1}{e^{i\theta}-1} = \frac{e^{i\theta/2}+e^{-i\theta/2}}{e^{i\theta/2}-e^{-i\theta/2}}$$

$$= \frac{2\cos(\theta/2)}{2i\sin(\theta/2)}$$

which is a nonzero imaginary number unless $\theta = \pi$, which is the same as $z=-1$

From the form of (68) it therefore seems clear that the matrix A cannot have real eigenvalues (this requires a proof!) when z is on the unit circle unless $z=-1$, and therefore that the only way to satisfy (67) for z on the unit circle is if $z=-1$.

In this way we come to the far-reaching conclusion that the stability threshold for $\mu > 0$ is exactly the same as the stability threshold for $\mu = 0$.

Thus, the sufficient condition (75) should be applicable not only to $\mu=0$ but to all $\mu \geq 0$, and moreover we can make the more precise statement that the system will be stable if and only if

(78) $$\frac{(\Delta t)^2 K}{\rho L_0}\left(\hat{\Phi}(m_1)\hat{\Phi}(m_2)\right)^2 \left\| \sum_{m_3 \in Z_n} \left(\hat{\Phi}(m_3)\right)^2 \hat{P}(\underline{m}) \right\|$$

$$\leq 4$$

for all $(m_1, m_2) \in Z_N^2$. Note that

$\sum_{m_3 \in Z_N} \left(\hat{\Phi}(m_3)\right)^2 \hat{P}(\underline{m})$ is a symmetric

positive definite 3×3 matrix for each (m_1, m_2), and its operator norm is simply its largest eigenvalue.