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Equilibrium of a Rigid Body with Applied Force and Torque Suspended by Linear Springs

Let the rigid body have n points at which springs of stiffness K are attached. The locations of these points are denoted

(1) $$\underline{Z}_1 \cdots \underline{Z}_n$$

in coordinates attached to the body, with the origin of these coordinates chosen so that

(2) $$\sum_{k=1}^{n} \underline{Z}_k = 0$$

Let the spring which is connected at one end to the point $\underline{Z}_k$ of the body be connected at its other end to the point $\underline{X}_k$ in physical space. The points

(3) $$\underline{X}_1 \cdots \underline{X}_n$$

therefore provide the framework within which the rigid body is suspended.

Let a force $\underline{F}$ and a torque $\underline{N}$ be externally applied to the rigid body. The equations of equilibrium are

$$(4) \qquad \underline{0} = \underline{F} + K \sum_{k=1}^{n} \left(\underline{X}_k - \left(\underline{Y} + R \underline{Z}_k \right) \right)$$

$$(5) \qquad \underline{0} = \underline{N} + K \sum_{k=1}^{n} \left(R \underline{Z}_k \right) \times \left(\underline{X}_k - \left(\underline{Y} + R \underline{Z}_k \right) \right)$$

where

$$(6) \qquad \underline{Z} \longrightarrow \underline{Y} + R \underline{Z}$$

is the rigid-body motion (rotation followed by translation) that takes the body from its reference configuration to its equilibrium configuration. In particular, R is a 3×3 orthogonal matrix

$$(7) \qquad R^T R = R R^T = I$$

Note that the force $\underline{F}$ is applied to the point of the rigid body which is located at the origin in body coordinates and at the point $\underline{Y}$ in physical coordinates. This is also the point about which we measure the torque in equation (5), so $\underline{F}$ itself generates no torque and therefore does not appear in equation (5). We could equivalently say that a force $\underline{F}/n$ is applied at each of the locations $\underline{Y} + R\underline{Z}_k$. Such a force would again contribute no torque about the point $\underline{Y}$ because of equation (2).

Our goal is to solve equations (4,5,7) for the unknowns $\underline{Y}$ and R. The given data are $\underline{Z}_1 \cdots \underline{Z}_n$, $\underline{X}_1 \cdots \underline{X}_n$, $\underline{F}$, $\underline{N}$ and K. We are especially interested in K large.

Note that equations (4) and (5) can both be simplified by making use of equation (2). In equation (5), moreover, we have the further simplification that $\underline{Z}_k \times \underline{Z}_k = 0$. The results are

(8) $$0 = \underline{F} + K \sum_{k=1}^{n} (\underline{X}_k - \underline{Y})$$

(9) $$0 = \underline{N} + K \sum_{k=1}^{n} (R\underline{Z}_k) \times \underline{X}_k$$

Note that the unknowns $\underline{Y}$ and R are now completely uncoupled. The solution of equation (8) for $\underline{Y}$ is

(10) $$\underline{Y} = \frac{\underline{F}}{Kn} + \frac{1}{n} \sum_{k=1}^{n} \underline{X}_k$$

We can make equation (9) into a matrix equation by taking the cross product of each term with an arbitrary vector $\underline{U}$. Thus

(11) $$0 = \underline{N} \times \underline{U} + K \sum_{k=1}^{n} \left((R\underline{Z}_k) \times \underline{X}_k \right) \times \underline{U}$$

Now let $(\underline{N}\times)$ denote the 3×3 antisymmetric matrix such that $(\underline{N}\times)\underline{U} = \underline{N}\times\underline{U}$, and make use of the vector identity

(12) $$(\underline{a}\times\underline{b})\times\underline{c} = \underline{b}(\underline{a}\cdot\underline{c}) - \underline{a}(\underline{b}\cdot\underline{c})$$

Then (11) becomes

(13) $$0 = \left((\underline{N}\times) + K \sum_{k=1}^{n} \left(\underline{X}_k (R\underline{Z}_k)^T - (R\underline{Z}_k) \underline{X}_k^T \right) \right) \underline{U}$$

Since $\underline{U}$ is arbitrary, this gives the matrix equation

(14) $$0 = (\underline{N}\times) + K \sum_{k=1}^{n} \left(\underline{X}_k \underline{Z}_k^T R^T - R \underline{Z}_k \underline{X}_k^T \right)$$

Let

$$(15) \qquad A = \sum_{k=1}^{n} \underline{Z}_k \underline{X}_k^T$$

Then (14) becomes

$$(16) \qquad 0 = (\underline{N}\times) + K\left((RA)^T - (RA)\right)$$

Note that both terms on the right-hand side are antisymmetric.

We first solve (16) in the special case $\underline{N} = 0$. The solution to this problem will be denoted R_0. Thus

$$(17) \qquad 0 = (R_0 A)^T - (R_0 A)$$

and

$$(18) \qquad R_0^T R_0 = R_0 R_0^T = I$$

Equation (17) states that $R_0 A$ is a symmetric matrix. We call this matrix M.

Thus,

(19) $$M = R_0 A$$

and $M^T = M$. Thus

(20) $$M^2 = M^T M = A^T R_0^T R_0 A = A^T A$$

Thus,

(21) $$M = (A^T A)^{1/2}$$

If A is non-singular, we can now solve for R_0 from (19) and (21), with the result that

(22) $$R_0 = (A^T A)^{1/2} A^{-1}$$

Sufficient conditions for A to be non-singular will be derived later. If A is non-singular, $A^T A$ is a 3×3 symmetric positive definite matrix. As such, it has 8 square roots which cannot be connected continuously to each other, since an eigenvalue would have to become zero to allow such a connection. In the special case $\underline{X}_k = \underline{Z}_k$, we want the solution $R_0 = I$, and it is easy to check that this will be the case if and only if we choose the principal square root, i.e., the one with positive eigenvalues.

Returning now to the general case of equation (16) with <u>N</u> nonzero, we look for a solution of the form

$$(23) \qquad R = R_1 R_0$$

This is a rotation R_0 to the torque-free configuration followed by a rotation R_1 to account for the torque. When K is large, we expect that R_1 will be close to the identity and thus instead of making R_1 be an orthogonal matrix we look for

$$(24) \qquad R_1 = I + \frac{1}{K}\Omega_1$$

where Ω_1 is an <u>antisymmetric</u> matrix. This makes

$$(25) \qquad R_1 R_1^T = I + \mathcal{O}(K^{-2})$$

so R_1 only approximately describes a rigid motion. Since $R_0 A = M$, we have

$$(26)\qquad RA = R_1 R_0 A = R_1 M$$
$$= \left(I + \frac{1}{K}\Omega_1\right) M$$

Substituting (26) into (16) and recalling that M is symmetric gives

$$(27)\qquad (\underline{N}\times) = \Omega_1 M + M\Omega_1$$

This is an equation stating that two antisymmetric matrices are equal, since $(\underline{N}\times)$ is antisymmetric, and, on the right-hand side, Ω_1 is antisymmetric and $M = (A^T A)^{1/2}$ is symmetric.

We can solve for Ω_1 by setting

$$(28)\qquad M = S^T D S$$

where S is orthogonal and D is diagonal. The diagonal elements of D, denoted D_1, D_2, D_3

are the positive square roots of the eigenvalues of $A^T A$. Then (27) becomes

(29) $$(Nx) = \Omega_1 S^T D S + S^T D S \Omega_1$$

Multiply on the left by S and on the right by S^T to obtain

(30) $$S(Nx)S^T = (S\Omega_1 S^T)D + D(S\Omega_1 S^T)$$

In components

$$(S(Nx)S^T)_{ij} = (S\Omega_1 S^T)_{ij}(D_i + D_j)$$

Thus

(31) $$(S\Omega_1 S^T)_{ij} = \frac{(S(Nx)S^T)_{ij}}{D_i + D_j}$$

Once $(S\Omega_1 S^T)$ is known, we can of course find Ω_1 by

(32) $$\Omega_1 = S^T(S\Omega_1 S^T)S$$

This completes the construction of our (approximate) solution.

Even though our solution defines only an approximately rigid motion

(33) $$\underline{Z} \longrightarrow \underline{Y} + R_1 R_0 \underline{Z}$$

where R_0 is orthogonal and R_1 is approximately orthogonal, it nevertheless defines a configuration in which the total torque and total force on the rigid body are <u>exactly</u> zero, since equations (4) and (5) are exactly satisfied.

It is also of interest to evaluate the total force and torque applied by the springs to the support points $\underline{X}_k$. The total force is given by

(34) $$K \sum_{k=1}^{n} \left(\left(\underline{Y} + R \underline{Z}_k \right) - \underline{X}_k \right) = \underline{F}$$

by equation (4), and the total torque is given by

(35) $$K \sum_{k=1}^{n} \left(\underline{X}_k - \underline{Y} \right) \times \left(\left(\underline{Y} + R \underline{Z}_k \right) - \underline{X}_k \right)$$

$$= K \sum_{k=1}^{n} \left(\underline{X}_k - \underline{Y}\right) \times \left(R\underline{Z}_k - \left(\underline{X}_k - \underline{Y}\right)\right)$$

$$= K \sum_{k=1}^{n} \left(\underline{X}_k - Y\right) \times \left(R\underline{Z}_k\right)$$

$$= K \sum_{k=1}^{n} \underline{X}_k \times \left(R\underline{Z}_k\right) \quad = \underline{N}$$

by equation (9).
Thus, the external force and torque applied to the rigid body are transmitted unchanged to the support points from which the body is suspended, and this is not affected by our approximation.

Appendix: Sufficient conditions for the existence of A^{-1}.

In this appendix we study matrices of the form

(1) $$A = \sum_{k=1}^{n} \underline{Z}_k \underline{X}_k^T$$

where $\underline{Z}_k \in \mathbb{R}^3$, $\underline{X}_k \in \mathbb{R}^3$ for $k = 1 \cdots n$

Throughout the appendix we make the assumption that

(2) $$(Z_1 \cdots Z_n) \text{ spans } \mathbb{R}^3$$

which indeed is necessary for the existence of A^{-1}. Thus $n \geq 3$.

We shall prove three sufficient conditions of increasing generality for the existence of A^{-1}. Each case includes the previous cases, and the proof of each case relies on the previous cases.

The first case is defined by

(3) $$\underline{X}_k = \underline{Z}_k \quad \text{for} \quad k = 1 \cdots n$$

In this case, we shall prove that the symmetric matrix

(4) $$A = \sum_{k=1}^{n} Z_k Z_k^T$$

is strictly positive definite and therefore has a smallest eigenvalue $\lambda_{min} > 0$. This eigenvalue will be used in the statement and proof of the subsequent sufficient conditions.

The second case is defined by

(5) $$\frac{\|\underline{X}_k - \underline{Z}_k\|}{\|\underline{Z}_k\|} < \frac{\lambda_{min}}{\sum_{k=1}^{n} \|\underline{Z}_k\|^2}$$

for $k = 1 \cdots n$. That is, the $\underline{X}_k$ are sufficiently close to the $\underline{Z}_k$ in a relative sense.

The third case is defined by the existence of a 3x3 nonsingular matrix Q (possibly a rotation, but not necessarily) such that

$$(6) \qquad \frac{\| Q\underline{X}_k - \underline{Z}_k \|}{\| \underline{Z}_k \|} < \frac{\lambda_{min}}{\sum_{k=1}^{n} \| \underline{Z}_k \|^2}$$

for $k = 1 \cdots n$. That is, there exists a nonsingular linear mapping that brings the $\underline{X}_k$ sufficiently close to the $\underline{Z}_k$ in a relative sense.

<u>Proof in Case 1</u>: Here, we assume that $\underline{X}_k = \underline{Z}_k$, so that

$$(7) \qquad A = \sum_{k=1}^{n} \underline{Z}_k \underline{Z}_k^T$$

which is symmetric and non-negative. Then

$$(8) \qquad \underline{U}^T A \underline{U} = \sum_{k=1}^{n} (\underline{U}^T \underline{Z}_k)(\underline{Z}_k^T \underline{U}) = \sum_{k=1}^{n} (\underline{Z}_k^T \underline{U})^2$$

If $\underline{U}^T A \underline{U} = 0$, then it follow from (8) that

(9) $$\underline{Z}_k^T \underline{U} = 0 \text{ for } k = 1 \cdots n$$

Since $(Z_1 \cdots Z_n)$ spans $\mathbb{R}^3$, this implies that

(10) $$\underline{V}^T \underline{U} = 0$$

for all $\underline{V} \in \mathbb{R}^3$, and thus $\underline{U} = 0$.

It follows that the smallest eigenvalue of

(11) $$\sum_{k=1}^{n} \underline{Z}_k \underline{Z}_k^T$$

is positive. Let this eigenvalue be denoted λ_{min}. We then have

(12) $$\underline{U}^T \sum_{k=1}^{n} \underline{Z}_k \underline{Z}_k^T \underline{U} \geq \lambda_{min} \|\underline{U}\|^2$$

for all $\underline{U} \in \mathbb{R}^3$. This completes the proof that A is nonsingular in case 1.

Proof in Case 2

Here we assume that $(\underline{X}_1 \cdots \underline{X}_n)$ satisfy (5). We rewrite A as

$$(13) \qquad A = \sum_{k=1}^{n} \underline{Z}_k \underline{X}_k^T = \sum_{k=1}^{n} \underline{Z}_k \underline{Z}_k^T + \sum_{k=1}^{n} \underline{Z}_k (\underline{X}_k - \underline{Z}_k)^T$$

So that

$$(14) \qquad \underline{U}^T A \underline{U} = \underline{U}^T \left(\sum_{k=1}^{n} Z_k Z_k^T \right) \underline{U} + \sum_{k=1}^{n} (\underline{U}^T \underline{Z}_k) \left((\underline{X}_k - \underline{Z}_k)^T \underline{U} \right)$$

$$\geq \lambda_{min} \|\underline{U}\|^2 - \left(\sum_{k=1}^{n} \|\underline{Z}_k\| \, \|\underline{X}_k - \underline{Z}_k\| \right) \|\underline{U}\|^2$$

$$\geq \left(\lambda_{min} - \theta_{max} \sum_{k=1}^{n} \|\underline{Z}_k\|^2 \right) \|\underline{U}\|^2$$

where

$$(15) \qquad \theta_{max} = \max_{1 \leq k \leq n} \frac{\|\underline{X}_k - \underline{Z}_k\|}{\|\underline{Z}_k\|}$$

Now our hypothesis (5) is equivalent to the statement that

$$(16) \qquad \lambda_{min} - \theta_{max} \sum_{k=1}^{n} \| \underline{z}_k \|^2 > 0$$

Thus

$$(17) \qquad A\underline{U} = 0 \Rightarrow \underline{U}^T A \underline{U} = 0 \Rightarrow \underline{U} = 0$$

by (14) and (16). This proves that A is non-singular in Case 2.

Proof in Case 3:

Here, we assume the existence of a nonsingular 3×3 matrix Q such that $(\underline{X}_1 \cdots X_n)$ satisfy (6). Then, by case 2, the matrix

$$(18) \qquad \tilde{A} = \sum_{k=1}^{n} Z_k (Q \underline{X}_k)^T$$

$$= \left(\sum_{k=1}^{n} \underline{Z}_k \underline{X}_k^T \right) Q$$

is non-singular. Since Q is also

nonsingular, it follows that

$$A = \sum_{k=1}^{n} \underline{Z}_k \underline{Y}_k^T \tag{19}$$

is nonsingular as well ∎