COMPUTING THE FIRST BETTI NUMBER AND DESCRIBING THE CONNECTED COMPONENTS OF SEMI-ALGEBRAIC SETS

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Abstract. In this paper we describe a singly exponential algorithm for computing the first Betti number of a given semi-algebraic set. Singly exponential algorithms for computing the zero-th Betti number, and the Euler-Poincaré characteristic, were known before. No singly exponential algorithm was known for computing any of the individual Betti numbers other than the zero-th one. We also give algorithms for obtaining semi-algebraic descriptions of the semi-algebraically connected components of any given real algebraic or semi-algebraic set in single-exponential time improving on previous results.

1. Introduction

Let R be a real closed field and $S \subset \mathrm{R}^k$ a semi-algebraic set defined by a quantifier-free Boolean formula with atoms of the form $P > 0, P < 0, P = 0$ for $P \in \mathcal{P} \subset \mathrm{R}[X_1, \ldots, X_k]$. We call S a $\mathcal{P}$-semi-algebraic set. If, instead, the Boolean formula has atoms of the form $P = 0, P \geq 0, P \leq 0,\ P \in \mathcal{P}$, and additionally contains no negation, then we will call S a $\mathcal{P}$-closed semi-algebraic set. It is well known [18, 19, 17, 21, 1, 12] that the topological complexity of S (measured by the various Betti numbers of S) is bounded by $O(sd)^k$, where $s = \#(\mathcal{P})$ and $d = \max_{P \in \mathcal{P}} \deg(P)$. More precise bounds on the individual Betti numbers of S appear in [2]. Even though the Betti numbers of S are bounded singly exponentially in k, there is no singly exponential algorithm for computing the Betti numbers of S. This absence is related to the fact that there is no known algorithm for producing a singly exponential sized triangulation of S (which would immediately imply a singly exponential algorithm for computing the Betti numbers of S). In fact, the existence of a singly exponential sized triangulation, is considered to be a major open question in algorithmic real algebraic geometry. Moreover, determining the exact complexity of computing the Betti numbers of semi-algebraic sets is an area of active research in computational complexity theory, for instance counting versions of complexity classes in the Blum-Shub-Smale model of computation (see [9]).

Doubly exponential algorithms (with complexity $(sd)^{2^{O(k)}}$) for computing all the Betti numbers are known, since it is possible to obtain a triangulation of S in doubly exponential time using cylindrical algebraic decomposition [11, 6]. In the absence of a singly exponential time algorithm for computing triangulations of

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semi-algebraic sets, algorithms with single exponential complexity are known only for the problems of testing emptiness [20, 4], computing the zero-th Betti number (i.e. the number of semi-algebraically connected components of S) [14, 10, 13, 5], as well as the Euler-Poincaré characteristic of S [1].

In this paper we describe the first singly exponential algorithm for computing the first Betti number of a given semi-algebraic set $S \subset \mathrm{R}^k$. In the process, we also give efficient algorithms for obtaining semi-algebraic descriptions of the semi-algebraically connected components of a given real algebraic or semi-algebraic set. These algorithms have complexity bounds which improve the complexity of the best previously known algorithm [16].

The rest of the paper is organized as follows.

There are several ideas involved in the design of our algorithm for computing the first Betti number of a given semi-algebraic set, which corresponds to the main steps in our algorithm. We describe each of them separately in different sections.

In Section 3 we recall the notion of a roadmap of an algebraic set [5] and indicate how to use it to construct connecting paths in basic semi-algebraic sets.

In Section 4 we define certain semi-algebraic sets which we call parametrized paths and prove that under a certain hypothesis these sets are semi-algebraically contractible. We also outline the input, output, and complexity of an algorithm computing a covering of a given basic semi-algebraic set, $S \subset \mathrm{R}^k$, by a singly exponential number of parametrized paths.

In Section 5, we use the properties of parametrized paths proved in Section 4 to give an algorithm (Algorithm 2) for computing a covering of a given closed and bounded semi-algebraic set by a single exponential sized family of closed, bounded as well as contractible semi-algebraic sets. The complexity of this algorithm is singly exponential.

In Section 6, we recall some results from algebraic topology which allows us to compute the first Betti number of a closed and bounded semi-algebraic set from a covering of the given set consisting of closed, bounded and contractible sets. The main tool here is a spectral sequence associated to the Mayer-Vietoris double complex. We show how to compute the first Betti number once we have computed a covering by closed contractible sets and the number of connected components of their pair-wise and triple-wise intersections of the sets in this covering and their incidences. If the size of the covering is singly exponential, this yields a singly exponential algorithm for computing the first Betti number. Extensions of these ideas for computing a fixed number of higher Betti numbers in singly exponential time is possible and is reported on in a subsequent paper [3]. In Section 7, we describe an algorithm for computing the first Betti number of a given $\mathcal{P}$-closed semi-algebraic set.

In Section 8 we recall a technique introduced by Gabrielov and Vorobjov [12], for replacing any given semi-algebraic set by one which is closed and bounded and has the same homotopy type as the given set. In fact, we prove a slight strengthening of the main result in [12], in that we prove that the new set has the same homotopy type as the given one, while the corresponding result (Lemma 5) in [12] states that only the sum of the Betti numbers is preserved. The above construction allows us to reduce the case of general semi-algebraic sets to ones which are closed and bounded treated in Section 6 without any significant worsening of complexity.

In Section 9, we describe an algorithm for computing the first Betti number of a general semi-algebraic set, using the construction described in Section 8 to first reduce the problem to the $\mathcal{P}$-closed case already treated in Section 7.

Finally, in Section 10 we indicate that the algorithms described in Section 4 actually produces descriptions of the connected components of a given algebraic or semi-algebraic set in an efficient manner.

2. Preliminaries

Let R be a real closed field. For an element $a \in \mathrm{R}$ we let

$$\mathrm{sign}(a) = \begin{cases} 0 & \text{if } a = 0, \\ 1 & \text{if } a > 0, \\ -1 & \text{if } a < 0. \end{cases}$$

If $\mathcal{P}$ is a finite subset of $\mathrm{R}[X_1, \ldots, X_k]$, we write the *set of zeros* of $\mathcal{P}$ in R^k as

$$\mathrm{Z}(\mathcal{P}, \mathrm{R}^k) = \{x \in \mathrm{R}^k \mid \bigwedge_{P \in \mathcal{P}} P(x) = 0\}.$$

We denote by $B(0, r)$ the open ball with center 0 and radius r.

Let $\mathcal{Q}$ and $\mathcal{P}$ be finite subsets of $\mathrm{R}[X_1, \ldots, X_k]$, $Z = \mathrm{Z}(\mathcal{Q}, \mathrm{R}^k)$, and $Z_r = Z \cap B(0, r)$. A *sign condition* on $\mathcal{P}$ is an element of $\{0, 1, -1\}^{\mathcal{P}}$. The *realization of the sign condition* σ *over* Z, $\mathcal{R}(\sigma, Z)$, is the basic semi-algebraic set

$$\{x \in \mathrm{R}^k \ \mid \ \bigwedge_{Q \in \mathcal{Q}} Q(x) = 0 \wedge \bigwedge_{P \in \mathcal{P}} \mathrm{sign}(P(x)) = \sigma(P)\}.$$

The *realization of the sign condition* σ *over* Z_r, $\mathcal{R}(\sigma, Z_r)$, is the basic semi-algebraic set $\mathcal{R}(\sigma, Z) \cap B(0, r)$. For the rest of the paper, we fix an open ball $B(0, r)$ with center 0 and radius r big enough so that, for every sign condition σ, $\mathcal{R}(\sigma, Z)$ and $\mathcal{R}(\sigma, Z_r)$ are homeomorphic. This is always possible by the local conical structure at infinity of semi-algebraic sets ([7], page 225).

A closed and bounded semi-algebraic set $S \subset \mathrm{R}^k$ is semi-algebraically triangulable (see [6]), and we denote by $H_i(S)$ the i-th simplicial homology group of S with rational coefficients. The groups $H_i(S)$ are invariant under semi-algebraic homeomorphisms and coincide with the corresponding singular homology groups when $\mathrm{R} = \mathbb{R}$. We denote by $b_i(S)$ the i-th Betti number of S (that is, the dimension of $H_i(S)$ as a vector space), and by $b(S)$ the sum $\sum_i b_i(S)$. For a closed but not necessarily bounded semi-algebraic set $S \subset \mathrm{R}^k$, we will denote by $H_i(S)$ the i-th simplicial homology group of $S \cap \overline{B(0, r)}$, where r is sufficiently large. The sets $S \cap \overline{B(0, r)}$ are semi-algebraically homeomorphic for all sufficiently large $r > 0$, by the local conical structure at infinity of semi-algebraic sets, and hence this definition makes sense.

The definition of homology groups of arbitrary semi-algebraic sets in R^k requires some care and several possibilities exist. In this paper, we define the homology groups of realizations of sign conditions as follows.

Let R denote a real closed field and R' a real closed field containing R. Given a semi-algebraic set S in R^k, the *extension* of S to R', denoted $\mathrm{Ext}(S, \mathrm{R}')$, is the semi-algebraic subset of ${\mathrm{R}'}^k$ defined by the same quantifier free formula that defines S. The set $\mathrm{Ext}(S, \mathrm{R}')$ is well defined (i.e. it only depends on the set S and not on the quantifier free formula chosen to describe it). This is an easy consequence of the transfer principle [6].

Now, let $S \subset \mathrm{R}^k$ be a $\mathcal{P}$-semi-algebraic set, where $\mathcal{P} = \{P_1, \ldots, P_s\}$ is a finite subset of $\mathrm{R}[X_1, \ldots, X_k]$. Let $\phi(X)$ be a quantifier-free formula defining S. Let $P_i = \sum_\alpha a_{i,\alpha} X^\alpha$ where the $a_{i,\alpha} \in \mathrm{R}$. Let $A = (\ldots, A_{i,\alpha}, \ldots)$ denote the vector of variables corresponding to the coefficients of the polynomials in the family $\mathcal{P}$, and let $a = (\ldots, a_{i,\alpha}, \ldots) \in \mathrm{R}^N$ denote the vector of the actual coefficients of the polynomials in $\mathcal{P}$. Let $\psi(A, X)$ denote the formula obtained from $\phi(X)$ by replacing each coefficient of each polynomial in $\mathcal{P}$ by the corresponding variable, so that $\phi(X) = \psi(a, X)$. It follows from Hardt's triviality theorem for semi-algebraic mappings [15], that there exists, $a' \in \mathbb{R}^N_{\mathrm{alg}}$ such that denoting by $S' \subset \mathbb{R}^k_{\mathrm{alg}}$ the semi-algebraic set defined by $\psi(a', X)$, the semi-algebraic set $\mathrm{Ext}(S', \mathrm{R})$, has the same homeomorphism type as S. We define the homology groups of S to be the singular homology groups of $\mathrm{Ext}(S', \mathbb{R})$. It follows from the Tarski-Seidenberg transfer principle, and the corresponding property of singular homology groups, that the homology groups defined this way are invariant under semi-algebraic homotopies. It is also clear that this definition is compatible with the simplicial homology for closed, bounded semi-algebraic sets, and the singular homology groups when the ground field is $\mathbb{R}$. Finally it is also clear that, the Betti numbers are not changed after extension: $b_i(S) = b_i(\mathrm{Ext}(S, \mathrm{R}'))$.

3. Roadmap of a semi-algebraic set

We first define a roadmap of a semi-algebraic set. Roadmaps are crucial ingredients in all singly exponential algorithms known for computing connectivity properties of semi-algebraic sets such as computing the number of connected components, as well as testing whether two points of a given semi-algebraic set belong to the same semi-algebraically connected component.

We use the following notations. Given $x = (x_1, \ldots, x_k)$ we write $\bar{x}_i$ for $(x_1, \ldots, x_i)$, and $\tilde{x}_i$ for $(x_{i+1}, \ldots, x_k)$. We also denote by $\pi_{1\ldots j}$ the projection, $x \mapsto \bar{x}_j$. Given a set $S \subset \mathrm{R}^k$, $y \in \mathrm{R}^j$ we denote by $S_y = S \cap \pi^{-1}_{1\ldots j}(y)$.

Let $S \subset \mathrm{R}^k$ be a semi-algebraic set. A *roadmap* for S is a semi-algebraic set M of dimension at most one contained in S which satisfies the following roadmap conditions:

- RM_1 For every semi-algebraically connected component D of S, $D \cap M$ is semi-algebraically connected.
- RM_2 For every $x \in \mathrm{R}$ and for every semi-algebraically connected component D' of S_x, $D' \cap M \neq \emptyset$.

We describe the construction of a roadmap M for a bounded algebraic set $\mathrm{Z}(Q, \mathrm{R}^k)$ which contains a finite set of points $\mathcal{N}$ of $\mathrm{Z}(Q, \mathrm{R}^k)$. A precise description of how the construction can be performed algorithmically can be found in [6].

A key ingredient of the roadmap is the construction of a particular finite set of points having the property that, they intersect every connected component of $\mathrm{Z}(Q, \mathrm{R}^k)$. We call them X_1-pseudo-critical points, since they are obtained as limits of the critical points of the projection to the X_1 coordinate of a bounded nonsingular algebraic hypersurface defined by a particular infinitesimal deformation of the polynomial Q. Their projections on the X_1-axis are called pseudo-critical values. These points are obtained as follows.

We denote by $\mathrm{R}\langle\zeta\rangle$ the real closed field of algebraic Puiseux series in ζ with coefficients in R [6]. The sign of a Puiseux series in $\mathrm{R}\langle\zeta\rangle$ agrees with the sign of the coefficient of the lowest degree term in ζ. This induces a unique order on $\mathrm{R}\langle\zeta\rangle$ which makes ζ infinitesimal: ζ is positive and smaller than any positive element of R. When $a \in \mathrm{R}\langle\zeta\rangle$ is bounded by an element of R, $\lim_\zeta(a)$ is the constant term of a, obtained by substituting 0 for ζ in a. We now define the deformation $\bar{Q}$ of Q as follows. Suppose that $\mathrm{Z}(Q, \mathrm{R}^k)$ is contained in the ball of center 0 and radius $1/c$. Let $\bar{d}$ be an even integer bigger than the degree d of Q,

$$G_k(\bar{d}, c) = c^{\bar{d}}(X_1^{\bar{d}} + \cdots + X_k^{\bar{d}} + X_2^2 + \cdots + X_k^2) - (2k-1), \tag{3.1}$$

$$\bar{Q} = \zeta G_k(\bar{d}, c) + (1-\zeta)Q. \tag{3.2}$$

The algebraic set $\mathrm{Z}(\bar{Q}, \mathrm{R}\langle\zeta\rangle^k)$ is a bounded and non-singular hypersurface lying infinitesimally close to $\mathrm{Z}(Q, \mathrm{R}^k)$, and the critical points of the projection map onto the X_1 co-ordinate restricted to $\mathrm{Z}(\bar{Q}, \mathrm{R}\langle\zeta\rangle^k)$ form a finite set of points. We take the images of these points under $\lim_\zeta$ and we call the points obtained in this manner the X_1-pseudo-critical points of $\mathrm{Z}(Q, \mathrm{R}^k)$. Their projections on the X_1-axis are called pseudo-critical values.

The construction of the roadmap of an algebraic set containing a finite number of input points $\mathcal{N}$ of this algebraic set is as follows. We first construct X_2-pseudo-critical points on $\mathrm{Z}(Q, \mathrm{R}^k)$ in a parametric way along the X_1-axis, by following continuously, as x varies on the X_1-axis, the X_2-pseudo-critical points on $\mathrm{Z}(Q, \mathrm{R}^k)_x$. This results in curve segments and their endpoints on $\mathrm{Z}(Q, \mathrm{R}^k)$. The curve segments are continuous semi-algebraic curves parametrized by open intervals on the X_1-axis, and their endpoints are points of $\mathrm{Z}(Q, \mathrm{R}^k)$ above the corresponding endpoints of the open intervals. Since these curves and their endpoints include, for every $x \in \mathrm{R}$, the X_2-pseudo-critical points of $\mathrm{Z}(Q, \mathrm{R}^k)_x$, they meet every connected component of $\mathrm{Z}(Q, \mathrm{R}^k)_x$. Thus the set of curve segments and their endpoints already satisfy RM_2. However, it is clear that this set might not be semi-algebraically connected in a semi-algebraically connected component, so RM_1 might not be satisfied. We add additional curve segments to ensure connectedness by recursing in certain distinguished hyperplanes defined by $X_1 = z$ for distinguished values z.

The set of *distinguished values* is the union of the X_1-pseudo-critical values, the first coordinates of the input points $\mathcal{N}$ and the first coordinates of the endpoints of the curve segments. A *distinguished hyperplane* is an hyperplane defined by $X_1 = v$, where v is a distinguished value. The input points, the endpoints of the curve segments and the intersections of the curve segments with the distinguished hyperplanes define the set of *distinguished points*. .

Let the distinguished values be $v_1 < \ldots < v_\ell$. Note that amongst these are the X_1-pseudo-critical values. Above each interval (v_i, v_{i+1}), we have constructed a collection of curve segments $\mathcal{C}_i$ meeting every semi-algebraically connected component of $\mathrm{Z}(Q, \mathrm{R}^k)_v$ for every $v \in (v_i, v_{i+1})$. Above each distinguished value v_i, we have a set of distinguished points $\mathcal{N}_i$. Each curve segment in $\mathcal{C}_i$ has an endpoint in $\mathcal{N}_i$ and another in $\mathcal{N}_{i+1}$. Moreover, the union of the $\mathcal{N}_i$ contains $\mathcal{N}$.

We then repeat this construction in each distinguished hyperplane H_i defined by $X_1 = v_i$ with input $Q(v_i, X_2, \ldots, X_k)$ and the distinguished points in $\mathcal{N}_i$. Thus, we construct distinguished values, $v_{i,1}, \ldots, v_{i,\ell(i)}$ of $\mathrm{Z}(Q(v_i, X_2, \ldots, X_k), \mathrm{R}^{k-1})$ (with the role of X_1 being now played by X_2) and the process is iterated until for $I =$

$(i_1, \ldots, i_{k-2}), 1 \leq i_1 \leq \ell, \ldots, 1 \leq i_{k-2} \leq \ell(i_1, \ldots, i_{k-3})$, we have distinguished values $v_{I,1} < \ldots < v_{I,\ell(I)}$ along the X_{k-1} axis with corresponding sets of curve segments and sets of distinguished points with the required incidences between them.

The following proposition is proved in [5] (see also [6]).

Proposition 1. The semi-algebraic set $\mathrm{RM}(\mathrm{Z}(Q, \mathrm{R}^k), \mathcal{N})$ obtained by this construction is a roadmap for $\mathrm{Z}(Q, \mathrm{R}^k)$ containing $\mathcal{N}$.

Note that if $x \in \mathrm{Z}(Q, \mathrm{R}^k)$, $\mathrm{RM}(\mathrm{Z}(Q, \mathrm{R}^k), \{x\})$ contains a path, $\gamma(x)$, connecting a distinguished point p of $\mathrm{RM}(\mathrm{Z}(Q, \mathrm{R}^k))$ to x.

Later in this paper we shall examine the properties of parametrized paths which are the unions of connecting paths starting at a given p and ending at x, where x varies over a certain semi-algebraic subset of $\mathrm{Z}(Q, \mathrm{R}^k)$. In order to do so it is useful to have a better understanding of the structure of these connecting paths – especially, of their dependence on x.

Recall that given $x = (x_1, \ldots, x_k)$ we write $\bar{x}_i$ for $(x_1, \ldots, x_i)$, and $\tilde{x}_i$ for $(x_{i+1}, \ldots, x_k)$. We first note that for any $x = (x_1, \ldots, x_k) \in \mathrm{Z}(Q, \mathrm{R}^k)$, we have by construction that, $\mathrm{RM}(\mathrm{Z}(Q, \mathrm{R}^k))$ is contained in $\mathrm{RM}(\mathrm{Z}(Q, \mathrm{R}^k), \{x\})$. In fact,

$$\mathrm{RM}(\mathrm{Z}(Q, \mathrm{R}^k), \{x\}) = \mathrm{RM}(\mathrm{Z}(Q, \mathrm{R}^k)) \cup \mathrm{RM}(\mathrm{Z}(Q, \mathrm{R}^k)_{x_1}, \mathcal{M}_{x_1}),$$

where $\mathcal{M}_{x_1}$ consists of $\tilde{x}_1$ and the finite set of points obtained by intersecting the curves in $\mathrm{RM}(\mathrm{Z}(Q, \mathrm{R}^k))$ parametrized by the X_1-coordinate, with the hyperplane $\pi_1^{-1}(x_1)$.

A connecting path $\gamma(x)$ (with non-self intersecting image) joining a distinguished point p of $\mathrm{RM}(\mathrm{Z}(Q, \mathrm{R}^k))$ to x can be extracted from $\mathrm{RM}(\mathrm{Z}(Q, \mathrm{R}^k), \{x\})$. The connecting path $\gamma(x)$ consists of two consecutive parts, $\gamma_0(x)$ and $\Gamma_1(x)$. The path $\gamma_0(x)$ is contained in $\mathrm{RM}(\mathrm{Z}(Q, \mathrm{R}^k))$ and the path $\Gamma_1(x)$ is contained in $\mathrm{Z}(Q, \mathrm{R}^k)_{x_1}$. The part $\gamma_0(x)$ consists of a sequence of sub-paths, $\gamma_{0,0}, \ldots, \gamma_{0,m}$. Each $\gamma_{0,i}$ is a semi-algebraic path parametrized by one of the co-ordinates $X_1, \ldots, X_k$, over some interval $[a_{0,i}, b_{0,i}]$ with $\gamma_{0,0}(a_{0,0}) = p$. The semi-algebraic maps, $\gamma_{0,0}, \ldots, \gamma_{0,m}$ and the end-points of their intervals of definition $a_{0,0}, b_{0,0}, \ldots, a_{0,m}, b_{0,m}$ are all independent of x (upto the discrete choice of the path $\gamma(x)$ in $\mathrm{RM}(\mathrm{Z}(Q, \mathrm{R}^k), \{x\})$), except $b_{0,m}$ which depends on x_1.

Moreover, $\Gamma_1(x)$ can again be decomposed into two parts, $\gamma_1(x)$ and $\Gamma_2(x)$ with $\Gamma_2(x)$ contained in $\mathrm{Z}(Q, \mathrm{R}^k)_{\bar{x}_2}$ and so on.

If $y = (y_1, \ldots, y_k) \in \mathrm{Z}(Q, \mathrm{R}^k)$ is another point such that $x_1 \neq y_1$, then since $\mathrm{Z}(Q, \mathrm{R}^k)_{x_1}$ and $\mathrm{Z}(Q, \mathrm{R}^k)_{y_1}$ are disjoint, it is clear that

$$\mathrm{RM}(\mathrm{Z}(Q, \mathrm{R}^k), \{x\}) \cap \mathrm{RM}(\mathrm{Z}(Q, \mathrm{R}^k), \{y\}) = \mathrm{RM}(\mathrm{Z}(Q, \mathrm{R}^k)).$$

Now consider a connecting path $\gamma(y)$ extracted from $\mathrm{RM}(\mathrm{Z}(Q, \mathrm{R}^k), \{y\})$. The images of $\Gamma_1(x)$ and $\Gamma_1(y)$ are disjoint. If the image of $\gamma_0(y)$ (which is contained in $\mathrm{RM}(\mathrm{Z}(Q, \mathrm{R}^k))$ follows the same sequence of curve segments as $\gamma_0(x)$ starting at p (that is, it consists of the same curves segments $\gamma_{0,0}, \ldots, \gamma_{0,m}$ as in $\gamma_0(x)$), then it is clear that the images of the paths $\gamma(x)$ and $\gamma(y)$ has the property that they are identical upto a point and they are disjoint after it. We call this the *divergence property*.

More generally, if the points x and y are such that, $x_i = y_i, 1 \leq i \leq j$ and $x_{j+1} \neq y_{j+1}$, then the paths $\Gamma_{j+1}(x)$ and $\Gamma_{j+1}(y)$, contained in $\mathrm{Z}(Q, \mathrm{R}^k)_{\bar{x}_{j+1}}$ and

$\mathrm{Z}(Q, \mathrm{R}^k)_{\bar{y}_{j+1}}$ respectively, will be disjoint. Moreover if the paths $\gamma_0(x), \ldots, \gamma_j(x)$ and $\gamma_0(y), \ldots, \gamma_j(y)$ are composed of the same sequence of curve segments, then $\gamma(x)$ and $\gamma(y)$ will also have the divergence property.

We now consider connecting paths in the semi-algebraic setting. We are given a polynomial $Q \in \mathrm{R}[X_1, \ldots, X_k]$ such that $\mathrm{Z}(Q, \mathrm{R}^k)$ is bounded and a finite set of polynomials $\mathcal{P} \subset \mathrm{D}[X_1, \ldots, X_k]$ in strong ℓ-general position with respect to Q. This means that any $\ell + 1$ polynomials belonging to $\mathcal{P}$ have no zeros in common with Q in R^k, and any ℓ polynomials belonging to $\mathcal{P}$ have at most a finite number of zeros in common with Q in R^k.

For every point x of $\mathrm{Z}(Q, \mathrm{R}^k)$, we denote by $\sigma(x)$ the sign condition on $\mathcal{P}$ at x. Let $\mathcal{R}(\overline{\sigma}(x), \mathrm{Z}(Q, \mathrm{R}^k)) = \{x \in \mathrm{Z}(Q, \mathrm{R}^k) \mid \bigwedge_{P \in \mathcal{P}} \ \mathrm{sign}(P(x)) \in \overline{\sigma}(x)(P)\}$, where $\overline{\sigma}$ is the relaxation of σ defined by

$$\begin{cases} \overline{\sigma} = \{0\} & \text{if} \quad \sigma = 0, \\ \overline{\sigma} = \{0, 1\} & \text{if} \quad \sigma = 1, \\ \overline{\sigma} = \{0, -1\} & \text{if} \quad \sigma = -1. \end{cases}$$

We say that $\overline{\sigma}(x)$ is the weak sign condition defined by x on $\mathcal{P}$. We denote by $\mathcal{P}(x)$ the union of $\{Q\}$ and the set of polynomials in $\mathcal{P}$ vanishing at x.

The connecting algorithm associates to $x \in \mathrm{Z}(Q, \mathrm{R}^k)$ a path entirely contained in the realization of $\overline{\sigma}(x)$ connecting x to a distinguished point of the roadmap of some $\mathrm{Z}(\mathcal{P}', \mathrm{R}^k)$, with $\mathcal{P}(x) \subset \mathcal{P}'$. The connecting algorithm proceeds as follows: construct a path γ connecting a distinguished point of $\mathrm{RM}(\mathrm{Z}(Q, \mathrm{R}^k))$ to x contained in $\mathrm{RM}(\mathrm{Z}(Q, \mathrm{R}^k), \{x\})$. If no polynomial of $\mathcal{P} \setminus \mathcal{P}(x)$ vanishes on γ, we are done. Otherwise let y be the last point of γ such that some polynomial of $\mathcal{P} \setminus \mathcal{P}(x)$ vanishes at y. Now keep the part of γ connecting y to x as end of the connecting path, and iterate the construction with y, noting that the realization of $\overline{\sigma}(y)$ is contained in the realization of $\overline{\sigma}(x)$, and $\mathcal{P} \setminus \mathcal{P}(y)$ is in $\ell - 1$ strong general position with respect to $\mathrm{Z}(\mathcal{P}(y), \mathrm{R}^k)$.

As in the algebraic case, two such connecting paths which start with the same sequence of curve segments will have the divergence property. This follows from the divergence property in the algebraic case and the recursive definition of connecting paths.

Formal descriptions and complexity analysis of the algorithms described above for computing roadmaps and connecting paths of algebraic and basic semi-algebraic sets can be found in [6] (Algorithm 15.12 and Algorithm 16.8).

4. Parametrized paths

We are given a polynomial $Q \in \mathrm{R}[X_1, \ldots, X_k]$ such that $\mathrm{Z}(Q, \mathrm{R}^k)$ is bounded and a finite set of polynomials $\mathcal{P} \subset \mathrm{D}[X_1, \ldots, X_k]$ in strong k'-general position with respect to Q, where k' is the dimension of $\mathrm{Z}(Q, \mathrm{R}^k)$.

We show how to obtain a covering of a given $\mathcal{P}$-closed semi-algebraic set contained in $\mathrm{Z}(Q, \mathrm{R}^k)$ by a family of semi-algebraically contractible subsets. The construction is based on a parametrized version of the connecting algorithm: we compute a family of polynomials such that for each realizable sign condition σ on this family, the description of the connecting paths of different points in the realization, $\mathcal{R}(\sigma, \mathrm{Z}(Q, \mathrm{R}^k))$, are uniform. We first define parametrized paths. A parametrized path is a semi-algebraic set which is a union of semi-algebraic paths having the divergence property.

More precisely,

Definition 4.1. A parametrized path γ is a continuous semi-algebraic mapping from $V \subset \mathrm{R}^{k+1} \to \mathrm{R}^k$, such that, denoting by $U = \pi_{1\ldots k}(V) \subset \mathrm{R}^k$, there exists a semi-algebraic continuous function $\ell : U \to [0, +\infty)$, and there exists a point a in R^k, such that

(1) $V = \{(x, t) \mid x \in U, 0 \le t \le \ell(x)\}$,
(2) $\forall\, x \in U,\ \gamma(x, 0) = a$,
(3) $\forall\, x \in U,\ \gamma(x, \ell(x)) = x$,
(4)

$$\begin{gathered}\forall\, x \in U, \forall\, y \in U, \forall\, s \in [0, \ell(x)], \forall\, t \in [0, \ell(y)] \\ (\gamma(x, s) = \gamma(y, t) \Rightarrow s = t)\,,\end{gathered}$$

(5)

$$\begin{gathered}\forall\, x \in U, \forall\, y \in U, \forall\, s \in [0, \min(\ell(x), \ell(y))] \\ (\gamma(x, s) = \gamma(y, s) \Rightarrow \forall\, t \le s\ \gamma(x, t) = \gamma(y, t))\,.\end{gathered}$$

Given a parametrized path, $\gamma : V \to \mathrm{R}^k$, we will refer to $U = \pi_{1\ldots k}(V)$ as its *base*. Also, any semi-algebraic subset $U' \subset U$ of the base of such a parametrized path, defines in a natural way the restriction of γ to the base U', which is another parametrized path, obtained by restricting γ to the set $V' \subset V$, defined by $V' = \{(x, t) \mid x \in U', 0 \le t \le \ell(x)\}$.

Proposition 2. Let $\gamma : V \to R^k$ be a parametrized path such that $U = \pi_{1\ldots k}(V)$ is closed and bounded. Then, the image of γ is semi-algebraically contractible.

We thank A. Gabrielov and N. Vorobjov for pointing out an error in a previous version of this paper, where the same proposition was stated without any extra condition on U. In fact, Proposition 2 is not true if we do not assume that U is closed and bounded.

Proof. (of Proposition 2) Let $W = \mathrm{Im}(\gamma)$ and $M = \sup_{x \in U} \ell(x)$. We prove that the semi-algebraic mapping $\phi : W \times [0, M] \to W$ sending

- $(\gamma(x, t), s)$ to $\gamma(x, s)$ if $t \ge s$,
- $(\gamma(x, t), s)$ to $\gamma(x, t)$ if $t < s$.

is continuous. Note that the map ϕ is well-defined, since $\gamma(x, t) = \gamma(x', t') \Rightarrow t = t'$, by condition (4).

Since ϕ satisfies

$$\begin{gathered}\phi(\gamma(x, t), 0) = a, \\ \phi(\gamma(x, t), M) = \gamma(x, t),\end{gathered}$$

this gives a semi-algebraic continuous contraction from W to $\{a\}$.

Let $w \in W, s \in [0, M]$. Let $\varepsilon > 0$ be an infinitesimal, and let $(w', s') \in \mathrm{Ext}(W \times [0, M], \mathrm{R}\langle\varepsilon\rangle)$ be such that $\lim_\varepsilon(w', s') = (w, s)$. In order to prove the continuity of ϕ at w it suffices to prove that $\lim_\varepsilon \mathrm{Ext}(\phi, \mathrm{R}\langle\varepsilon\rangle)(w', s') = \phi(w, s)$.

Let $w = \gamma(x, t)$ for some $x \in U, t \in [0, \ell(x)]$, and similarly let $w' = (x', t')$ for some $x' \in \mathrm{Ext}(U, \mathrm{R}\langle\varepsilon\rangle)$ and $t' \in [0, \mathrm{Ext}(\ell, \mathrm{R}\langle\varepsilon\rangle)(x')]$. Note that $\lim_\varepsilon(x') \in U$ since U is closed and bounded and $\lim_\varepsilon t' \in [0, \ell(\lim_\varepsilon x')]$.

Now,

$$\begin{array}{rcl}\gamma(x,t) & = & w \\ & = & \lim_\varepsilon(w') \\ & = & \lim_\varepsilon \operatorname{Ext}(\gamma, \mathrm{R}\langle\varepsilon\rangle)(x',t') \\ & = & \gamma(\lim_\varepsilon x', \lim_\varepsilon t').\end{array}$$

Condition (4) now implies that $\lim_\varepsilon t' = t$.

Without loss of generality let $t' \geq t$. The other case is symmetric. We have the following two sub-cases.

Case $s' > t'$: Since $s, t \in \mathrm{R}$ and $\lim_\varepsilon s' = s$ and $\lim_\varepsilon t' = t$, we must have that $s \geq t$. In this case $\operatorname{Ext}(\phi, \mathrm{R}\langle\varepsilon\rangle)(w', s') = \operatorname{Ext}(\gamma, \mathrm{R}\langle\varepsilon\rangle)(x', t')$. Then,

$$\begin{array}{rcl}\lim_\varepsilon \operatorname{Ext}(\phi, \mathrm{R}\langle\varepsilon\rangle)(w',s') & = & \lim_\varepsilon \operatorname{Ext}(\gamma, \mathrm{R}\langle\varepsilon\rangle)(x',t') \\ & = & \lim_\varepsilon w' \\ & = & w \\ & = & \phi(w,s).\end{array}$$

Case $s' \leq t'$: Again, since $s, t \in \mathrm{R}$ and $\lim_\varepsilon s' = s$ and $\lim_\varepsilon t' = t$, we must have that $s \leq t$.

In this case we have,

$$\begin{array}{rcl}\lim_\varepsilon \phi(w',s') & = & \lim_\varepsilon \operatorname{Ext}(\gamma, \mathrm{R}\langle\varepsilon\rangle)(x',s') \\ & = & \gamma(\lim_\varepsilon x', \lim_\varepsilon s') \\ & = & \gamma(\lim_\varepsilon x', s).\end{array}$$

Now,

$$\begin{array}{rcl}\gamma(\lim_\varepsilon x', t) & = & \gamma(\lim_\varepsilon x', \lim_\varepsilon t') \\ & = & \lim_\varepsilon \operatorname{Ext}(\gamma, \mathrm{R}\langle\varepsilon\rangle)(x',t') \\ & = & \lim_\varepsilon w' \\ & = & w \\ & = & \gamma(x,t).\end{array}$$

Thus, by condition (5) we have that $\gamma(\lim_\varepsilon x', s'') = \gamma(x, s'')$ for all $s'' \leq t$. Since, $s \leq t$, this implies,

$$\begin{array}{rcl}\lim_\varepsilon \operatorname{Ext}(\phi, \mathrm{R}\langle\varepsilon\rangle)(w',s') & = & \lim_\varepsilon \operatorname{Ext}(\gamma, \mathrm{R}\langle\varepsilon\rangle)(w',s') \\ & = & \gamma(\lim_\varepsilon x', \lim_\varepsilon s') \\ & = & \gamma(x,s) \\ & = & \phi(w,s).\end{array}$$

This proves the continuity of ϕ. □

We now describe how to compute parametrized paths in single exponential time using a parametrized version of the connecting algorithm. We describe the input, output and complexity of the algorithm which appears in [6] (Algorithm 16.15).

Algorithm 1. (Parametrized Bounded Connecting)

Input.

- a polynomial $Q \in \mathrm{D}[X_1, \ldots, X_k]$, such that $\mathrm{Z}(Q, \mathrm{R}^k) \subset B(0, 1/c)$,
- a finite set of polynomials $\mathcal{P} \subset \mathrm{D}[X_1, \ldots, X_k]$ in strong k'-general position with respect to Q, where k' is the dimension of $\mathrm{Z}(Q, \mathrm{R}^k)$.

Output.

- a finite set of polynomials $\mathcal{A} \subset \mathrm{R}[X_1, \ldots, X_k]$,

– a finite set Θ of quantifier free formulas, with atoms of the form $P = 0, P > 0, P < 0,\ P \in \mathcal{A}$, such that for every semi-algebraically connected component S of the realization of every weak sign condition on $\mathcal{P}$ on $\mathrm{Z}(Q, \mathrm{R}^k)$, there exists a subset $\Theta(S) \subset \Theta$ such that $S = \bigcup_{\theta \in \Theta(S)} \mathcal{R}(\theta, \mathrm{Z}(Q, \mathrm{R}^k))$,
– for every $\theta \in \Theta$, a parametrized path

$$\gamma_\theta : V_\theta \to \mathrm{R}^k,$$

with base $U_\theta = \mathcal{R}(\theta, \mathrm{Z}(Q, \mathrm{R}^k))$, such that for each $y \in \mathcal{R}(\theta, \mathrm{Z}(Q, \mathrm{R}^k))$, Im $\gamma_\theta(y, \cdot)$ is a semi-algebraic path which connects the point y to a distinguished point a_θ of some roadmap $\mathrm{RM}(\mathrm{Z}(\mathcal{P}' \cup \{Q\}, \mathrm{R}^k))$ where $\mathcal{P}' \subset \mathcal{P}$, staying inside $\mathcal{R}(\overline{\sigma}(y), \mathrm{Z}(Q, \mathrm{R}^k))$.

Complexity. $s^{k'+1} d^{O(k^4)}$, where s is a bound on the number of elements of $\mathcal{P}$ and d is a bound on the degrees of Q and the elements of $\mathcal{P}$.

Proof of correctness. Given the proof of correctness of Algorithm 16.15 (Parametrized Bounded Connecting) in [6], the only extra property that we need to prove is that for each $\theta \in \Theta$, γ_θ is a parametrized path. It is easy to see that γ_θ satisfies the conditions of Definition 4.1, using the divergence property of the paths $\gamma(y, \cdot)$ (see discussion in Section 3). □

5. Constructing coverings of closed semi-algebraic sets by closed contractible sets

We are again given a polynomial $Q \in \mathrm{R}[X_1, \ldots, X_k]$ such that $\mathrm{Z}(Q, \mathrm{R}^k)$ is bounded and a finite set of polynomials $\mathcal{P} \subset \mathrm{D}[X_1, \ldots, X_k]$ in strong k'-general position with respect to Q, where k' is the dimension of $\mathrm{Z}(Q, \mathrm{R}^k)$. We describe an algorithm for computing closed contractible coverings of $\mathcal{P}$-closed semi-algebraic sets, using the results of Section 4.

For the rest of this section we fix a $\mathcal{P}$-closed semi-algebraic set S contained in $\mathrm{Z}(Q, \mathrm{R}^k)$ and let $\#\mathcal{A} = t$. We denote by $\mathrm{Sign}(\mathcal{A}, S)$ the set of realizable sign conditions of $\mathcal{A}$ on $\mathrm{Z}(Q, \mathrm{R}^k)$ whose realizations are contained in S. We continue to follow the notations of Algorithm 1. For each $\sigma \in \mathrm{Sign}(\mathcal{A}, S)$ $\mathcal{R}(\sigma, \mathrm{Z}(Q, \mathrm{R}^k))$ is contained in $\mathcal{R}(\theta, \mathrm{Z}(Q, \mathrm{R}^k))$ for some $\theta \in \Theta$. We denote by γ_σ the restriction of γ_θ to the base $\mathcal{R}(\sigma, \mathrm{Z}(Q, \mathrm{R}^k))$. Since $\mathcal{R}(\sigma, \mathrm{Z}(Q, \mathrm{R}^k))$ is not necessarily closed and bounded, Im γ_σ might not be contractible. In order to ensure contractibility, we restrict the base of γ_σ to a slightly smaller set which is closed, using infinitesimals.

We introduce infinitesimals $\varepsilon_{2t} \gg \varepsilon_{2t-1} \gg \cdots \gg \varepsilon_2 \gg \varepsilon_1 > 0$. For $i = 1, \ldots, 2t$ we will denote by R_i the field $\mathrm{R}\langle\varepsilon_{2t}\rangle \cdots \langle\varepsilon_i\rangle$ and denote by R' the field R_1.

For $\sigma \in \mathrm{Sign}(\mathcal{A}, S)$ we define the level of σ by,

$$\mathrm{level}(\sigma) = \#\{P \in \mathcal{A} \mid \sigma(P) = 0\}.$$

Given $\sigma \in \mathrm{Sign}(\mathcal{A}, S)$, with $\mathrm{level}(\sigma) = j$, we denote by $\mathcal{R}(\sigma_-)$ the set defined on $\mathrm{Z}(Q, \mathrm{R}_{2j}^k)$ by the formula σ_- obtained by taking the conjunction of

$$\begin{array}{l} P = 0,\ \text{for each } P \in \mathcal{A} \text{ such that } \sigma(P) = 0, \\ P \geq \varepsilon_{2j},\ \text{for each } P \in \mathcal{A} \text{ such that } \sigma(P) = 1, \\ P \leq -\varepsilon_{2j},\ \text{for each } P \in \mathcal{A} \text{ such that } \sigma(P) = -1. \end{array}$$

Notice that $\mathcal{R}(\sigma_-)$ is closed, bounded and contained in $\mathcal{R}(\sigma, \mathrm{Z}(Q, \mathrm{R}^k_{2j}))$. Proposition 2 implies,

Proposition 3. $\gamma_\sigma(\mathcal{R}(\sigma_-))$ is semi-algebraically contractible.

Note that the sets $\gamma_\sigma(\mathcal{R}(\sigma_-))$ do not necessarily cover S. So we are going to enlarge them, preserving contractibility, to obtain a covering of S.

Given $\sigma \in \mathrm{Sign}(\mathcal{A}, S)$, with $\mathrm{level}(\sigma) = j$, we denote by $\mathcal{R}(\sigma^+_-)$ the set defined on $\mathrm{Z}(Q, \mathrm{R}^k_{2j-1})$, by the formula σ^+_- obtained by taking the conjunction of

$$\begin{array}{l} -\varepsilon_{2j-1} \leq P \leq \varepsilon_{2j-1}, \text{ for each } P \in \mathcal{A} \text{ such that } \sigma(P) = 0, \\ P \geq \varepsilon_{2j}, \text{ for each } P \in \mathcal{A} \text{ such that } \sigma(P) = 1, \\ P \leq -\varepsilon_{2j}, \text{ for each } P \in \mathcal{A} \text{ such that } \sigma(P) = -1. \end{array}$$

with the formula ϕ defining S. Let C_σ be the set defined by,

$$C_\sigma = \gamma_\sigma(\mathcal{R}(\sigma_-)) \cup \mathcal{R}(\sigma^+_-).$$

We now prove that

Proposition 4. C_σ is semi-algebraically contractible.

Let C be a closed and bounded semi-algebraic set contained in $\mathrm{R}\langle\varepsilon\rangle^k$. We can suppose without loss of generality that C is defined over $\mathrm{R}[\varepsilon]$ (see for example Proposition 2.80 in [6]). We denote by C_t the semi-algebraic subset of R^k defined by replacing ε by t in the definition of C. Note that C_ε is nothing but C.

We are going to use the following lemma.

Lemma 5.1. *Let B be a closed and bounded semi-algebraic set contained in R^k and let C be a closed and bounded semi-algebraic set contained in $\mathrm{R}\langle\varepsilon\rangle^k$. If there exists t_0 such that for every $t < t' < t_0$, $C_t \subset C_{t'}$ and $\lim_\varepsilon(C) = B$, then* $\mathrm{Ext}(B, R\langle\varepsilon\rangle)$ *has the same homotopy type as C.*

Proof. Hardt's Triviality theorem implies that there exists $t_0 > 0$, and a homeomorphism

$$\phi_{t_0} : C_{t_0} \times (0, t_0] \to \cup_{0<t\leq t_0} C_t$$

which preserves C_{t_0}. Replacing t_0 by ε gives a homeomorphism

$$\phi_\varepsilon : C \times (0, \varepsilon] \to \cup_{0<t\leq\varepsilon} C_t.$$

Defining

$$\psi : C \times [0, \varepsilon] \to C$$

by

$$\begin{cases} \psi(x, s) = \pi_{1\ldots k} \circ \phi(x, s), \text{ if } s > 0 \\ \psi(x, 0) = \lim_{s\to 0_+} \pi_{1\ldots k} \circ \phi(x, s), \end{cases}$$

it is clear that ψ is a semi-algebraic retraction of C to $\mathrm{Ext}(B, \mathrm{R}\langle\varepsilon\rangle)$. □

We now prove Proposition 4.

Proof. (of Proposition 4) Apply Lemma 5.1 to C_σ and $\mathrm{Ext}(\gamma_\sigma(\mathcal{R}(\sigma_-)), \mathrm{R}_{2j-1})$: thus C_σ can be semi-algebraically retracted to $\mathrm{Ext}(\gamma_\sigma(\mathcal{R}(\sigma_-)), \mathrm{R}_{2j-1})$.

Since $\mathrm{Ext}(\gamma_\sigma(\mathcal{R}(\sigma_-)), \mathrm{R}_{2j-1})$ is semi-algebraically contractible, so is C_σ. □

We now prove that the sets $\mathrm{Ext}(C_\sigma, \mathrm{R}')$ form a covering of $\mathrm{Ext}(S, \mathrm{R}')$.

Proposition 5. (Covering property)

$$\mathrm{Ext}(S,\mathrm{R}') = \bigcup_{\sigma\in\mathrm{Sign}(\mathcal{A},S)} \mathrm{Ext}(C_\sigma,\mathrm{R}').$$

The proposition is an immediate consequence of the following stronger result.

Proposition 6.

$$\mathrm{Ext}(S,\mathrm{R}') = \bigcup_{\sigma\in\mathrm{Sign}(\mathcal{A},S)} \mathcal{R}(\sigma_-^+,\mathrm{R}'^k).$$

Proof. By definition, $\mathrm{Ext}(S,\mathrm{R}') \supset \bigcup_{\sigma\in\mathrm{Sign}(\mathcal{A},S)} \mathcal{R}(\sigma_-^+,\mathrm{R}'^k)$. We now prove the reverse inclusion. Clearly, we have that $S = \bigcup_{\sigma\in\mathrm{Sign}(\mathcal{A},S)} \mathcal{R}(\sigma,\mathrm{R})$. Let $x \in \mathrm{Ext}(S,\mathrm{R}')$ and σ be the sign condition of the family $\mathcal{A}$ at x and let $\mathrm{level}(\sigma) = j$. If $x \in \mathcal{R}(\sigma_-^+,\mathrm{R}'^k)$, we are done. Otherwise, there exists $P \in \mathcal{A}$, such that x satisfies either $0 < P(x) < \varepsilon_{2j}$ or $-\varepsilon_{2j} < P(x) < 0$. Let $\mathcal{B} = \{P \in \mathcal{A} \mid \lim_{\varepsilon_{2j}} P(x) = 0\}$. Clearly $\#\mathcal{B} = j' > j$. Let $y = \lim_{\varepsilon_{2j}} x$. Since, $\mathrm{Ext}(S,\mathrm{R}')$ is closed and bounded and $x \in \mathrm{Ext}(S,\mathrm{R}')$, y is also in $\mathrm{Ext}(S,\mathrm{R}')$. Let τ be the sign condition of $\mathcal{A}$ at y with $\mathrm{level}(\tau) = j' > j$. If $x \in \mathcal{R}(\tau_-^+,\mathrm{R}'^k)$ we are done. Otherwise, for every $P \in \mathcal{A}$ such that $P(y) = 0$, we have that $-\varepsilon_{2j'-1} \le P(x) \le \varepsilon_{2j'-1}$, since $\lim_{\varepsilon_{2j}}(P(x)) = P(y) = 0$ and $\varepsilon_{2j'-1} \gg \varepsilon_{2j}$. So there exists $P \in \mathcal{A}$ such that x satisfies either $0 < P(x) < \varepsilon_{2j'}$ or $-\varepsilon_{2j'} < P(x) < 0$, and we replace $\mathcal{B}$ by $\{P \in \mathcal{A} \mid \lim_{\varepsilon_{2j'}} P(x) = 0\}$, and y by $y = \lim_{\varepsilon_{2j'}} x$. This process must terminate after at most t steps. □

Algorithm 2. (Covering by Contractible Sets)

Input.
- a finite set of s polynomials $\mathcal{P} \subset \mathrm{D}[X_1,\ldots,X_k]$ in strong k-general position on R^k, with $\deg(P_i) \le d$ for $1 \le i \le s$,
- a $\mathcal{P}$-closed semi-algebraic set S, contained in the sphere of center 0 and radius r, defined by a $\mathcal{P}$-closed formula ϕ.

Output. a set of formulas $\{\phi_1,\ldots,\phi_M\}$ such that
- each $\mathcal{R}(\phi_i,\mathrm{R}'^k)$ is semi-algebraically contractible, and
- $\bigcup_{1\le i\le M} \mathcal{R}(\phi_i,\mathrm{R}'^k) = \mathrm{Ext}(S,\mathrm{R}')$.

Procedure.

Step 1 Let $Q = X_1^2 + \ldots + X_k^2 - r^2$. Call Algorithm 1 (Parametrized Bounded Connecting) with input $Q,\mathcal{P}$. Let $\mathcal{A}$ be the family of polynomials output.

Step 2 Compute the set of realizable sign conditions $\mathrm{Sign}(\mathcal{A},S)$ using Algorithm 13.37 (Sampling on an Algebraic Set) in [6].

Step 3 Using Algorithm 14.21 (Quantifier Elimination) in [6], eliminate one variable to compute the image of the semi-algebraic map γ_{σ_-}. Finally, output the set of formulas $\{\phi_\sigma \mid \sigma \in \mathrm{Sign}(\mathcal{A},S)\}$ describing the semi-algebraic set C_σ.

Complexity. The complexity of the algorithm is bounded by $s^{(k+1)^2} d^{O(k^5)}$.

Proof of correctness. The correctness of the algorithm is a consequence of Propositions 4 and 5 and the correctness of Algorithm 1 (Parametrized Bounded Connecting), as well as the correctness of Algorithms 13.37 and 14.20 in [6]. □

Complexity analysis. The complexity of Step 1 of the algorithm is bounded by $s^{k+1}d^{O(k^4)}$, where s is a bound on the number of elements of $\mathcal{P}$ and d is a bound on the degrees of the elements of $\mathcal{P}$, using the complexity analysis of Algorithm 1 (Parametrized Bounded Connecting). The number of polynomials in $\mathcal{A}$ is $s^{k+1}d^{O(k^4)}$ and their degrees are bounded by $d^{O(k^3)}$. Thus the complexity of computing $\mathrm{Sign}(\mathcal{A}, S)$ is bounded by $s^{(k+1)^2}d^{O(k^5)}$ using Algorithm 13.37 (Sampling on an Algebraic Set) in [6]. In Step 3 of the algorithm there is a call to Algorithm 14.21 (Quantifier Elimination) in [6]. There are two blocks of variables of size k and 2 respectively. The number and degrees of the input polynomials are bounded by $s^{k+1}d^{O(k^4)}$ and $d^{O(k^3)}$ respectively. Moreover, observe that even though we introduced $2s$ infinitesimals, each arithmetic operation is performed in the ring D adjoined with at most $O(k)$ infinitesimals since the polynomials $\{P, P \pm \varepsilon_{2j}, P \pm \varepsilon_{2j-1}, P \in \mathcal{P}, 1 \leq j \leq s\}$ are in strong general position. Thus, the complexity of this step is bounded by $s^{(k+1)^2}d^{O(k^5)}$ using the complexity analysis of Algorithm 14.21 (Quantifier Elimination) in [6] and the fact that each arithmetic operation costs at most $d^{O(k^5)}$ in terms of arithmetic operations in the ring D. □

6. Topological Preliminaries

We first recall some results from algebraic topology which enable us to compute the first Betti number of any given closed and bounded semi-algebraic set, from the inclusion relationships amongst the connected components of the various pair-wise and triple-wise intersections of the elements of a covering of the given set by a finite number of contractible sets.

6.1. **Generalized Mayer-Vietoris exact sequence.** Let $A_1, \ldots, A_n$ be sub-complexes of a finite simplicial complex A such that $A = A_1 \cup \cdots \cup A_n$. Note that the intersections of any number of the sub-complexes, A_i, is again a sub-complex of A. We will denote by $A_{i_0,\ldots,i_p}$ the sub-complex $A_{i_0} \cap \cdots \cap A_{i_p}$.

Let $C^i(A)$ denote the $\mathbb{Q}$-vector space of i co-chains of A, and $C^\bullet(A)$, the complex

$$\cdots \to C^{q-1}(A) \xrightarrow{d} C^q(A) \xrightarrow{d} C^{q+1}(A) \to \cdots$$

where $d : C^q(A) \to C^{q+1}(A)$ are the usual co-boundary homomorphisms. More precisely, given $\omega \in C^q(A)$, and a $q+1$ simplex $[a_0, \ldots, a_{q+1}] \in A$,

$$d\omega([a_0, \ldots, a_{q+1}]) = \sum_{0 \leq i \leq q+1} (-1)^i \omega([a_0, \ldots, \hat{a}_i, \ldots, a_{q+1}]) \tag{6.1}$$

(here and everywhere else in the paper ˆ denotes omission). Now extend $d\omega$ to a linear form on all of $C_{q+1}(A)$ by linearity, to obtain an element of $C^{q+1}(A)$.

The generalized Mayer-Vietoris sequence is the following:

$$0 \longrightarrow C^\bullet(A) \xrightarrow{r} \prod_{i_0} C^\bullet(A_{i_0}) \xrightarrow{\delta_1} \prod_{i_0 < i_1} C^\bullet(A_{i_0,i_1})$$

$$\cdots \xrightarrow{\delta_{p-1}} \prod_{i_0 < \cdots < i_p} C^\bullet(A_{i_0,\ldots,i_p}) \xrightarrow{\delta_p} \prod_{i_0 < \cdots < i_{p+1}} C^\bullet(A_{i_0,\ldots,i_{p+1}}) \cdots$$

where r is induced by restriction and the connecting homomorphisms δ are described below.

Given an $\omega \in \prod_{i_0<\cdots<i_p} C^q(A_{i_0,\ldots,i_p})$ we define $\delta(\omega)$ as follows:
first note that $\delta(\omega) \in \prod_{i_0<\cdots<i_{p+1}} C^q(A_{i_0,\ldots,i_{p+1}})$, and it suffices to define $\delta(\omega)_{i_0,\ldots,i_{p+1}}$ for each $(p+2)$-tuple $0 \le i_0 < \cdots < i_{p+1} \le n$. Note that, $\delta(\omega)_{i_0,\ldots,i_{p+1}}$ is a linear form on the vector space, $C_q(A_{i_0,\ldots,i_{p+1}})$, and hence is determined by its values on the q-simplices in the complex $A_{i_0,\ldots,i_{p+1}}$. Furthermore, each q-simplex, $s \in A_{i_0,\ldots,i_{p+1}}$ is automatically a simplex of the complexes $A_{i_0,\ldots,\hat{i_i},\ldots i_{p+1}}$, $0 \le i \le p+1$.

We define,

$$(\delta\omega)_{i_0,\ldots,i_{p+1}}(s) = \sum_{0\le i\le p+1} (-1)^i \omega_{i_0,\ldots,\hat{i_i},\ldots,i_{p+1}}(s),$$

(here and everywhere else in the paper $\hat{}$ denotes omission). The fact that the generalized Mayer-Vietoris sequence is exact is classical (see [2] for example).

The cohomology groups $H^0(A_{i_0,\ldots,i_p})$ are isomorphic to the $\mathbb{Q}$-vector space of locally constant functions on $A_{i_0,\ldots,i_p}$ and the induced homomorphisms,

$$\delta_p : H^*(A_{i_0,\ldots,i_p}) \to H^*(A_{i_0,\ldots,i_{p+1}})$$

are then given by generalized restrictions, i.e. for

$$\phi \in \oplus_{1\le i_0<\cdots<i_p\le n} H^0(A_{i_0,\ldots,i_p}),$$

a locally constant function on $A_{i_0,\ldots,i_p}$,

$$\delta_p(\phi)_{i_0,\ldots,i_{p+1}} = \sum_{i=0}^{p} (-1)^i \phi_{i_0,\ldots,\hat{i_i},\ldots,i_{p+1}}|_{A_{i_0,\ldots,i_{p+1}}}.$$

The following proposition provides the key tool for computing the first Betti number.

Proposition 7. Let $A_1,\ldots,A_n$ be sub-complexes of a finite simplicial complex A such that $A = A_1 \cup \cdots \cup A_n$ and each A_i is acyclic, that is $H^0(A_i) = \mathbb{Q}$ and $H^q(A_i) = 0$ for all $q > 0$. Then, $b_1(A) = \dim(\mathrm{Ker}(\delta_2)) - \dim(\mathrm{Im}(\delta_1))$, with

$$\prod_i H^0(A_i) \xrightarrow{\delta_1} \prod_{i<j} H^0(A_{i,j}) \xrightarrow{\delta_2} \prod_{i<j<\ell} H^0(A_{i,j,\ell})$$

To prove Proposition 7, we consider the following bi-graded double complex $\mathcal{M}^{p,q}$, with total differential $D = \delta + (-1)^p d$, where

$$\mathcal{M}^{p,q} = \prod_{i_0,\ldots,i_p} C^q(A_{i_0,\ldots,i_p}).$$

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow d & & \uparrow d & & \uparrow d & & \\
0 & \longrightarrow & \prod_{i_0} C^3(A_{i_0}) & \xrightarrow{\delta} & \prod_{i_0<i_1} C^3(A_{i_0,i_1}) & \xrightarrow{\delta} & \prod_{i_0<i_1<i_2} C^3(A_{i_0,i_1,i_2}) & \longrightarrow & \\
 & & \uparrow d & & \uparrow d & & \uparrow d & & \\
0 & \longrightarrow & \prod_{i_0} C^2(A_{i_0}) & \xrightarrow{\delta} & \prod_{i_0<i_1} C^2(A_{i_0,i_1}) & \xrightarrow{\delta} & \prod_{i_0<i_1<i_2} C^2(A_{i_0,i_1,i_2}) & \longrightarrow & \\
 & & \uparrow d & & \uparrow d & & \uparrow d & & \\
0 & \longrightarrow & \prod_{i_0} C^1(A_{i_0}) & \xrightarrow{\delta} & \prod_{i_0<i_1} C^1(A_{i_0,i_1}) & \xrightarrow{\delta} & \prod_{i_0<i_1<i_2} C^1(A_{i_0,i_1,i_2}) & \longrightarrow & \\
 & & \uparrow d & & \uparrow d & & \uparrow d & & \\
0 & \longrightarrow & \prod_{i_0} C^0(A_{i_0}) & \xrightarrow{\delta} & \prod_{i_0<i_1} C^0(A_{i_0,i_1}) & \xrightarrow{\delta} & \prod_{i_0<i_1<i_2} C^0(A_{i_0,i_1,i_2}) & \longrightarrow & \\
 & & \uparrow d & & \uparrow d & & \uparrow d & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

There are two spectral sequences (corresponding to taking horizontal or vertical filtrations respectively) associated with $\mathcal{M}^{p,q}$ both converging to $H^*_D(\mathcal{M})$. The first terms of these are ${}'E_1 = H_\delta \mathcal{M}, {}'E_2 = H_d H_\delta \mathcal{M}$, and ${}''E_1 = H_d \mathcal{M}, {}''E_2 = H_\delta H_d \mathcal{M}$. Because of the exactness of the generalized Mayer-Vietoris sequence, we have that,

$$
{}'E_1 = \left|
\begin{array}{cccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \\
\uparrow d & \uparrow 0 & \uparrow 0 & \uparrow 0 & \uparrow 0 & \\
C^3(A) & 0 & 0 & 0 & 0 & \cdots \\
\uparrow d & \uparrow 0 & \uparrow 0 & \uparrow 0 & \uparrow 0 & \\
C^2(A) & 0 & 0 & 0 & 0 & \cdots \\
\uparrow d & \uparrow 0 & \uparrow 0 & \uparrow 0 & \uparrow 0 & \\
C^1(A) & 0 & 0 & 0 & 0 & \cdots \\
\uparrow d & \uparrow 0 & \uparrow 0 & \uparrow 0 & \uparrow 0 & \\
C^0(A) & 0 & 0 & 0 & 0 & \cdots \\
\hline
\end{array}
\right.
$$

and

$$
{}'E_2 = \left|
\begin{array}{ccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
H^3(A) & 0 & 0 & 0 & 0 & 0 & \cdots \\
H^2(A) & 0 & 0 & 0 & 0 & 0 & \cdots \\
H^1(A) & 0 & 0 & 0 & 0 & 0 & \cdots \\
H^0(A) & 0 & 0 & 0 & 0 & 0 & \cdots \\
\hline
\end{array}
\right.
$$

The degeneration of this sequence at E_2 shows that $H^*_D(\mathcal{M}) \cong H^*(A)$.

The initial term $''E_1$ of the second spectral sequence is given by,

$$''E_1 = \begin{array}{|ccccccc} \vdots & & \vdots & & \vdots & & \\ \prod_{i_0} H^3(A_{i_0}) & \xrightarrow{\delta} & \prod_{i_0<i_1} H^3(A_{i_0,i_1}) & \xrightarrow{\delta} & \prod_{i_0<i_1<i_2} H^3(A_{i_0,i_1,i_2}) & \longrightarrow \\ \prod_{i_0} H^2(A_{i_0}) & \xrightarrow{\delta} & \prod_{i_0<i_1} H^2(A_{i_0,i_1}) & \xrightarrow{\delta} & \prod_{i_0<i_1<i_2} H^2(A_{i_0,i_1,i_2}) & \longrightarrow \\ \prod_{i_0} H^1(A_{i_0}) & \xrightarrow{\delta} & \prod_{i_0<i_1} H^1(A_{i_0,i_1}) & \xrightarrow{\delta} & \prod_{i_0<i_1<i_2} H^1(A_{i_0,i_1,i_2}) & \longrightarrow \\ \prod_{i_0} H^0(A_{i_0}) & \xrightarrow{\delta} & \prod_{i_0<i_1} H^0(A_{i_0,i_1}) & \xrightarrow{\delta} & \prod_{i_0<i_1<i_2} H^0(A_{i_0,i_1,i_2}) & \longrightarrow \\ \hline \end{array}$$

The cohomology groups $H^0(A_{i_0,\ldots,i_p})$ occurring as summands in the bottom row of $''E_1$ are isomorphic to the $\mathbb{Q}$-vector space of locally constant functions on $A_{i_0,\ldots,i_p}$ and the homomorphisms, $''d_1 : ''E_1^{p,0} \to ''E_1^{p+1,0}$ are then given by generalized restrictions, i.e. for

$$\phi \in \oplus_{1\leq i_0<\cdots<i_p\leq n} H^0(A_{i_0,\ldots,i_p}),$$

with each $\phi_{i_0,\ldots,i_{p+1}}$ a locally constant function on $A_{i_0,\ldots,i_p}$,

$$''d_1(\phi)_{i_0,\ldots,i_{p+1}} = \sum_{i=0}^{p}(-1)^i \phi_{i_0,\ldots,\hat{i_i},\ldots,i_{p+1}}|_{A_{i_0,\ldots,i_{p+1}}}.$$

Proof. (Proof of Proposition 7) Since, $H^q(A_i) = 0$ for all $q > 0$, all the terms in the first column of $''E_1$ are zero except the bottom term, and clearly $''d_2^{0,1} = 0$. Thus, $''E_\infty^{1,0} = ''E_2^{1,0}$ and $''E_\infty^{0,1} = 0$. Thus, $H^1(A) \cong ''E_\infty^{1,0} \oplus ''E_\infty^{0,1} \cong ''E_2^{1,0}$. □

7. Computing the first Betti number in the $\mathcal{P}$-closed case

Let S be a $\mathcal{P}$-closed semi-algebraic set. We first replace S by a $\mathcal{P}^\star$-closed and bounded semi-algebraic set, where the elements of $\mathcal{P}^\star$ are slight modifications of the elements of $\mathcal{P}$, and the family $\mathcal{P}^\star$ is in general position and $b_i(S^\star) = b_i(S), 0 \leq i \leq k$.

Define

$$H_i = 1 + \sum_{1\leq j\leq k} i^j X_j^{d'}.$$

where d' is the smallest number strictly bigger than the degree of all the polynomials in $\mathcal{P}$. Using arguments similar to the proof of Proposition 13.7 in [6], it is easy to see that the family $\mathcal{P}^\star$ of polynomials $P_i - \delta H_i, P_i + \delta H_i$, with $P_i \in \mathcal{P}$. is in general position in $\mathrm{R}\langle\delta\rangle^k$.

Lemma 7.1. *Denote by $S^\star$ the set obtained by replacing any $P_i \geq 0$ in the definition of S by $P_i \geq -\delta H_i$ and every $P_i \leq 0$ in the definition of S by $P_i \leq \delta H_i$. If S is bounded, the set* $\mathrm{Ext}(S, \mathrm{R}\langle\delta\rangle^k$ *is semi-algebraically homotopy equivalent to $S^\star$.*

Proof. Note that S is closed and bounded, $\lim_\delta S^\star = S$,and $S_t \subset S_{t'}$. The claim follows by Lemma 5.1. □

Algorithm 3. (First Betti Number of a $\mathcal{P}$-closed Semi-algebraic Set)

Input.

– a finite set of polynomials $\mathcal{P} \subset \mathrm{D}[X_1, \ldots, X_k]$,
– a formula defining a $\mathcal{P}$-closed semi-algebraic set, S.

OUTPUT. $b_1(S)$.

PROCEDURE.

Step 1 Let ε be an infinitesimal. Replace S by the semi-algebraic set T defined as the intersection of the cylinder $S \times \mathrm{R}\langle\varepsilon\rangle$ with the upper hemisphere defined by $\varepsilon^2(X_1^2 + \ldots + X_k^2 + X_{k+1}^2) = 1, X_{k+1} \geq 0$.

Step 2 Replace T by $T^\star$ using the notation of Lemma 7.1.

Step 3 Use Algorithm 2 (Covering by Contractible Sets) with input $\varepsilon^2(X_1^2 + \ldots + X_k^2 + X_{k+1}^2) - 4$ and $\mathcal{P}^\star$, to compute a covering of $T^\star$ by closed, bounded and contractible sets, T_i, described by formulas ϕ_i.

Step 4 Use Algorithm 16.27 (General Roadmap) in [6] to compute exactly one sample point of each connected component of the pairwise and triplewise intersections of the T_i's. For every pair i, j and every k compute the incidence relation between the connected components of $T^\star_{ijk}$ and $T^\star_{ij}$ as follows: compute a roadmap of $T^\star_{ij}$, containing the sample points of the connected components of $T^\star_{ijk}$ using Algorithm 16.27 (General Roadmap).

Step 5 Using linear algebra compute $b_1(T^\star) = \dim(\mathrm{Ker}(\delta_2)) - \dim(\mathrm{Im}(\delta_1))$, with

$$\prod_i H^0(T^\star_i) \xrightarrow{\delta_1} \prod_{i<j} H^0(T^\star_{ij}) \xrightarrow{\delta_2} \prod_{i<j<\ell} H^0(T^\star_{ij\ell})$$

COMPLEXITY. The complexity of the algorithm is bounded by $(sd)^{k^{O(1)}}$, where $s = \#\mathcal{P}$ and $d = \max_{P \in \mathcal{P}} \deg(P)$.

PROOF OF CORRECTNESS. First note that T is closed and bounded and has the same Betti numbers as S, using the local conical structure at infinity. It follows from Lemma 7.1 that T and $T^\star$ have the same Betti numbers. The correctness of the algorithm is a consequence of the correctness of Algorithm 2 (Covering by Contractible Sets), Algorithm 16.27 (General Roadmap) in [6], and Proposition 7. □

COMPLEXITY ANALYSIS. The complexity of Step 3 of the algorithm is bounded by $s^{(k+1)^2} d^{O(k^6)}$ using the complexity analysis of Algorithm 2 (Covering by Contractible Sets) and noticing that each arithmetic operation takes place a ring consisting of D adjoined with at most k infinitesimals. Finally, the complexity of Step 4 is also bounded by $(sd)^{k^{O(1)}}$, using the complexity analysis of Algorithm 16.27 (General Roadmap) in [6]. □

8. Replacement by closed sets without changing homology

In this section, we describe a construction due to Gabrielov and Vorobjov [12] for replacing any given semi-algebraic subset of a bounded semi-algebraic set by a closed bounded semi-algebraic subset and strengthen the result in [12] to prove that the new set has the same homotopy type as the original one. Moreover, the polynomials defining the bounded closed semi-algebraic subset are closely related (by infinitesimal perturbations) to the polynomials defining the original subset. In particular, their degrees do not increase, while the number of polynomials used in the definition of the new set is at most twice the square of the number used in the definition of the original set.

Let $\mathcal{C} \subset \mathrm{R}[X_1, \ldots, X_k]$ be a finite set of polynomials with t elements, and let S be a bounded $\mathcal{C}$-closed set. We denote by $\mathrm{Sign}(\mathcal{C}, S)$ the set of realizable sign conditions of $\mathcal{C}$ whose realizations are contained in S.

Recall that, for $\sigma \in \mathrm{Sign}(\mathcal{C})$ we define the level of σ as $\#\{P \in \mathcal{C} | \sigma(P) = 0\}$. As before let, $\varepsilon_{2t} \gg \varepsilon_{2t-1} \gg \cdots \gg \varepsilon_2 \gg \varepsilon_1 > 0$ be infinitesimals, and we will denote by R_i the field $\mathrm{R}\langle\varepsilon_{2t}\rangle \cdots \langle\varepsilon_i\rangle$ and denote by R' the field R_1. For $i > 2t$, $\mathrm{R}_i = \mathrm{R}$ and for $i < 0, \mathrm{R}_i = \mathrm{R}'$.

We now describe the construction due to Gabrielov and Vorobjov. For each level $m, 0 \leq m \leq t$, we denote by $\mathrm{Sign}_m(\mathcal{C}, S)$ the subset of $\mathrm{Sign}(\mathcal{C}, S)$ of elements of level m.

Given $\sigma \in \mathrm{Sign}_m(\mathcal{C}, S)$ denote by $\mathcal{R}(\sigma_+^c)$ the intersection of $\mathrm{Ext}(S, \mathrm{R}_{2m})$ with the closed semi-algebraic set defined by the conjunction of the inequalities,

$$\begin{cases} -\varepsilon_{2m} \leq P \leq \varepsilon_{2m} \text{ for each } P \in \mathcal{C} \text{ such that } \sigma(P) = 0, \\ P \geq 0, \text{ for each } P \in \mathcal{C} \text{ such that } \sigma(P) = 1, \\ P \leq 0, \text{ for each } P \in \mathcal{C} \text{ such that } \sigma(P) = -1. \end{cases}$$

and denote by, $\mathcal{R}(\sigma_+^o)$ the intersection of $\mathrm{Ext}(S, \mathrm{R}_{2m-1})$ with the open semi-algebraic set defined by the conjunction of the inequalities,

$$\begin{cases} -\varepsilon_{2m-1} < P < \varepsilon_{2m-1} \text{ for each } P \in \mathcal{C} \text{ such that } \sigma(P) = 0, \\ P > 0, \text{ for each } P \in \mathcal{C} \text{ such that } \sigma(P) = 1, \\ P < 0, \text{ for each } P \in \mathcal{C} \text{ such that } \sigma(P) = -1. \end{cases}$$

Notice that,

$$\mathrm{Ext}(\mathcal{R}(\sigma), \mathrm{R}_{2m}) \subset \mathcal{R}(\sigma_+^c),$$
$$\mathrm{Ext}(\mathcal{R}(\sigma), \mathrm{R}_{2m-1}) \subset \mathcal{R}(\sigma_+^o).$$

Let $X \subset S$ be a $\mathcal{C}$-semi-algebraic set such that $X = \bigcup_{\sigma \in \Sigma} \mathcal{R}(\sigma)$ with $\Sigma \subset \mathrm{Sign}(\mathcal{C}, S)$. We denote $\Sigma_m = \Sigma \cap \mathrm{Sign}_m(\mathcal{C}, S)$ and define a sequence of sets, $X^m \subset \mathrm{R}'^k$, $0 \leq m \leq t$ inductively.

- Define $X^0 = \mathrm{Ext}(X, \mathrm{R}')$.
- For $0 \leq m \leq t$, we define

$$X^{m+1} = \left(X^m \cup \bigcup_{\sigma \in \Sigma_m} \mathrm{Ext}(\mathcal{R}(\sigma_+^c), \mathrm{R}') \right) \setminus \bigcup_{\sigma \in \mathrm{Sign}_m(\mathcal{C}, S) \setminus \Sigma_m} \mathrm{Ext}(\mathcal{R}(\sigma_+^o), \mathrm{R}')$$

We denote by X' the set X^{t+1}.

The following theorem is a slight strengthening of a result in [12] (where it is shown that the sum of the Betti numbers of X and X' are equal) and the proof is very close in spirit to the one in [12].

Theorem 8.1. *The sets* $\mathrm{Ext}(X, R')$ *and* X' *are semi-algebraically homotopy equivalent. In particular,*

$$H_*(X) \cong H_*(X').$$

For the purpose of the proof we introduce several new families of sets defined inductively.

For each p, $0 \leq p \leq t+1$ we define sets, $Y_p \subset \mathrm{R}_{2p}^k, Z_p \subset \mathrm{R}_{2p-1}^k$ as follows.

- We define

$$Y_p^p = \mathrm{Ext}(X, \mathrm{R}_{2p}) \cup \bigcup_{\sigma \in \Sigma_p} \mathcal{R}(\sigma_+^c),$$

$$Z_p^p = \mathrm{Ext}(Y_p^p, \mathrm{R}_{2p-1}) \setminus \bigcup_{\sigma \in \mathrm{Sign}_p(\mathcal{C},S)\setminus \Sigma_p} \mathcal{R}(\sigma_+^o).$$

- For $p \le m \le t$, we define

$$Y_p^{m+1} = \left(Y_p^m \cup \bigcup_{\sigma \in \Sigma_m} \mathrm{Ext}(\mathcal{R}(\sigma_+^c), \mathrm{R}_{2p}) \right) \setminus \bigcup_{\sigma \in \mathrm{Sign}_m(\mathcal{C},S)\setminus \Sigma_m} \mathrm{Ext}(\mathcal{R}(\sigma_+^o), \mathrm{R}_{2p})$$

$$Z_p^{m+1} = \left(Z_p^m \cup \bigcup_{\sigma \in \Sigma_m} \mathrm{Ext}(\mathcal{R}(\sigma_+^c), \mathrm{R}_{2p-1}) \right) \setminus \bigcup_{\sigma \in \mathrm{Sign}_m(\mathcal{C},S)\setminus \Sigma_m} \mathrm{Ext}(\mathcal{R}(\sigma_+^o), \mathrm{R}_{2p-1}).$$

We denote by $Y_p \subset \mathrm{R}_{2p}^k$ (respectively, $Z_p \subset \mathrm{R}_{2p-1}^k$) the set Y_p^{t+1} (respectively, Z_p^{t+1}).

Note that

- $X = Y_{t+1} = Z_{t+1}$, and
- $Z_0 = X'$.

Notice also that for each $p, 0 \le p \le t$,

(1) $\mathrm{Ext}(Z_{p+1}^{p+1}, \mathrm{R}_{2p}) \subset Y_p^p$,
(2) $Z_p^p \subset \mathrm{Ext}(Y_p^p, \mathrm{R}_{2p-1})$.

The following inclusions now follow directly from the definitions of Y_p and Z_p.

Lemma 8.2. *For each* $p, 0 \le p \le t$,

(1) $\mathrm{Ext}(Z_{p+1}, \mathrm{R}_{2p}) \subset Y_p$,
(2) $Z_p \subset \mathrm{Ext}(Y_p, \mathrm{R}_{2p-1})$.

We now prove that in both the inclusions in Lemma 8.2 above, the pairs of sets are in fact semi-algebraically homotopy equivalent. These suffice to prove Theorem 8.1.

Lemma 8.3. *For* $1 \le p \le t$, Y_p *is semi-algebraically homotopy equivalent to* $\mathrm{Ext}(Z_{p+1}, \mathrm{R}_{2p})$.

Proof. Let $Y_p(u) \subset \mathrm{R}_{2p+1}^k$ denote set obtained by replacing the infinitesimal ε_{2p} in the definition of Y_p by u, and for $u_0 > 0$, we will denote by Let $Y_p((0, u_0]) \subset \mathrm{R}_{2p+1}^{k+1}$ the set $\{(x, u) | x \in Y_p(u), u \in (0, u_0]\}$.

By Hardt's triviality theorem there exist $u_0 \in \mathrm{R}_{2p+1},\ u_0 > 0$ and a homeomorphism,

$$\psi : Y_p(u_0) \times (0, u_0] \to Y_p((0, u_0]),$$

such that

(1) $\pi_{k+1}(\phi(x, u)) = u$,
(2) $\psi(x, u_0) = (x, u_0)$ for $x \in Y_p(u_0)$, and
(3) for all $u \in (0, u_0]$, and for every sign condition σ of the family, $\cup_{P \in \mathcal{C}} \{P, P \pm \varepsilon_{2t}, \ldots, P \pm \varepsilon_{2p+1}\}$, $\psi(\cdot, u)$ restricts to a homeomorphism of $\mathcal{R}(\sigma, Y_p(u_0))$ to $\mathcal{R}(\sigma, Y_p(u))$.

Now, we specialize u_0 to ε_{2p} and denote the map corresponding to ψ by ϕ.

For $\sigma \in \Sigma_p$, we define, $\mathcal{R}(\sigma^o_{++})$ to be the set defined by,

$$\begin{cases} -2\varepsilon_{2p} < P < 2\varepsilon_{2p}, \text{ for each } P \in \mathcal{C} \text{ such that } \sigma(P) = 0, \\ P > -\varepsilon_{2p}, \text{ for each } P \in \mathcal{C} \text{ such that } \sigma(P) = 1, \\ P < \varepsilon_{2p}, \text{ for each } P \in \mathcal{C} \text{ such that } \sigma(P) = -1. \end{cases}$$

Let $\lambda : Y_p \to \mathrm{R}_{2p}$ be a semi-algebraic continuous function such that,

$$\begin{cases} \lambda(x) = 1, \text{ on } Y_p \cap \cup_{\sigma \in \Sigma_p} \mathcal{R}(\sigma^c_+), \\ \lambda(x) = 0, \text{ on } Y_p \setminus \cup_{\sigma \in \Sigma_p} \mathcal{R}(\sigma^o_{++}), \\ 0 < \lambda(x) < 1, \text{ else.} \end{cases}$$

We now construct a semi-algebraic homotopy,

$$h : Y_p \times [0, \varepsilon_{2p}] \to Y_p,$$

by defining,

$$\begin{aligned} h(x,t) &= \pi_{1\ldots k} \circ \phi(x, \lambda(x)t + (1-\lambda(x))\varepsilon_{2p}), \\ &\quad \text{for } 0 < t \le \varepsilon_{2p}, \\ h(x,0) &= \lim_{t \to 0+} h(x,t), \text{ else.} \end{aligned}$$

Note that the last limit exists since S is closed and bounded. We now show that, $h(x,0) \in \mathrm{Ext}(Z_{p+1}, \mathrm{R}_{2p})$ for all $x \in Y_p$.

Let $x \in Y_p$ and $y = h(x,0)$.

There are two cases to consider.

$\lambda(x) < 1$: In this case, $x \in \mathrm{Ext}(Z_{p+1}, \mathrm{R}_{2p})$ and by property (3) of ϕ and the fact that $\lambda(x) < 1$, $y \in \mathrm{Ext}(Z_{p+1}, \mathrm{R}_{2p})$.

$\lambda(x) = 1$: Let σ_y be the sign condition of $\mathcal{C}$ at y and suppose that $y \notin \mathrm{Ext}(Z_{p+1}, \mathrm{R}_{2p})$. There are two cases to consider.

- $\sigma_y \in \Sigma$: In this case, $y \in X$ and hence there must exist $\tau \in \mathrm{Sign}_m(\mathcal{C}, S) \setminus \Sigma_m$, with $m > p$ such that $y \in \mathcal{R}(\tau^o_+)$.
- $\sigma_y \notin \Sigma$: In this case, taking $\tau = \sigma_y$, $\mathrm{level}(\tau) > p$ and $y \in \mathcal{R}(\tau^o_+)$.

It follows from the definition of y, and property (3) of ϕ, that for any $m > p$, and every $\rho \in \mathrm{Sign}_m(\mathcal{C}, S)$,
- $y \in \mathcal{R}(\rho^o_+)$ implies that $x \in \mathcal{R}(\rho^o_+)$, and
- $x \in \mathcal{R}(\rho^c_+)$ implies that $y \in \mathcal{R}(\rho^c_+)$.

Thus, $x \notin Y_p$ which is a contradiction.

It follows that,

(1) $h(\cdot, \varepsilon_{2p}) : Y_p \to Y_p$ is the identity map,
(2) $h(Y_p, 0) = \mathrm{Ext}(Z_{p+1}, \mathrm{R}_{2p})$, and
(3) $h(\cdot, t)$ restricted to $\mathrm{Ext}(Z_{p+1}, \mathrm{R}_{2p})$ gives a semi-algebraic homotopy between $h(\cdot, \varepsilon_{2p})|_{\mathrm{Ext}(Z_{p+1}, \mathrm{R}_{2p})} = \mathrm{id}_{\mathrm{Ext}(Z_{p+1}, \mathrm{R}_{2p})}$ and $h(\cdot, 0)|_{\mathrm{Ext}(Z_{p+1}, \mathrm{R}_{2p})}$.

Thus, Y_p is semi-algebraically homotopy equivalent to $\mathrm{Ext}(Z_{p+1}, \mathrm{R}_{2p})$. □

Lemma 8.4. *For each* $p, 0 \le p \le t$, Z_p *is semi-algebraically homotopy equivalent to* $\mathrm{Ext}(Y_p, \mathrm{R}_{2p-1})$.

Proof. For the purpose of the proof we define the following new sets for $u \in \mathrm{R}_{2p}$.

(1) Let $Z'_p(u) \subset \mathrm{R}_{2p}^k$ be the set obtained by replacing in the definition of Z_p, ε_{2j} by $\varepsilon_{2j} - u$ and ε_{2j-1} by $\varepsilon_{2j-1} + u$ for all $j > p$, and ε_{2p} by $\varepsilon_{2p} - u$, and ε_{2p-1} by u. For $u_0 > 0$ we will denote by $Z'_p((0, u_0])$ the set $\{(x, u) \mid x \in Z'_p(u), u \in (0, u_0]\}$.
(2) Let $Y'_p(u) \subset \mathrm{R}_{2p}^k$ be the set obtained by replacing in the definition of Y_p, ε_{2j} by $\varepsilon_{2j} - u$ and ε_{2j-1} by $\varepsilon_{2j-1} + u$ for all $j > p$ and ε_{2p} by by $\varepsilon_{2p} - u$.
(3) For $\sigma \in \mathrm{SIGN}_m(\mathcal{C}, S)$, with $m \geq p$, let $\mathcal{R}(\sigma_+^c)(u) \subset \mathrm{R}_{2p}^k$ denote the set obtained by replacing ε_{2m} by $\varepsilon_{2m} - u$ in the definition of $\mathcal{R}(\sigma_+^c)$.
(4) For $\sigma \in \mathrm{SIGN}_m(\mathcal{C}, S)$, with $m > p$, let $\mathcal{R}(\sigma_+^o)(u) \subset \mathrm{R}_{2p}^k$ denote the set obtained by replacing ε_{2m-1} by $\varepsilon_{2m-1} + u$ in the definition of $\mathcal{R}(\sigma_+^o)$.
(5) Finally, for $\sigma \in \mathrm{SIGN}_p(\mathcal{C}, S)$ let $\mathcal{R}(\sigma_+^o)(u) \subset \mathrm{R}_{2p-1}^k$ denote the set obtained by replacing in the definition of $\mathcal{R}(\sigma_o^c)$, ε_{2p-1} by u.

Notice that by definition, for any $u, v \in \mathrm{R}_{2p}$ with $0 < u \leq v$, $Z'_p(u) \subset Y'_p(u)$, $Z'_p(v) \subset Z'_p(u)$, $Y'_p(v) \subset Y'_p(u)$, and

$$\bigcup_{0<s\leq u} Y'_p(s) = \bigcup_{0<s\leq u} Z'_p(s).$$

We denote by Z'_p (respectively, Y'_p) the set $Z'_p(\varepsilon_{2p-1})$ (respectively, $Y'_p(\varepsilon_{2p-1})$). It is easy to see that Y'_p is semi-algebraically homotopy equivalent to $\mathrm{Ext}(Y_p, \mathrm{R}_{2p-1})$, and Z'_p is semi-algebraically homotopy equivalent to Z_p. We now prove that, Y'_p is semi-algebraically homotopy equivalent to Z'_p, which suffices to prove the lemma.

Let $\mu : Y'_p \to \mathrm{R}_{2p-1}$ be the semi-algebraic map defined by

$$\mu(x) = \sup_{u \in (0, \varepsilon_{2p-1}]} \{u \mid x \in Z'_p(u)\}.$$

We prove separately (Lemma 8.5 below) that μ is continuous. Note that the definition of the set $Z'_p(u)$ (as well as the set $Y'_p(u)$) is more complicated than the more natural one consisting of just replacing ε_{2p-1} in the definition of Z_p by u, is due to the fact that with the latter definition the map μ defined below is not necessarily continuous.

We now construct a continuous semi-algebraic map,

$$h : Y'_p \times [0, \varepsilon_{2p-1}] \to Y'_p$$

as follows.

By Hardt's triviality theorem there exist $u_0 \in \mathrm{R}_{2p}$, with $u_0 > 0$ and a semi-algebraic homeomorphism,

$$\psi : Z'_p(u_0) \times (0, u_0] \to Z'_p((0, u_0]),$$

such that

(1) $\pi_{k+1}(\psi(x, u)) = u$,
(2) $\psi(x, u_0) = (x, u_0)$ for $x \in Z'_p(u_0)$, and
(3) $\psi(\cdot, u)$ restricts to a homeomorphism of $\mathcal{R}(\sigma, Z'_p(u_0))$ to $\mathcal{R}(\sigma, Z'_p(u))$, for every sign condition σ of the family, $\cup_{P \in \mathcal{C}} \{P, P \pm \varepsilon_{2t}, \ldots, P \pm \varepsilon_{2p+1}\}$, for all $u \in (0, u_0]$.

We now specialize u_0 to ε_{2p-1} and denote by ϕ the corresponding map,

$$\phi : Z'_p \times (0, \varepsilon_{2p-1}] \to Z'_p((0, \varepsilon_{2p-1}]).$$

Note, that for every u, $0 < u \leq \varepsilon_{2p-1}$, ϕ gives a homeomorphism, $\phi_u : Z'_p(u) \to Z'_p$. Hence, for every pair, u, u', $0 < u \leq u' \leq \varepsilon_{2p-1}$, we have a homeomorphism, $\theta_{u,u'} : Z'_p(u) \to \mathbb{Z}'_p(u')$ obtained by composing ϕ_u with $\phi_{u'}^{-1}$. For $0 \leq u' < u \leq \varepsilon_{2p-1}$, we let $\theta_{u,u'}$ be the identity map. It is clear that $\theta_{u,u'}$ varies continuously with u and u'.

For $x \in Y'_p, t \in [0, \varepsilon_{2p-1}]$ we now define,

$$h(x,t) = \theta_{\mu(x),t}(x).$$

It is easy to verify from the definition of h and the properties of ϕ listed above that, h is continuous and satisfies the following.

(1) $h(\cdot, 0) : Y'_p \to Y'_p$ is the identity map,
(2) $h(Y'_p, \varepsilon_{2p-1}) = Z'_p$, and
(3) $h(\cdot, t)$ restricts to a homeomorphism $Z'_p \times t \to Z'_p$ for every $t \in [0, \varepsilon_{2p-1}]$.

This proves the required homotopy equivalence. □

We now prove that the function μ used in the proof above is continuous.

Lemma 8.5. *The semi-algebraic map* $\mu : Y'_p \to \mathrm{R}_{2p-1}$ *defined by*

$$\mu(x) = \sup_{u \in (0, \varepsilon_{2p-1}]} \{u \mid x \in Z'_p(u)\}$$

is continuous.

Proof. Let $0 < \delta \ll \varepsilon_{2p-1}$ be a new infinitesimal. In order to prove the continuity of μ (which is a semi-algebraic function defined over R_{2p-1}), it suffices to show that

$$\lim_\delta \mathrm{Ext}(\mu, \mathrm{R}_{2p-1}\langle\delta\rangle)(x') = \lim_\delta \mathrm{Ext}(\mu, \mathrm{R}_{2p-1}\langle\delta\rangle)(x)$$

for every pair of points $x, x' \in \mathrm{Ext}(Y'_p, \mathrm{R}_{2p-1}\langle\delta\rangle)$ such that $\lim_\delta x = \lim_\delta x'$.

Consider such a pair of points $x, x' \in \mathrm{Ext}(Y'_p, \mathrm{R}_{2p-1}\langle\delta\rangle)$. Let $u \in (0, \varepsilon_{2p-1}]$ be such that $x \in Z'_p(u)$. We show below that this implies $x' \in Z'_p(u')$ for some u' satisfying $\lim_\delta u' = \lim_\delta u$.

Let m be the largest integer such that there exists $\sigma \in \Sigma_m$ with $x \in \mathcal{R}(\sigma_+^c)(u)$. Since $x \in Z'_p(u)$ such an m must exist.

We have two cases:

(1) $m > p$: Let $\sigma \in \Sigma_m$ with $x \in \mathcal{R}(\sigma_+^c)(u)$. Then, by the maximality of m, we have that for each $P \in \mathcal{C}$, $\sigma(P) \neq 0$ implies that $\lim_\delta P(x) \neq 0$. As a result, we have that $x' \in \mathcal{R}(\sigma_+^c)(u')$ for all $u' < u - \max_{P \in \mathcal{P}, \sigma(P)=0} |P(x) - P(x')|$, and hence we can choose u' such that $x' \in \mathcal{R}(\sigma_+^c)(u')$ and $\lim_\delta u' = \lim_\delta u$.
(2) $m \leq p$: If $x' \notin Z'_p(u)$ then since $x' \in Y'_p \subset Y'_p(u)$,

$$x' \in \cup_{\sigma \in \mathrm{Sign}_p(\mathcal{C}, S) \setminus \Sigma_p} \mathcal{R}(\sigma_+^o)(u).$$

Let $\sigma \in \mathrm{Sign}_p(\mathcal{C}, S) \setminus \Sigma_p$ be such that $x' \in \mathcal{R}(\sigma_+^o)(u)$. We prove by contradiction that $\lim_\delta \max_{P \in \mathcal{P}, \sigma(P)=0} |P(x')| = u$.

Assume that

$$\lim_\delta \max_{P \in \mathcal{P}, \sigma(P)=0} |P(x')| \neq u.$$

Since, $x \notin \mathcal{R}(\sigma_+^o)(u)$ by assumption, and $\lim_\delta x' = \lim_\delta x$, there must exist $P \in \mathcal{C}$, $\sigma(P) \neq 0$, and $\lim_\delta P(x) = 0$. Letting τ denote the sign condition defined by $\tau(P) = 0$ if $\lim_\delta P(x) = 0$ and $\tau(P) = \sigma(P)$ else, we have that $\mathrm{level}(\tau) > p$ and x belongs to both $\mathcal{R}(\tau_+^o)(u)$ as well as $\mathcal{R}(\tau_+^c)(u)$.

Now there are two cases to consider depending on whether τ is in Σ or not. If $\tau \in \Sigma$, then the fact that $x \in \mathcal{R}(\tau_+^c)(u)$ contradicts the choice of m, since $m \leq p$ and $\mathrm{level}(\tau) > p$. If $\tau \notin \Sigma$ then x gets removed at the level of τ in the construction of $Z_p'(u)$, and hence $x \in \mathcal{R}(\rho_+^c)(u)$ for some $\rho \in \Sigma$ with $\mathrm{level}(\rho) > \mathrm{level}(\tau) > p$. This again contradicts the choice of m. Thus, $\lim_\delta \max_{P\in\mathcal{P},\sigma(P)=0} |P(x')| = u$ and since $x' \notin \cup_{\sigma\in\mathrm{Sign}_p(\mathcal{C},S)\setminus\Sigma_p}\mathcal{R}(\sigma_+^o)(u')$ for all $u' < \max_{P\in\mathcal{P},\sigma(P)=0} |P(x')|$, we can choose u' such that $\lim_\delta u' = \lim_\delta u$, and $x' \notin \cup_{\sigma\in\mathrm{Sign}_p(\mathcal{C},S)\setminus\Sigma_p}\mathcal{R}(\sigma_+^o)(u')$.

In both cases we have that $x' \in Z_p'(u')$ for some u' satisfying $\lim_\delta u' = \lim_\delta u$, showing that $\lim_\delta \mu(x') \geq \lim_\delta \mu(x)$. The reverse inequality follows by exchanging the roles of x and x' in the previous argument. Hence, $\lim_\delta \mu(x') = \lim_\delta \mu(x)$, proving the continuity of μ. □

Proof. (Proof of Theorem 8.1) The theorem follows immediately from Lemmas 8.3 and 8.4. □

9. Computing the first Betti number of a general semi-algebraic set

In this section we describe the algorithm for computing the first Betti number of a general semi-algebraic set. We first replace the given set by a closed and bounded one, using the construction described in the previous section. We then apply Algorithm 3.

Algorithm 4. (First Betti Number of a $\mathcal{P}$- Semi-algebraic Set)

Input.
- a finite set of polynomials $\mathcal{P} \subset \mathrm{D}[X_1, \ldots, X_k]$,
- a formula defining a $\mathcal{P}$-semi-algebraic set, S.

Output. $b_1(T)$.

Procedure.

Step 1 Let ε be an infinitesimal. Define $\tilde{S}$ as the intersection of $\mathrm{Ext}(S, \langle\varepsilon\rangle)$ with the ball of center 0 and radius $1/\varepsilon$. Define $\mathcal{Q}$ as $\mathcal{P} \cup \{\varepsilon^2(X_1^2 + \ldots + X_k^2 + X_{k+1}^2) - 4, X_{k+1}\}$ Replace $\tilde{S}$ by the $\mathcal{Q}$- semi-algebraic set S defined as the intersection of the cylinder $\tilde{S} \times \mathrm{R}\langle\varepsilon\rangle$ with the upper hemisphere defined by $\varepsilon^2(X_1^2 + \ldots + X_k^2 + X_{k+1}^2) = 4, X_{k+1} \geq 0$.

Step 2 Using the Gabrielov-Vorobjov construction described above, replace T by a $\mathcal{Q}'$-closed set, T'.

Step 3 Use Algorithm 3 to compute the first Betti number of T'.

Complexity. The complexity of the algorithm is bounded by $(sd)^{k^{O(1)}}$, where $s = \#\mathcal{P}$ and $d = \max_{P\in\mathcal{P}} \deg(P)$.

Proof of correctness. The correctness of the algorithm is a consequence of Theorem 8.1 and the correctness of Algorithm 3. □

Complexity analysis. In Step 2 of the algorithm the cardinality of $\mathcal{Q}'$ is $2(s+1)^2$ and the degrees of the polynomials in $\mathcal{Q}'$ are still bounded by d. The complexity of

Step 3 of the algorithm is then bounded by $(sd)^{k^{O(1)}}$ using the complexity analysis of Algorithm 3. □

10. Computing Connected Components

If one is interested in computing semi-algebraic descriptions of the connected components of a given semi-algebraic set, then using Algorithm 1 (Parametrized Bounded Connecting) it is possible to do so with a complexity making precise the one of previously known algorithms, whose complexities were of the form $(sd)^{k^{O(1)}}$ (see [16]). We have the following theorems (we refer the reader to [6] for details of the proof).

Theorem 10.1. *If $\mathrm{Z}(Q, \mathrm{R}^k)$ is an algebraic set defined as the zero set of a polynomial $Q \in \mathrm{D}[X_1, \ldots, X_k]$ of degree $\leq d$, then there is an algorithm that outputs quantifier free formulas whose realizations are the semi-algebraically connected components of $\mathrm{Z}(Q, \mathrm{R}^k)$. The complexity of the algorithm in the ring generated by the coefficients of Q is bounded by $d^{O(k^3)}$ and the degrees of the polynomials that appear in the output are bounded by $O(d)^{k^2}$. Moreover, if $\mathrm{D} = \mathbb{Z}$, and the bitsizes of the coefficients of the polynomials are bounded by τ, then the bitsizes of the integers appearing in the intermediate computations and the output are bounded by $\tau d^{O(k^2)}$.*

Theorem 10.2. *Let $\mathcal{P} = \{P_1, \ldots, P_s\} \subset \mathrm{D}[X_1, \ldots, X_k]$ with $\deg(P_i) \leq d, 1 \leq i \leq s$ and a semi-algebraic set S defined by a $\mathcal{P}$ quantifier-free formula. There exists an algorithm that outputs quantifier-free semi-algebraic descriptions of all the semi-algebraically connected components of S. The complexity of the algorithm is bounded by $s^{k+1} d^{O(k^4)}$. The degrees of the polynomials that appear in the output are bounded by $d^{O(k^3)}$. Moreover, if the input polynomials have integer coefficients whose bitsize is bounded by τ the bitsize of coefficients output is $d^{O(k^3)}\tau$.*

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